Advances in Difference Equations
a SpringerOpen Journal

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Existence of solutions for a coupled system of fractional p-Laplacian equations at resonance

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Abstract

In this paper, by using the extension of Mawhin's continuation theorem due to Ge, we study the existence of solutions for a coupled system of fractional p-Laplacian equations at resonance. A new result on the existence of solutions for a fractional boundary value problem is obtained.

MSC: 34B15

Keywords: fractional p-Laplacian equation; coupled system; boundary value problem; degree theory; resonance

1 Introduction

In the recent years, fractional differential equations have played an important role in many fields such as physics, electrical circuits, biology, control theory, *etc.* (see [1–9]). Recently, many scholars have paid more attention to boundary value problems for fractional differential equations (see [10–25]).

In [10], by means of a fixed point theorem on a cone, Agarwal *et al.* considered a two-point boundary value problem at nonresonance given by

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t, x(t), D_{0^+}^{\mu}x(t)) = 0, \\ x(0) = x(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $\mu > 0$ are real numbers, $\alpha - \mu \geq 1$ and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative.

By using the coincidence degree theory, Bai (see [20]) considered m-point fractional boundary value problems at resonance in the form

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1, \\ I_{0^+}^{2-\alpha}u(t)|_{t=0} = 0, \qquad u(1) = \sum_{i=1}^{m-2}\beta_i u(\eta_i), \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, $\beta_i \in \mathbb{R}$, $\eta_i \in (0,1)$ are given constants such that $\sum_{i=1}^{m-2}\beta_i\eta_i^{m-1} = 1$, and $D_{0^+}^{\alpha}$, $I_{0^+}^{\alpha}$ are the Riemann-Liouville differentiation and integration.

Moreover, the existence of solutions to a coupled system of fractional differential equations have been studied by many authors (see [26–33]).

In [28], relying on Schauder's fixed point theorem, Ahmad *et al.* considered a three-point boundary value problem for a coupled system of nonlinear fractional differential equations at nonresonance given by

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, v(t), D_{0^+}^{p} v(t)), & 0 < t < 1, \\ D_{0^+}^{\beta} v(t) = g(t, u(t), D_{0^+}^{q} u(t)), & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \gamma u(\eta), \quad v(0) = 0, \quad v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q \geq 1$, $\beta - p \geq 1$, $\gamma \eta^{\alpha-1} < 1$, $\gamma \eta^{\beta-1} < 1$, D is the standard Riemann-Liouville differentiation and $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In [33], by using the coincidence degree theory due to Mawhin, Jiang discussed the existence of solutions to a coupled system of fractional differential equations at resonance

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, v(t), D_{0^+}^{\delta} v(t)), \quad u(0) = 0, \quad D_{0^+}^{\gamma} u(1) = \sum_{i=1}^{n} a_i D_{0^+}^{\gamma} u(\xi_i), \\ D_{0^+}^{\beta} v(t) = g(t, u(t), D_{0^+}^{\gamma} u(t)), \quad v(0) = 0, \quad D_{0^+}^{\delta} v(1) = \sum_{i=1}^{n} a_i D_{0^+}^{\delta} v(\eta_i), \end{cases}$$

where $t \in [0,1]$, $1 < \alpha, \beta \leq 2$, $0 < \gamma \leq \alpha - 1$, $0 < \delta \leq \beta - 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson (see [34]) introduced the p-Laplacian equation as follows:

$$\big(\phi_p\big(x(t)\big)\big) = f\big(t, x(t), x(t)\big),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

In the past few decades, many important results relative to a p-Laplacian equation with certain boundary value conditions have been obtained. We refer the reader to [35–38] and the references cited therein. We noticed that ϕ_p is a quasi-linear operator. So, Mawhin's continuation theorem is not suitable for a p-Laplacian operator. In [39], Ge and Ren extended Mawhin's continuation theorem, which is used to deal with more general abstract operator equations.

Motivated by all the works above, in this paper, we consider the following boundary value problem (BVP for short) for a coupled system of fractional p-Laplacian equations given by

$$\begin{cases} D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} u(t)) = f(t, v(t), D_{0^+}^{\delta} v(t)), & t \in (0,1), \\ D_{0^+}^{\gamma} \phi_p(D_{0^+}^{\delta} v(t)) = g(t, u(t), D_{0^+}^{\alpha} u(t)), & t \in (0,1), \\ D_{0^+}^{\alpha} u(0) = D_{0^+}^{\alpha} u(1) = D_{0^+}^{\delta} v(0) = D_{0^+}^{\delta} v(1) = 0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$, $D_{0^+}^{\delta}$ are the standard Caputo fractional derivatives, $0 < \alpha, \delta, \beta, \gamma \leq 1$, $1 < \alpha + \beta < 2$, $1 < \delta + \gamma < 2$ and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on the existence of solutions for BVP (1.1) under nonlinear growth restriction of f and g, based on the extension of Mawhin's continuation theorem due to Ge (see [39]). Finally, in Section 4, an example is given to illustrate the main result.

2 Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper.

Definition 2.1 Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A continuous operator

$$M : X \cap \operatorname{dom} M \to Y$$

is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y,
(ii) $\operatorname{Ker} M := \{X \cap \operatorname{dom} M : Mu = 0\}$ is linearly homeomorphic to R^n, $n < \infty$.

Definition 2.2 Let X be a real Banach space and $\widehat{X} \subset X$. The operator $P : X \to \widehat{X}$ is said to be a projector provided $P^2 = P$, $P(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 P(x_1) + \lambda_2 P(x_2)$ for $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in \mathbb{R}$. The operator $Q : X \to \widehat{X}$ is said to be a semi-projector provided $Q^2 = Q$.

Definition 2.3 ([39]) Let $\widehat{X} = \operatorname{Ker} M$ and $\widetilde{X}$ be the complement space of $\widehat{X}$ in X, then $X = \widehat{X} \oplus \widetilde{X}$. On the other hand, suppose that $\widehat{Y}$ is a subspace of Y and $\widetilde{Y}$ is the complement space of $\widehat{Y}$ in Y so that $Y = \widehat{Y} \oplus \widetilde{Y}$. Let $P : X \to \widehat{X}$ be a projector and $Q : Y \to \widehat{Y}$ be a semi-projector, and let $\Omega \subset X$ be an open and bounded set with origin $\theta \in \Omega$, where θ is the origin of a linear space.

Suppose that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there is $\widehat{Y} \subset Y$ with $\dim \widehat{Y} = \dim \widehat{X}$ and an operator $R : \overline{\Omega} \times [0,1] \to X$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I - Q)Y, \tag{2.1}$$

$$QN_\lambda x = \theta, \quad \lambda \in (0,1) \quad \Leftrightarrow \quad QNx = \theta, \tag{2.2}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}, \tag{2.3}$$

$$M\big[P + R(\cdot, \lambda)\big] = (I - Q)N_\lambda. \tag{2.4}$$

Lemma 2.1 ([39], Ge-Mawhin's continuation theorem) *Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. $\Omega \subset X$ is an open and bounded nonempty set. Suppose that*

$$M : X \cap \operatorname{dom} M \to Y$$

is a quasi-linear operator and

$$N_\lambda : \overline{\Omega} \to Y, \quad \lambda \in [0,1]$$

is M-compact in $\overline{\Omega}$. In addition, if

(C_1) $Mx \neq N_\lambda x$, $\forall (x, \lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$,
(C_2) $QNx \neq 0$, *for* $x \in \operatorname{dom} M \cap \partial\Omega$,
(C_3) $\deg(JQN, \operatorname{Ker} M \cap \Omega, 0) \neq 0$,

where $N = N_1$ and $J : \widehat{Y} \to \widehat{X}$ is a homeomorphism with $J(\theta) = \theta$, then the equation $Mu = Nu$ has at least one solution in $\overline{\Omega}$.

Definition 2.4 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function x is given by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Definition 2.5 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is given by

$$D_{0^+}^{\alpha} x(t) = I_{0^+}^{n-\alpha} \frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds,$$

where n is the smallest integer greater than or equal to α, provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.2 [40] *Assume that $D_{0^+}^{\alpha} x \in C[0,1]$, $\alpha > 0$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i = -\frac{x^{(i)}(0)}{i!}$, $i = 0, 1, 2, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

Lemma 2.3 [40] *Assume that $\alpha > 0$ and $x \in C[0,1]$. Then*

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} x(t) = x(t).$$

In this paper, we denote $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, $X_1 = \{x | x, D_{0^+}^{\alpha} x \in Y\}$ with the norm $\|x\|_{X_1} = \max\{\|x\|_\infty, \|D_{0^+}^{\alpha} x\|_\infty\}$ and $X_2 = \{x | x, D_{0^+}^{\delta} x \in Y\}$ with the norm $\|x\|_{X_2} = \max\{\|x\|_\infty, \|D_{0^+}^{\delta} x\|_\infty\}$, where $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$. Then we denote $\overline{X} = X_1 \times X_2$ with the norm $\|(u,v)\|_{\overline{X}} = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}$ and $\overline{Y} = Y \times Y$ with the norm $\|(x,y)\|_{\overline{Y}} = \max\{\|x\|_Y, \|y\|_Y\}$. Obviously, both $\overline{X}$ and $\overline{Y}$ are Banach spaces.

Define the operator $M_1 : \operatorname{dom} M_1 \subset X_1 \to Y$ by

$$M_1 u = D_{0^+}^{\beta} \phi_p\bigl(D_{0^+}^{\alpha} u\bigr),$$

where

$$\operatorname{dom} M_1 = \bigl\{u \in X | D_{0^+}^{\beta} \phi_p\bigl(D_{0^+}^{\alpha} u\bigr) \in Y, D_{0^+}^{\alpha} u(0) = D_{0^+}^{\alpha} u(1) = 0\bigr\}.$$

Define the operator $M_2 : \operatorname{dom} M_2 \subset X_2 \to Y$ by

$$M_2 v = D_{0^+}^{\gamma} \phi_p\bigl(D_{0^+}^{\delta} v\bigr),$$

where

$$\operatorname{dom} M_2 = \{v \in X_2 | D_{0^+}^{\gamma} \phi_p(D_{0^+}^{\delta} v) \in Y, D_{0^+}^{\delta} v(0) = D_{0^+}^{\delta} v(1) = 0\}.$$

Define the operator $M : \operatorname{dom} M \subset \overline{X} \to \overline{Y}$ by

$$M(u,v) = (M_1 u, M_2 v), \tag{2.5}$$

where

$$\operatorname{dom} M = \{(u,v) \in \overline{X} | u \in \operatorname{dom} M_1, v \in \operatorname{dom} M_2\}.$$

Define the operator $N : \overline{X} \to \overline{Y}$ by

$$N(u,v) = (N^1 v, N^2 u),$$

where $N^1 : X_2 \to Y$

$$N^1 v(t) = f(t, v(t), D_{0^+}^{\delta} v(t))$$

and $N^2 : X_1 \to Y$

$$N^2 u(t) = g(t, u(t), D_{0^+}^{\alpha} u(t)).$$

Then BVP (1.1) is equivalent to the operator equation

$$M(u,v) = N(u,v), \quad (u,v) \in \operatorname{dom} M.$$

3 Main result

In this section, a theorem on the existence of solutions for BVP (1.1) will be given.

Theorem 3.1 *Let $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H$_1$) *there exist nonnegative functions $p_i, q_i, r_i \in C[0,1]$ $(i = 1,2)$ with*

$$\frac{1}{\Gamma(\beta+1)\Gamma(\gamma+1)} \left(\frac{2^{p-1} Q_1}{(\Gamma(\delta+1))^{p-1}} + R_1 \right) \left(\frac{2^{p-1} Q_2}{(\Gamma(\alpha+1))^{p-1}} + R_2 \right) < 1 \tag{3.1}$$

such that for all $(u,v) \in \mathbb{R}^2$, $t \in [0,1]$,

$$|f(t,u,v)| \le p_1(t) + q_1(t)|u|^{p-1} + r_1(t)|v|^{p-1}$$

and

$$|g(t,u,v)| \le p_2(t) + q_2(t)|u|^{p-1} + r_2(t)|v|^{p-1},$$

where $P_i = \|p_i\|_\infty$, $Q_i = \|q_i\|_\infty$, $R_i = \|r_i\|_\infty$ $(i = 1,2)$;

(H_2) *there exists a constant* $B > 0$ *such that for all* $t \in [0,1]$, $|u| > B$, $v \in \mathbb{R}$ *either*

$$uf(t,u,v) > 0, \qquad ug(t,u,v) > 0$$

or

$$uf(t,u,v) < 0, \qquad ug(t,u,v) < 0.$$

Then BVP (1.1) *has at least one solution.*

In order to prove Theorem 3.1, we need to prove some lemmas below.

Lemma 3.1 *Let* M *be defined by* (2.5), *then*

$$\operatorname{Ker} M = (\operatorname{Ker} M_1, \operatorname{Ker} M_2) = \{(u,v) \in \overline{X} | (u,v) = (a,b), a, b \in \mathbb{R}\}, \tag{3.2}$$

$$\begin{aligned} \operatorname{Im} M &= (\operatorname{Im} M_1, \operatorname{Im} M_2) \\ &= \left\{(x,y) \in \overline{Y} \,\middle|\, \int_0^1 (1-s)^{\beta-1} x(s)\,ds = 0, \int_0^1 (1-s)^{\gamma-1} y(s)\,ds = 0\right\}, \end{aligned} \tag{3.3}$$

and M *is a quasi-linear operator.*

Proof By Lemma 2.2, $M_1 u = D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} u) = 0$ has the solution

$$u(t) = u(0) + I_{0^+}^{\alpha} \phi_q(c_0) = u(0) + \frac{\phi_q(c_0)}{\Gamma(\alpha+1)} t^{\alpha}, \qquad c_0 = \phi_p\big(D_{0^+}^{\alpha} u(0)\big),$$

which satisfies

$$D_{0^+}^{\alpha} u(t) = \phi_q(c_0).$$

Combining with the boundary value condition $D_{0^+}^{\alpha} u(0) = 0$, we have

$$\operatorname{Ker} M_1 = \{u \in X_1 | u = a, a \in \mathbb{R}\}.$$

For $x \in \operatorname{Im} M_1$, there exists $u \in \operatorname{dom} M_1$ such that $x = M_1 u \in Y$. By Lemma 2.2, we have

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= \phi_q\big(I_{0^+}^{\beta} x(t) + c_0\big) \\ &= \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)\,ds + c_0\right). \end{aligned}$$

From the condition $D_{0^+}^{\alpha} u(0) = 0$, one has $c_0 = 0$. By the condition $D_{0^+}^{\alpha} u(1) = 0$, we obtain that

$$\int_0^1 (1-s)^{\beta-1} x(s)\,ds = 0. \tag{3.4}$$

On the other hand, suppose that $x \in Y$ and satisfies $\int_0^1 (1-s)^{\beta-1} x(s)\,ds = 0$. Let $u(t) = I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}x(t))$, then $u \in \operatorname{dom} M_1$. By Lemma 2.3, we have $D_{0^+}^{\alpha}u(t) = x(t)$. So that $x \in \operatorname{Im} M_1$. Then we have

$$\operatorname{Im} M_1 = \left\{ x \in Y \,\middle|\, \int_0^1 (1-s)^{\beta-1} x(s)\,ds = 0 \right\}.$$

Then we have $\dim \operatorname{Ker} M_1 = 1$ and $M_1(\operatorname{dom} M_1 \cap X_1) \subset Y$ closed. Therefore, M_1 is a quasi-linear operator. Similarly, we can get

$$\operatorname{Ker} M_2 = \{ v \in X_2 \mid v = b, b \in \mathbb{R} \},$$
$$\operatorname{Im} M_2 = \left\{ y \in Y \,\middle|\, \int_0^1 (1-s)^{\gamma-1} y(s)\,ds = 0 \right\},$$

and M_2 is a quasi-linear operator. Then the proof is complete. □

Lemma 3.2 *Let $\Omega \subset \overline{X}$ be an open and bounded set, then N_λ is M-compact in $\overline{\Omega}$.*

Proof Define the continuous projector $P : \overline{X} \to \widehat{X}$ and the semi-projector $Q : \overline{Y} \to \widehat{Y}$

$$P(u,v) = (P_1 u, P_2 v) = \big(u(0), v(0)\big),$$
$$Q(x,y) = (Q_1 x, Q_2 y) = \left(\beta \int_0^1 (1-s)^{\beta-1} x(s)\,ds, \gamma \int_0^1 (1-s)^{\gamma-1} y(s)\,ds \right),$$

where $\widehat{X} = \operatorname{Ker} M$ and $\widehat{Y} = \operatorname{Im} Q$.

Obviously, $\operatorname{Im} P = \operatorname{Ker} M$ and $P^2(u,v) = P(u,v)$. It follows from $(u,v) = ((u,v) - P(u,v)) + P(u,v)$ that $\overline{X} = \operatorname{Ker} P + \operatorname{Ker} M$. By a simple calculation, we can get that $\operatorname{Ker} M \cap \operatorname{Ker} P = \{(0,0)\}$. Then we get

$$\overline{X} = \operatorname{Ker} M \oplus \operatorname{Ker} P = \widehat{X} \oplus \widetilde{X}.$$

For $(x,y) \in \overline{Y}$, we have

$$Q^2(x,y) = Q(Q_1 x, Q_2 y) = \big(Q_1^2 x, Q_2^2 y\big).$$

By the definition of Q_1, we can get

$$Q_1^2 x = Q_1 x \cdot \beta \int_0^1 (1-s)^{\beta-1}\,ds = Q_1 x.$$

Similar proof can show that $Q_2^2 y = Q_2 y$. Thus, we have $Q^2(x,y) = Q(x,y)$.

Let $(x,y) = ((x,y) - Q(x,y)) + Q(x,y)$, where $(x,y) - Q(x,y) \in \operatorname{Ker} Q = \operatorname{Im} M$, $Q(x,y) \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} M$ and $Q^2(x,y) = Q(x,y)$ that $\operatorname{Im} Q \cap \operatorname{Im} M = \{(0,0)\}$. Then we have

$$\overline{Y} = \operatorname{Im} Q \oplus \operatorname{Im} M = \widehat{Y} \oplus \widetilde{Y}.$$

Thus

$$\dim \widehat{X} = \dim \operatorname{Ker} M = \dim \operatorname{Im} Q = \dim \widehat{Y}.$$

Let $\Omega \subset \overline{X}$ be an open and bounded set with $(\theta,\theta) \in \Omega$. For each $(u,v) \in \overline{\Omega}$, we can get $Q[(I-Q)N_\lambda(u,v)] = 0$. Thus, $(I-Q)N_\lambda(u,v) \in \operatorname{Im} M = \operatorname{Ker} Q$. Take any $(x,y) \in \operatorname{Im} M$ in the type $(x,y) = ((x,y) - Q(x,y)) + Q(x,y)$. Since $Q(x,y) = 0$, we can get $(I-Q)(x,y) \in \overline{Y}$. So (2.1) holds. It is easy to verify (2.2).

Furthermore, define $R = (R_1, R_2) : \overline{\Omega} \times [0,1] \to \widetilde{X}$ by

$$\begin{aligned} R_1(u,\lambda)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}\big((I-Q_1)N_\lambda^1 v(\tau)\big)\,d\tau\right) ds, \\ R_2(v,\lambda)(t) &= \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \phi_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1}\big((I-Q_2)N_\lambda^2 u(\tau)\big)\,d\tau\right) ds. \end{aligned}$$

By the continuity of f and g, it is easy to get that $R(u,v,\lambda)$ is continuous on $\overline{\Omega} \times [0,1]$. Moreover, for all $(u,v) \in \overline{\Omega}$, there exists a constant $T > 0$ such that $\max\{|I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(\tau)|, |I_{0^+}^\gamma (I-Q_2)N_\lambda^2 u(\tau)|\} \le T$, so we can easily obtain that $R(\overline{\Omega},\lambda)$ is uniformly bounded. By the Arzela-Ascoli theorem, we just need to prove that $R : \overline{\Omega} \times [0,1] \to \widetilde{X}$ is equicontinuous. Furthermore, for $0 \le t_1 < t_2 \le 1$, $(u,v,\lambda) \in \overline{\Omega} \times [0,1] = (\overline{\Omega}_1, \overline{\Omega}_2) \times [0,1]$, we have

$$\begin{aligned} &\big|R(u,v,\lambda)(t_2) - R(u,v,\lambda)(t_1)\big| \\ &\quad = \big|\big(I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_2)\big), I_{0^+}^\delta \phi_q\big(I_{0^+}^\gamma (I-Q_2)N_\lambda^2 u(t_2)\big)\big) \\ &\qquad - \big(I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_1)\big), I_{0^+}^\delta \phi_q\big(I_{0^+}^\gamma (I-Q_2)N_\lambda^2 u(t_1)\big)\big)\big| \\ &\quad = \big|\big(I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_2)\big) - I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_1)\big), \\ &\qquad I_{0^+}^\delta \phi_q\big(I_{0^+}^\gamma (I-Q_2)N_\lambda^2 u(t_2)\big) - I_{0^+}^\delta \phi_q\big(I_{0^+}^\gamma (I-Q_2)N_\lambda^2 u(t_1)\big)\big)\big|. \end{aligned}$$

By $|I_{0^+}^\beta (I-Q_1)N_\lambda^1 v| \le T$, we have

$$\begin{aligned} &\big|I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_2)\big) - I_{0^+}^\alpha \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(t_1)\big)\big| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \bigg| \int_0^{t_2} (t_2-s)^{\alpha-1} \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(s)\big)\,ds \\ &\qquad - \int_0^{t_1} (t_1-s)^{\alpha-1} \phi_q\big(I_{0^+}^\beta (I-Q_1)N_\lambda^1 v(s)\big)\,ds \bigg| \\ &\quad \le \frac{\phi_q(T)}{\Gamma(\alpha)} \left[\int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\,ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds\right] \\ &\quad = \frac{\phi_q(T)}{\Gamma(\alpha+1)} \big(t_2^\alpha - t_1^\alpha\big). \end{aligned}$$

Since t^α is uniformly continuous on $[0,1]$, so $R_1(\overline{\Omega}_1,\lambda)$ is equicontinuous. Similarly, we can get $I_{0^+}^\beta((I-Q_1)N_\lambda^1 v(\tau)) \subset C[0,1]$ is equicontinuous. Considering that $\phi_q(s)$ is uniformly continuous on $[-T,T]$, we have $D_{0^+}^\alpha R_1(\overline{\Omega}_1,\lambda) = I_{0^+}^\beta((I-Q_1)N_\lambda^1(\overline{\Omega}))$ is also equicontinuous. So, we can obtain that $R_1(\overline{\Omega}_1,\lambda) \to \widetilde{X}_1$ is compact.

Similarly, we can get that $R_2(\overline{\Omega}_2,\lambda) \to \widetilde{X}_2$ is compact. So, we can obtain that $R : \overline{\Omega} \times [0,1] \to \widetilde{X}$ is compact.

For each $(u,v) \in \Sigma_\lambda = \{(u,v) \in \overline{\Omega} : M(u,v) = N_\lambda(u,v)\}$, we have $(D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}u(t)), D_{0^+}^{\gamma}\phi_p(D_{0^+}^{\delta}v(t))) = N_\lambda(u(t),v(t)) \in \operatorname{Im} M$. Thus,

$$\begin{aligned} R_1(u,\lambda)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\big((I-Q_1)N_\lambda^1 v(\tau)\big)\,d\tau\right)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha}u(\tau)\big)\,d\tau\right)ds, \end{aligned}$$

which together with $D_{0^+}^{\alpha}u(0) = 0$ yields that

$$R_1(u,\lambda)(t) = u(t) - u(0) = \big[(I-P_1)u\big](t).$$

It is easy to verify that $R_1(u,0)(t)$ is the zero operator. Similarly, we can get $R_2(v,\lambda)(t) = [(I-P_2)v](t)$ and $R_2(v,0)(t)$ is the zero operator. So (2.3) holds.

On the other hand,

$$\begin{aligned} &M_1\big[P_1u + R_1(u,\lambda)\big](t) \\ &\quad = M_1\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\big((I-Q_1)N_\lambda^1 v(\tau)\big)\,d\tau\right)ds + u(0)\right] \\ &\quad = \big[\big((I-Q_1)N_\lambda^1\big)v\big](t). \end{aligned}$$

Similarly, we have $M_2[P_2v + R_2(v,\lambda)](t) = [((I-Q_2)N_\lambda^2)u](t)$. So, (2.4) holds. Then we have that N_λ is M-compact in $\overline{\Omega}$. The proof is complete. □

Lemma 3.3 *Suppose that* (H_1), (H_2) *hold, then the set*

$$\Omega_1 = \big\{(u,v) \in \operatorname{dom} M \setminus \operatorname{Ker} M \mid M(u,v) = \lambda N(u,v), \lambda \in (0,1)\big\}$$

is bounded.

Proof Take $(u,v) \in \Omega_1$, then $N(u,v) \in \operatorname{Im} M$. By (3.3), we have

$$\begin{aligned} &\int_0^1 (1-s)^{\beta-1} f\big(s, v(s), D_{0^+}^{\delta}v(s)\big)\,ds = 0, \\ &\int_0^1 (1-s)^{\gamma-1} g\big(s, u(s), D_{0^+}^{\alpha}u(s)\big)\,ds = 0. \end{aligned}$$

Then, by the integral mean value theorem, there exist constants $\xi, \eta \in (0,1)$ such that $f(\xi, v(\xi), D_{0^+}^{\delta}v(\xi)) = 0$ and $g(\eta, u(\eta), D_{0^+}^{\alpha}u(\eta)) = 0$. So, from (H_2), we get $|v(\xi)| \le B$ and $|u(\eta)| \le B$.

By Lemma 2.2,

$$\begin{aligned} v(t) &= v(0) + I_{0^+}^{\delta}D_{0^+}^{\delta}v(t) \\ &= v(0) + \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}D_{0^+}^{\delta}v(s)\,ds. \end{aligned}$$

Take $t = \xi$, we have

$$v(\xi) = v(0) + \frac{1}{\Gamma(\delta)} \int_0^{\xi} (\xi - s)^{\delta-1} D_{0^+}^{\delta} v(s)\, ds.$$

Then we have

$$\begin{aligned} |v(0)| &\le |v(\xi)| + \frac{1}{\Gamma(\delta)} \int_0^{\xi} (\xi - s)^{\delta-1} |D_{0^+}^{\delta} v(s)|\, ds \\ &\le |v(\xi)| + \frac{1}{\Gamma(\delta)} \|D_{0^+}^{\delta} v\|_\infty \cdot \frac{1}{\delta} \xi^{\delta} \\ &\le B + \frac{1}{\Gamma(\delta+1)} \|D_{0^+}^{\delta} v\|_\infty. \end{aligned}$$

So, we get

$$\begin{aligned} |v(t)| &\le |v(0)| + \frac{1}{\Gamma(\delta)} \int_0^{t} (t - s)^{\delta-1} |D_{0^+}^{\delta} v(s)|\, ds \\ &\le |v(0)| + \frac{1}{\Gamma(\delta)} \|D_{0^+}^{\delta} v\|_\infty \cdot \frac{1}{\delta} t^{\delta} \\ &\le B + \frac{2}{\Gamma(\delta+1)} \|D_{0^+}^{\delta} v\|_\infty, \quad \forall t \in [0,1]. \end{aligned}$$

That is,

$$\|v\|_\infty \le B + \frac{2}{\Gamma(\delta+1)} \|D_{0^+}^{\delta} v\|_\infty. \tag{3.5}$$

Similarly, we can get

$$\|u\|_\infty \le B + \frac{2}{\Gamma(\alpha+1)} \|D_{0^+}^{\alpha} u\|_\infty. \tag{3.6}$$

By $M(u,v) = \lambda N(u,v)$ and $D_{0^+}^{\alpha} u(0) = D_{0^+}^{\delta} v(0) = 0$, we get

$$\begin{aligned} \phi_p\big(D_{0^+}^{\alpha} u(t)\big) &= \lambda I_{0^+}^{\beta} N^1 v(t) \\ &= \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, v(s), D_{0^+}^{\delta} v(s)\big)\, ds. \end{aligned}$$

So, from (H_1), we have

$$\begin{aligned} \big|\phi_p\big(D_{0^+}^{\alpha} u(t)\big)\big| &\le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big|f\big(s, v(s), D_{0^+}^{\delta} v(s)\big)\big|\, ds \\ &\le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big(p_1(s) + q_1(s)|v(s)|^{p-1} \\ &\quad + r_1(s)|D_{0^+}^{\delta} v(s)|^{p-1}\big)\, ds \\ &\le \frac{1}{\Gamma(\beta)} \big(\|p_1\|_\infty + \|q_1\|_\infty \|v\|_\infty^{p-1} + \|r_1\|_\infty \|D_{0^+}^{\delta} v\|_\infty^{p-1}\big) \cdot \frac{1}{\beta} t^{\beta} \\ &\le \frac{1}{\Gamma(\beta+1)} \big(P_1 + Q_1 \|v\|_\infty^{p-1} + R_1 \|D_{0^+}^{\delta} v\|_\infty^{p-1}\big), \end{aligned}$$

which together with $|\phi_p(D^\alpha_{0^+}u(t))| = |D^\alpha_{0^+}u(t)|^{p-1}$ and (3.5) yields that

$$\|D^\alpha_{0^+}u\|_\infty^{p-1} \le \frac{1}{\Gamma(\beta+1)}\left[P_1 + Q_1\left(B + \frac{2}{\Gamma(\delta+1)}\|D^\delta_{0^+}v\|_\infty\right)^{p-1} + R_1\|D^\delta_{0^+}v\|_\infty^{p-1}\right]. \tag{3.7}$$

Similarly, we can get

$$\|D^\delta_{0^+}v\|_\infty^{p-1} \le \frac{1}{\Gamma(\gamma+1)}\left[P_2 + Q_2\left(B + \frac{2}{\Gamma(\alpha+1)}\|D^\alpha_{0^+}u\|_\infty\right)^{p-1} + R_2\|D^\alpha_{0^+}u\|_\infty^{p-1}\right]. \tag{3.8}$$

Then from (3.1), (3.7) and (3.8), we can see that there exists a constant $M_1 > 0$ such that

$$\|D^\alpha_{0^+}u\|_\infty, \|D^\delta_{0^+}v\|_\infty \le M_1. \tag{3.9}$$

Thus, from (3.5) and (3.6), we get

$$\|u\|_\infty, \|v\|_\infty \le \max\left\{B + \frac{2M_1}{\Gamma(\alpha+1)}, B + \frac{2M_1}{\Gamma(\delta+1)}\right\} := M_2. \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\|(u,v)\|_{\overline{X}} \le \max\{M_1, M_2\} := M.$$

So, Ω_1 is bounded. The proof is complete. □

Lemma 3.4 *Suppose that* (H_3) *holds, then the set*

$$\Omega_2 = \{(u,v) | (u,v) \in \operatorname{Ker} M, N(u,v) \in \operatorname{Im} M\}$$

is bounded.

Proof For $(u,v) \in \Omega_2$, we have $(u,v) = (a,b)$. Then, from $N(u,v) \in \operatorname{Im} M$, we get

$$\int_0^1 (1-s)^{\beta-1} f(s,b,0)\,ds = 0,$$
$$\int_0^1 (1-s)^{\gamma-1} g(s,a,0)\,ds = 0,$$

which together with (H_2) implies $|a|, |b| \le B$. Thus, we have

$$\|(u,v)\|_{\overline{X}} \le B.$$

Hence, Ω_2 is bounded. The proof is complete. □

Lemma 3.5 *Suppose that the first part of* (H_2) *holds, then the set*

$$\Omega_3 = \{(u,v) \in \operatorname{Ker} M | \lambda J^{-1}(u,v) + (1-\lambda) QN(u,v) = (0,0), \lambda \in [0,1]\}$$

is bounded, where $J^{-1}: \operatorname{Ker} M \to \operatorname{Im} Q$ *is a homeomorphism defined by*

$$J^{-1}(a,b) = (b,a), \quad a,b \in \mathbf{R}.$$

Proof For $(u,v) \in \Omega_3$, we have $(u,v) = (a,b)$ and

$$\lambda b + (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} f(s,b,0)\,ds = 0, \tag{3.11}$$

$$\lambda a + (1-\lambda)\gamma \int_0^1 (1-s)^{\gamma-1} g(s,a,0)\,ds = 0. \tag{3.12}$$

If $\lambda = 1$, then $a = b = 0$. For $\lambda \in [0,1)$, we can obtain $|a|, |b| \le B$. Otherwise, if $|a|$ or $|b| > B$, in view of the first part of (H_2), one has

$$\lambda b^2 + (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} bf(s,b,0)\,ds > 0,$$

or

$$\lambda a^2 + (1-\lambda)\gamma \int_0^1 (1-s)^{\gamma-1} ag(s,a,0)\,ds > 0,$$

which contradicts (3.11) or (3.12). Therefore, Ω_3 is bounded. The proof is complete. □

Remark 3.1 If the second part of (H_2) holds, then the set

$$\Omega_3' = \bigl\{(u,v) \in \operatorname{Ker} M \mid -\lambda J^{-1}(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\bigr\}$$

is bounded.

Proof of Theorem 3.1 Set $\Omega = \{(u,v) \in \overline{X} \mid \|(u,v)\|_{\overline{X}} < \max\{M,B\} + 1\}$. It follows from Lemmas 3.1 and 3.2 that M is a quasi-linear operator and N_λ is M-compact on $\overline{\Omega}$. By Lemmas 3.3 and 3.4, we get that the following two conditions are satisfied:

(C_1) $Mx \neq N_\lambda x$, $\forall (x,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$,
(C_2) $QNx \neq 0$, for $x \in \operatorname{dom} M \cap \partial\Omega$.

Take

$$H\bigl((u,v),\lambda\bigr) = \pm\lambda(u,v) + (1-\lambda)JQN(u,v).$$

According to Lemma 3.5 (or Remark 3.1), we know that $H((u,v),\lambda) \neq 0$ for $(u,v) \in \operatorname{Ker} M \cap \partial\Omega$. Therefore

$$\begin{aligned}\deg\bigl(JQN|_{\operatorname{Ker} M}, \Omega \cap \operatorname{Ker} M, (0,0)\bigr) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} M, (0,0)\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} M, (0,0)\bigr) \\ &= \deg\bigl(\pm I, \Omega \cap \operatorname{Ker} M, (0,0)\bigr) \neq 0.\end{aligned}$$

So, condition (C_3) of Lemma 2.1 is satisfied. By Lemma 2.1, we can get that $M(u,v) = N(u,v)$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$. Therefore BVP (1.1) has at least one solution. The proof is complete. □

4 Example

Example 4.1 Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{3}{4}}\phi_3(D_{0^+}^{\frac{1}{2}}u(t)) = -\frac{25}{16} + \frac{1}{16}v^2(t) + te^{-|D_{0^+}^{\frac{4}{5}}v(t)|}, & t \in (0,1), \\ D_{0^+}^{\frac{1}{4}}\phi_3(D_{0^+}^{\frac{4}{5}}v(t)) = \frac{30}{17} + \frac{1}{17}u^2(t) + \sin^2(D_{0^+}^{\frac{1}{2}}u(t)), & t \in (0,1), \\ D_{0^+}^{\frac{1}{2}}u(0) = D_{0^+}^{\frac{1}{2}}u(1) = D_{0^+}^{\frac{4}{5}}v(0) = D_{0^+}^{\frac{4}{5}}v(1) = 0. \end{cases} \tag{4.1}$$

Corresponding to BVP (1.1), we have that $p = 3$, $\alpha = \frac{1}{2}$, $\delta = \frac{4}{5}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{4}$ and

$$f(t,u,v) = -\frac{25}{16} + \frac{1}{16}u^2 + te^{-|v|},$$
$$g(t,u,v) = \frac{30}{17} + \frac{1}{17}u^2 + \sin^2 v.$$

Choose $p_1(t) = p_2(t) = 10$, $q_1(t) = \frac{1}{16}$, $q_2(t) = \frac{1}{17}$, $r_1(t) = r_2(t) = 0$, $B = 5$. Then we have $P_1 = P_2 = 10$, $Q_1 = \frac{1}{16}$, $Q_2 = \frac{1}{17}$, $R_1(t) = R_2(t) = 0$. By a simple calculation, we get

$$\frac{1}{\Gamma(\frac{3}{4}+1)\Gamma(\frac{1}{4}+1)}\left(\frac{2^2\frac{1}{16}}{(\Gamma(\frac{4}{5}+1))^2}\right)\left(\frac{2^2\frac{1}{17}}{(\Gamma(\frac{1}{2}+1))^2}\right) < 1.$$

Then (H_1) and the first part of (H_2) hold.

By Theorem 3.1, we obtain that BVP (4.1) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. All authors read and approved the final manuscript.

Acknowledgements
The authors are grateful to those who gave useful suggestions about the original manuscript. This research was supported by the Fundamental Research Funds for the Central Universities (2013QNA33).

Received: 24 May 2013 Accepted: 2 October 2013 Published: 08 Nov 2013

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10.1186/1687-1847-2013-312
Cite this article as: Hu et al.: **Existence of solutions for a coupled system of fractional *p*-Laplacian equations at resonance.** *Advances in Difference Equations* 2013, 2013:312