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On a model of magnetization dynamics with vertical spin stiffness

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Abstract
We consider a mathematical model describing magnetization dynamics with vertical spin stiffness. The model consists of a modified form of the Landau-Lifshitz-Gilbert equation for the evolution of the magnetization vector in a rigid ferromagnet. The modification lies in the presence in the effective field of a nonlinear term describing vertical spin stiffness. We prove the global existence of weak solutions to the model by using the Faedo-Galerkin method and discuss the limit of the obtained solutions as the vertical spin stiffness parameter tends to zero.

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1 Introduction and preliminary result

Ferromagnetic systems have attracted much interest for a long time because of the intriguing physics and applications [1]. Because of the development of information technology, the research on magnetization dynamics in micro-magnets has become an active field [2]. Great efforts have been devoted to this field by aiming to manipulate magnetization more efficiently [3]. In [4], it is showed that vertical spin stiffness can significantly modify the domain-wall structure in ferromagnetic semiconductors and hence should be included in the Landau-Lifshitz-Gilbert (LLG) equation in studying the magnetization dynamics.

The present work deals with magnetization dynamics in the presence of vertical spin stiffness. We shall adopt the model derived in [4], which consists of a modified LLG equation where the modification lies in the presence of a second-order gradient term in the effective field. To describe the model equations, we consider $\Omega \subset \mathbb{R}^3$ a bounded and regular open set of $\mathbb{R}^3$. The generic point of $\mathbb{R}^3$ is denoted by $x = (x_1, x_2, x_3)$. We assume that a ferromagnetic material occupies the domain Ω.

The magnetization field of the ferromagnetic material which belongs to S^2 (the unit sphere of $\mathbb{R}^3$) almost everywhere is denoted by $\mathbf{m}(x,t)$. Its evolution is governed by the following modified LLG equation (see [4]):

$$\begin{cases} \partial_t \mathbf{m} = \alpha \mathbf{m} \times \partial_t \mathbf{m} - (1+\alpha^2)\mathbf{m} \times \mathcal{H}_{\text{eff}}(\mathbf{m}) \quad \text{in } Q = (0,T) \times \Omega, \\ \mathbf{m}(\cdot, 0) = \mathbf{m}_0, \qquad |\mathbf{m}_0| = 1 \quad \text{in } \Omega, \\ \partial_\nu \mathbf{m} = 0 \quad \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1}$$

where $T > 0$, the symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$ and $\partial_\nu \mathbf{m}$ denotes the outward normal derivative of $\mathbf{m}$ on the boundary of Ω. The positive constant α represents the damping parameter. The effective magnetic field $\mathcal{H}_{\text{eff}}$ depends on $\mathbf{m}$ and is given by

$$\mathcal{H}_{\text{eff}}(\mathbf{m}) = a\Delta\mathbf{m} + b\mathbf{m} \times \Delta\mathbf{m}. \tag{2}$$

The first term on the right-hand side of (2) is called the exchange magnetic field, where the positive constant a is the exchange coefficient. The term parameterized by the positive constant b stands for vertical spin stiffness field.

Remark 1 Since we will focus on only the new term parameterized by b and for the sake of simplicity, anisotropy field (which is generally taken linear with respect to $\mathbf{m}$) and demagnetizing field are neglected. However, we note that these simplifications do not limit the proposed analysis.

Throughout, we make use of the following notation. For Ω an open bounded domain of $\mathbb{R}^3$, we denote by $\mathbb{L}^p(\Omega) = (L^p(\Omega))^3$ and $\mathbb{H}^1(\Omega) = (H^1(\Omega))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $\|\cdot\|_{\mathbb{L}^p(\Omega)}$ and $\|\cdot\|_{\mathbb{H}^1(\Omega)}$.

Lemma 1 *If* $\mathbf{m}$ *is a regular solution of the problem* (1) *then we have for all* $t \in (0,T)$ *the following energy estimate*:

$$\frac{\beta + \alpha\lambda}{2} \int_\Omega \left|\nabla\mathbf{m}(t)\right|^2 \mathrm{d}x + \alpha \int_0^t \int_\Omega |\partial_t \mathbf{m}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{\beta + \alpha\lambda}{2} \int_\Omega |\nabla\mathbf{m}_0|^2 \,\mathrm{d}x,$$

where $\beta = a(1+\alpha^2)$ *and* $\lambda = b(1+\alpha^2)$.

Proof By using the saturation constraint, the LLG equation (1) can be written in the following form:

$$\alpha\partial_t\mathbf{m} + \mathbf{m} \times \partial_t\mathbf{m} - \beta\Delta\mathbf{m} - \lambda\mathbf{m} \times \Delta\mathbf{m} - \beta|\nabla\mathbf{m}|^2\mathbf{m} = 0. \tag{3}$$

Taking the inner product of (3) by $\partial_t\mathbf{m}$ and $\Delta\mathbf{m}$, respectively, we get

$$\alpha \int_\Omega |\partial_t\mathbf{m}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla\mathbf{m}|^2 \,\mathrm{d}x - \lambda \int_\Omega \mathbf{m} \times \Delta\mathbf{m} \cdot \partial_t\mathbf{m} \,\mathrm{d}x = 0 \tag{4}$$

and

$$\begin{aligned} &-\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla\mathbf{m}|^2 \,\mathrm{d}x + \int_\Omega \mathbf{m} \times \partial_t\mathbf{m} \cdot \Delta\mathbf{m} \,\mathrm{d}x \\ &-\beta \int_\Omega |\Delta\mathbf{m}|^2 \,\mathrm{d}x + \beta \int_\Omega (\mathbf{m} \cdot \Delta\mathbf{m})^2 \,\mathrm{d}x = 0. \end{aligned} \tag{5}$$

Combining (4) and (5), we get

$$\frac{\beta + \alpha\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla\mathbf{m}|^2 \,\mathrm{d}x + \alpha \int_\Omega |\partial_t\mathbf{m}|^2 \,\mathrm{d}x + \lambda\beta \int_\Omega |\Delta\mathbf{m}|^2 \,\mathrm{d}x = \lambda\beta \int_\Omega (\mathbf{m} \cdot \Delta\mathbf{m})^2 \,\mathrm{d}x.$$

Since

$$\int_\Omega (\mathbf{m}\cdot\Delta\mathbf{m})^2\,\mathrm{d}x \le \int_\Omega |\Delta\mathbf{m}|^2\,\mathrm{d}x,$$

integrating from 0 to t, we get

$$\frac{\beta+\alpha\lambda}{2}\int_\Omega \bigl|\nabla\mathbf{m}(t)\bigr|^2\,\mathrm{d}x + \alpha\int_0^t\int_\Omega |\partial_t\mathbf{m}|^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{\beta+\alpha\lambda}{2}\int_\Omega |\nabla\mathbf{m}_0|^2\,\mathrm{d}x$$

for all $t\in(0,T)$. This completes the proof of the lemma. □

Before dealing with the existence of finite energy global weak solutions to the problem (1), let us first review some previous results. We limit ourselves to mentioning a handful of references concerning the existence and we refer to the survey [5] for a more detailed bibliographical account. The general framework (although without vertical spin stiffness, *i.e.* the case where $b = 0$) has been established in earlier papers; see for instance [6–8], using the FGP method. This method gives an approximate sequence of solutions converging to a global solution of the problem. The next results concern systems with further dissipation terms. For example, in [9], the LLG equation with a regularizing term of the type $\Delta\partial_t\mathbf{m}$ is considered and an existence theorem which rests on a preliminary penalty/regularization is proved. The modification considered in [10] consists in adding to the standard dissipation term in the LLG equation another higher-order term of the type $\Delta\Delta\mathbf{m}$. The FGP method is also used to solve the problem. In [11], a model with dry-friction dissipation which is accounted by adding a dry-friction-like term to the standard Gilbert damping is studied. Using the notion of subdifferential of a convex function, this dissipation is written as $r\in\partial R_{\alpha,\beta}(\partial_t\mathbf{m})$ where $R_{\alpha,\beta}(a) := \frac{\alpha}{2}|a|^2 + \beta|a|$ for all $a\in\mathbb{R}^3$. To prove the existence of weak solutions, a strategy slightly different from [9] is adopted. It consists of a penalization of the saturation constraint, adding (for regularization) an exchange-type dissipation $\varepsilon\Delta\partial_t\mathbf{m}$ to the effective field and passing to the limit as $\varepsilon\to 0$. Another LLG model with inertial effects was considered in [12] and global existence established. In this model the modification lies in the presence of a second-order time derivative of magnetization in the effective field. Let us mention that in the framework of current-induced magnetization switching, [13] addresses the global existence of weak solutions to LLG model with a transport-type term in the effective field. All these proofs are based on some penalization and using various kind of regularizations. What is new in this work is the last term in equation (2), which has never been previously treated. It represents a perpendicular magnetic field to the spin stiffness (see [4]). This term can significantly alter the structure of domain walls in ferromagnetic semiconductors. The main problem here is that this stiffness of rotation cannot be written in terms of free energy, and therefore cannot be established from a functional derivative from the free energy with respect to the local magnetization. We finally mention that significant progress was made to design schemes constructing the weak solutions to the general LLG equation. Several schemes were proposed and their convergence to weak solutions was proved. A significant step forward in the convergence theory of numerical schemes has been made recently; see [14–16]. This will be helpful to give a strategy for efficient computer implementation which may reflect the true nature of the augmentation of the LLG model considered in this paper.

The rest of the paper is divided as follows. In the next section we prove the global existence of weak solutions to the model by using the Faedo-Galerkin method. The last section reveals the relationships between the LLG equation we have studied in this paper and the classical LLG equation *i.e.*, without vertical spin stiffness field.

2 Global existence of weak solutions

Let us first give the definition of weak solutions to problem (1).

Definition 1 Let $\mathbf{m}_0 \in \mathbb{H}^1(\Omega)$ with $|\mathbf{m}_0| = 1$ a.e., we say that a three dimensional vector $\mathbf{m} = (m_1, m_2, m_3)$ is a weak solution of problem (1) if

- for all $T > 0$, $\mathbf{m} \in \mathbb{H}^1(Q)$, $\partial_t \mathbf{m} \in \mathbb{L}^2(Q)$, and $|\mathbf{m}| = 1$ a.e. in Q;
- for all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$ with $\boldsymbol{\phi}(\cdot, 0) = \boldsymbol{\phi}(\cdot, T) = 0$, we have

$$\begin{aligned}&\int_Q \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t - \alpha \int_Q \mathbf{m} \times \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \beta \int_Q \mathbf{m} \times \nabla \mathbf{m} \cdot \nabla \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t + \lambda \int_Q \mathbf{m} \times \Delta \mathbf{m} \cdot \mathbf{m} \times \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t;\end{aligned} \tag{6}$$

- $\mathbf{m}(x, 0) = \mathbf{m}_0(x)$ in the trace sense;
- for all $t \in (0, T)$, there holds

$$\begin{aligned}&\frac{\beta + \alpha\lambda}{2} \int_\Omega \left|\nabla \mathbf{m}(t)\right|^2 \mathrm{d}x + \alpha \int_0^t \int_\Omega |\partial_t \mathbf{m}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \frac{\beta + \alpha\lambda}{2} \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x.\end{aligned} \tag{7}$$

Remark 2 We will show in Section 2.2 that $\mathbf{m} \times \Delta \mathbf{m}$ makes sense in $\mathbb{L}^2(Q)$, and for this reason, it will be clear that (6) makes sense.

To prove the global existence of weak solutions of the problem (1) we proceed as in [6–8].

2.1 The penalty problem

Let $\varepsilon > 0$. We introduce the following penalty problem.

For initial datum $\mathbf{m}_0 \in \mathbb{H}^1(\Omega)$, and for each positive number T, find a vector field $\mathbf{m}_\varepsilon$ in Q such as to satisfy the equation

$$\partial_t \mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon + \beta \Delta \mathbf{m}^\varepsilon + \lambda \mathbf{m}^\varepsilon \times \Delta \mathbf{m}^\varepsilon - \alpha \partial_t \mathbf{m}^\varepsilon - \frac{1}{\varepsilon}\left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right)\mathbf{m}^\varepsilon = 0 \tag{8}$$

subject to the initial and boundary conditions

$$\begin{cases} \mathbf{m}^\varepsilon(\cdot, 0) = \mathbf{m}_0, \qquad |\mathbf{m}_0| = 1 \quad \text{in } \Omega, \\ \partial_\nu \mathbf{m}^\varepsilon = 0 \quad \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{9}$$

The last term of equation (8) was introduced at the end to represent the constraint $|\mathbf{m}| = 1$.

We have the following result.

Theorem 1 *For each fixed positive* ε, *there is a weak solution* $\mathbf{m}^\varepsilon$ *of Problem* (8)-(9) *such that*

$$\int_Q \partial_t \mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \beta \int_Q \nabla \mathbf{m}^\varepsilon \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \lambda \int_Q \mathbf{m}^\varepsilon \times \nabla \mathbf{m}^\varepsilon \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$
$$- \alpha \int_Q \partial_t \mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon} \int_Q \left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right) \mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = 0$$

for any $\boldsymbol{\varphi}$ *in* $\mathbb{H}^1(Q)$. *Moreover, the following energy estimate holds*:

$$\frac{\beta + \alpha\lambda}{2} \int_\Omega \left|\nabla \mathbf{m}^\varepsilon\right|^2(t) \,\mathrm{d}x + \alpha \int_0^t \int_\Omega \left|\partial_t \mathbf{m}^\varepsilon\right|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{4\varepsilon} \left(1 + \frac{\alpha\lambda}{\beta}\right) \int_\Omega \left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right)^2(t) \,\mathrm{d}x \le \frac{\beta + \alpha\lambda}{2} \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x \tag{10}$$

for all $t \in (0, T)$.

Proof To show the existence of a solution for the penalized problem using the method of Faedo-Galerkin and since $\mathbb{H}^1(\Omega)$ is a separable Hilbert space we can approximate $\mathbf{m}$ by $\mathbf{m}^{\varepsilon,N}$. We set

$$\mathbf{m}^{\varepsilon,N}(x,t) = \sum_{i=1}^N \mathbf{a}_i(t) f_i(x),$$

where $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and orthogonal in $H^1(\Omega)$ consisting of eigenfunctions of $-\Delta$, *i.e.*,

$$\begin{cases} -\Delta f_i = \lambda_i f_i, & i = 1, 2, \dots, \\ \partial_\nu f_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $\mathbf{a}_i(t)$ are $\mathbb{R}^3$-valued vectors.

We obtain the following approximate problem:

$$\partial_t \mathbf{m}^{\varepsilon,N} \times \mathbf{m}^{\varepsilon,N} + \beta \Delta \mathbf{m}^{\varepsilon,N} + \lambda \mathbf{m}^{\varepsilon,N} \times \Delta \mathbf{m}^{\varepsilon,N}$$
$$- \alpha \partial_t \mathbf{m}^{\varepsilon,N} - \frac{1}{\varepsilon} \left(\left|\mathbf{m}^{\varepsilon,N}\right|^2 - 1\right) \mathbf{m}^{\varepsilon,N} = 0, \tag{12}$$

on Q with the following initial and boundary conditions:

$$\mathbf{m}^{\varepsilon,N}(\cdot,0) = \mathbf{m}^N(\cdot,0) \quad \text{in } \Omega,$$
$$\partial_\nu \mathbf{m}^{\varepsilon,N} = 0 \quad \text{on } \partial\Omega,$$

and

$$\int_\Omega \mathbf{m}^N(\cdot,0) f_i \,\mathrm{d}x = \int_\Omega \mathbf{m}_0 f_i \,\mathrm{d}x.$$

Multiplying equation (12) by f_i and integrating over Ω, we get an ordinary differential system. In fact, we have

$$\partial_t \mathbf{m}^{\varepsilon,N} \times \mathbf{m}^{\varepsilon,N} - \alpha \partial_t \mathbf{m}^{\varepsilon,N} = \mathbb{M}\left(\mathbf{m}^{\varepsilon,N}\right) \partial_t \mathbf{m}^{\varepsilon,N},$$

where

$$\mathbb{M}(\mathbf{m}^{\varepsilon,N}) = \begin{pmatrix} -\alpha & m_3^{\varepsilon,N} & -m_2^{\varepsilon,N} \\ -m_3^{\varepsilon,N} & -\alpha & m_1^{\varepsilon,N} \\ m_2^{\varepsilon,N} & -m_1^{\varepsilon,N} & -\alpha \end{pmatrix}.$$

We can write equation (12) in the form

$$\mathbb{M}(\mathbf{m}^{\varepsilon,N})\partial_t \mathbf{m}^{\varepsilon,N} = -\beta\Delta\mathbf{m}^{\varepsilon,N} - \lambda\mathbf{m}^{\varepsilon,N} \times \Delta\mathbf{m}^{\varepsilon,N} + \frac{1}{\varepsilon}\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2 - 1\right)\mathbf{m}^{\varepsilon,N}.$$

Note that

$$\det\mathbb{M}(\mathbf{m}^{\varepsilon,N}) = -\alpha\left(\alpha^2 + \left|\mathbf{m}^{\varepsilon,N}\right|^2\right) \neq 0.$$

Hence $\mathbb{M}(\mathbf{m}^{\varepsilon,N})$ is invertible.

The resulting system is then locally Lipschitz. There exists a unique local solution for the approximate problem that can extend on $[0,T]$ using an *a priori* estimate.

To get bounds on the solutions, we multiply equation (12) by $\partial_t\mathbf{m}^{\varepsilon,N}$ and integrate over Ω to obtain

$$\begin{aligned} &\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x - \lambda\int_\Omega \mathbf{m}^{\varepsilon,N}\times\Delta\mathbf{m}^{\varepsilon,N}\cdot\partial_t\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x \\ &\quad + \alpha\int_\Omega\left|\partial_t\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2\mathrm{d}x = 0. \end{aligned} \tag{13}$$

Multiply again equation (12) with $\Delta\mathbf{m}^{\varepsilon,N}$ and integrate over Ω; we get

$$\begin{aligned} &\int_\Omega\partial_t\mathbf{m}^{\varepsilon,N}\times\mathbf{m}^{\varepsilon,N}\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x + \beta\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x \\ &\quad + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x = 0. \end{aligned} \tag{14}$$

Multiplying (14) by λ and taking the sum with (13)

$$\begin{aligned} &\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x + \alpha\int_\Omega\left|\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2\mathrm{d}x \\ &\quad + \lambda\beta\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x + \frac{\lambda\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x \\ &= \frac{\lambda}{\varepsilon}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x. \end{aligned} \tag{15}$$

On the other hand, the Young inequality gives

$$\begin{aligned} &\frac{\lambda}{\varepsilon}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)\mathbf{m}^{\varepsilon,N}\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x \\ &\quad\le \frac{\lambda}{2d\varepsilon^2}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x + \frac{\lambda d}{2}\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x \end{aligned} \tag{16}$$

for any constant $d > 0$.

We multiply (12) by $(|\mathbf{m}^{\varepsilon,N}|^2-1)\mathbf{m}^{\varepsilon,N}$ and integrate over Ω to get

$$\beta\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)\mathbf{m}^{\varepsilon,N}\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x-\frac{\alpha}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x$$
$$-\frac{1}{\varepsilon}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x=0.$$

Hence

$$\frac{\lambda}{\varepsilon}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)\mathbf{m}^{\varepsilon,N}\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x$$
$$=\frac{\alpha\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x+\frac{\lambda}{\beta\varepsilon^2}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x.$$

Therefore

$$\frac{\alpha\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x+\frac{\lambda}{\beta\varepsilon^2}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x$$
$$\le\frac{\lambda}{2d\varepsilon^2}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x+\frac{\lambda d}{2}\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x.$$

That is,

$$\frac{\alpha\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x+\frac{\lambda}{\varepsilon^2}\left(\frac{1}{\beta}-\frac{1}{2d}\right)\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x$$
$$\le\frac{\lambda d}{2}\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x.$$

So, for $d>\frac{\beta}{2}$,

$$\frac{\lambda}{2d\beta\varepsilon^2}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x$$
$$\le\frac{\lambda d}{2(2d-\beta)}\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x-\frac{\alpha\lambda}{4\beta\varepsilon(2d-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x.$$

Therefore from (16), we have

$$\frac{\lambda}{\varepsilon}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)\mathbf{m}^{\varepsilon,N}\cdot\Delta\mathbf{m}^{\varepsilon,N}\,\mathrm{d}x$$
$$\le\frac{\lambda d}{2}\left(1+\frac{\beta}{2d-\beta}\right)\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x-\frac{\alpha\lambda}{4\varepsilon(2d-\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x$$

and then from (15)

$$\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x+\alpha\int_\Omega\left|\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x+\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\bigr)^2\,\mathrm{d}x$$
$$+\lambda\beta\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x+\frac{\lambda\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x\le\frac{\lambda d^2}{2d-\beta}\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\,\mathrm{d}x.$$

That is,

$$\frac{\beta+\alpha\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x+\alpha\int_\Omega\left|\partial_t\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x+\lambda\left(\beta-\frac{d^2}{2d-\beta}\right)\int_\Omega\left|\Delta\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x$$
$$+\frac{1}{4\varepsilon}\left(1+\frac{\alpha\lambda}{2d-\beta}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2\mathrm{d}x\le 0.$$

Taking $d=\beta$ we get $\beta-\frac{d^2}{2d-\beta}=0$ and therefore

$$\frac{\beta+\alpha\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x+\alpha\int_\Omega\left|\partial_t\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x$$
$$+\frac{1}{4\varepsilon}\left(1+\frac{\alpha\lambda}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2\mathrm{d}x\le 0.$$

We integrate from 0 to t to get

$$\begin{aligned}&\frac{\beta+\alpha\lambda}{2}\int_\Omega\left|\nabla\mathbf{m}^{\varepsilon,N}\right|^2(t)\,\mathrm{d}x+\alpha\int_0^t\int_\Omega\left|\partial_t\mathbf{m}^{\varepsilon,N}\right|^2\mathrm{d}x\,\mathrm{d}t\\&\quad+\frac{1}{4\varepsilon}\left(1+\frac{\alpha\lambda}{\beta}\right)\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2(t)\,\mathrm{d}x\\&\le\frac{\beta+\alpha\lambda}{2}\int_\Omega\left|\nabla\mathbf{m}^{N}\right|^2(0)\,\mathrm{d}x+\frac{1}{4\varepsilon}\left(1+\frac{\alpha\lambda}{\beta}\right)\int_\Omega\left(\left|\mathbf{m}^{N}\right|^2-1\right)^2(0)\,\mathrm{d}x\end{aligned}\tag{17}$$

for all $t\in(0,T)$.

The right-hand side is uniformly bounded. Indeed $\mathbb{H}^1(\Omega)\hookrightarrow\mathbb{L}^4(\Omega)$ with continuous embedding, therefore

$$\begin{aligned}\int_\Omega\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)^2(0)\,\mathrm{d}x&=\int_\Omega\left|\mathbf{m}^N(0)\right|^4\mathrm{d}x-2\int_\Omega\left|\mathbf{m}^N(0)\right|^2\mathrm{d}x+\mathrm{meas}(\Omega)\\&\le\left\|\mathbf{m}^N(0)\right\|^4_{\mathbb{L}^4(\Omega)}+\mathrm{meas}(\Omega)\\&\le C_1\left\|\mathbf{m}^N(0)\right\|^4_{\mathbb{H}^1(\Omega)}+C_2,\end{aligned}$$

where C_1 et C_2 are two constants independent of ε and N.

Furthermore, note that $\mathbf{m}^{\varepsilon,N}(0)=\mathbf{m}^N(0)$, and since $\mathbf{m}^N(0)$ has the same components as $\mathbf{m}_0$ in the basis $\{f_i\}_{i\in\mathbb{N}}$ and $\mathbf{m}_0\in\mathbb{H}^1(\Omega)$, $\|\mathbf{m}_0\|_{\mathbb{H}^1(\Omega)}\le C_3$ with $C_3>0$ is a constant independent of ε and N. Hence

$$\left\|\mathbf{m}^N(0)\right\|_{\mathbb{H}^1(\Omega)}\le C_3.$$

Therefore

$$\left\|\nabla\mathbf{m}^N(0)\right\|_{\mathbb{L}^2(\Omega)}\le C_3.$$

Thus for ε fixed, we have

$$\begin{aligned}&\left(\left|\mathbf{m}^{\varepsilon,N}\right|^2-1\right)_N\quad\text{is bounded in }L^\infty\left(0,T,L^2(\Omega)\right),\\&\left(\nabla\mathbf{m}^{\varepsilon,N}\right)_N\quad\text{is bounded in }L^\infty\left(0,T,\mathbb{L}^2(\Omega)\right).\end{aligned}$$

By the Young inequality

$$\int_\Omega |\mathbf{m}^{\varepsilon,N}|^2\,\mathrm{d}x \le C + \int_\Omega \bigl(|\mathbf{m}^{\varepsilon,N}|^2 - 1\bigr)^2\,\mathrm{d}x,$$

where C is a constant which does not depend on N. Therefore

$$\begin{aligned}
&\bigl(\mathbf{m}^{\varepsilon,N}\bigr)_N \quad \text{is bounded in } L^\infty\bigl(0,T,\mathbb{H}^1(\Omega)\bigr),\\
&\bigl(\partial_t\mathbf{m}^{\varepsilon,N}\bigr)_N \quad \text{is bounded in } L^2\bigl(0,T,\mathbb{L}^2(\Omega)\bigr) := \mathbb{L}^2(Q).
\end{aligned}$$

Then we have the following convergences for a subsequence further noted $\mathbf{m}^{\varepsilon,N}$ for any $1 < p < \infty$:

$$\mathbf{m}^{\varepsilon,N} \rightharpoonup \mathbf{m}^\varepsilon \quad \text{weakly in } L^p\bigl(0,T,\mathbb{H}^1(\Omega)\bigr), \tag{18}$$

$$\mathbf{m}^{\varepsilon,N} \longrightarrow \mathbf{m}^\varepsilon \quad \text{strongly in } L^2\bigl(0,T,\mathbb{L}^2(\Omega)\bigr) \text{ and a.e.}, \tag{19}$$

$$\partial_t\mathbf{m}^{\varepsilon,N} \rightharpoonup \partial_t\mathbf{m}^\varepsilon \quad \text{weakly in } L^2\bigl(0,T,\mathbb{L}^2(\Omega)\bigr), \tag{20}$$

$$\bigl|\mathbf{m}^{\varepsilon,N}\bigr|^2 - 1 \rightharpoonup \zeta \quad \text{weakly in } L^p\bigl(0,T,L^2(\Omega)\bigr). \tag{21}$$

The convergence (19) is a consequence of (18) and the compactness embedding of $L^2(0,T,\mathbb{H}^1(\Omega))$ in $L^2(0,T,\mathbb{L}^2(\Omega))$. On the other hand $\zeta = |\mathbf{m}^\varepsilon|^2 - 1$. This is provided by the following lemma.

Lemma 2 *Let Θ be a bounded open subset of $\mathbb{R}^d_x \times \mathbb{R}_t$, h_n, and h are functions of $L^q(\Theta)$ with $1 < q < \infty$ such as $\|h_n\|_{L^q(\Theta)} \le C$, $h_n \longrightarrow h$ a.e. in Θ; then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

The proof of Lemma 2 can be found in [17]. In our case $\Theta = Q$, $h_N = |\mathbf{m}^{\varepsilon,N}|^2 - 1$, $h = |\mathbf{m}^\varepsilon|^2 - 1$, and $q = 2$, and from (19) $|\mathbf{m}^{\varepsilon,N}|^2 - 1 \longrightarrow |\mathbf{m}^\varepsilon|^2 - 1$ a.e., and we have in particular $|\mathbf{m}^{\varepsilon,N}|^2 - 1 \in L^2(\Theta)$, $|\mathbf{m}^\varepsilon|^2 - 1 \in L^2(\Theta)$, and $\||\mathbf{m}^\varepsilon|^2 - 1\|_{L^2(\Theta)} \le C$.

Now, we pass to the limit as $N \to \infty$. Multiplying the equation (12) by $\boldsymbol{\varphi} \in \mathcal{C}^\infty(\overline{Q})$ and integrating on Q,

$$\begin{aligned}
&\int_Q \partial_t\mathbf{m}^{\varepsilon,N} \times \mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \beta \int_Q \nabla\mathbf{m}^{\varepsilon,N} \cdot \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \alpha \int_Q \partial_t\mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \lambda \int_Q \mathbf{m}^{\varepsilon,N} \times \nabla\mathbf{m}^{\varepsilon,N} \cdot \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon}\int_Q \bigl(\bigl|\mathbf{m}^{\varepsilon,N}\bigr|^2 - 1\bigr)\mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{22}$$

We have

$$\partial_t\mathbf{m}^{\varepsilon,N} \rightharpoonup \partial_t\mathbf{m}^\varepsilon \quad \text{weakly in } \mathbb{L}^2(Q)$$

and

$$\mathbf{m}^{\varepsilon,N} \longrightarrow \mathbf{m}^\varepsilon \quad \text{strongly in } L^2\bigl(0,T,\mathbb{L}^2(\Omega)\bigr).$$

Thus

$$\int_Q \partial_t\mathbf{m}^{\varepsilon,N} \times \mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_Q \partial_t\mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand

$$\nabla \mathbf{m}^{\varepsilon,N} \rightharpoonup \nabla \mathbf{m}^{\varepsilon} \quad \text{weakly in } \mathbb{L}^2(Q).$$

Therefore

$$\int_Q \nabla \mathbf{m}^{\varepsilon,N} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_Q \nabla \mathbf{m}^{\varepsilon} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_Q \mathbf{m}^{\varepsilon,N} \times \nabla \mathbf{m}^{\varepsilon,N} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \longrightarrow -\int_Q \mathbf{m}^{\varepsilon} \times \nabla \mathbf{m}^{\varepsilon} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t,$$

and from (20)

$$\int_Q \partial_t \mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_Q \partial_t \mathbf{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t.$$

Taking into account (21), we have

$$\int_Q \left(\left|\mathbf{m}^{\varepsilon,N}\right|^2 - 1\right) \mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_Q \left(\left|\mathbf{m}^{\varepsilon}\right|^2 - 1\right) \mathbf{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t.$$

Using the previous convergences and passing to the limit ($N \to \infty$) in (22), we get

$$\begin{aligned} &\int_Q \partial_t \mathbf{m}^{\varepsilon} \times \mathbf{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \beta \int_Q \nabla \mathbf{m}^{\varepsilon} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \lambda \int_Q \mathbf{m}^{\varepsilon} \times \nabla \mathbf{m}^{\varepsilon} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \alpha \int_Q \partial_t \mathbf{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon} \int_Q \left(\left|\mathbf{m}^{\varepsilon}\right|^2 - 1\right) \mathbf{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = 0 \end{aligned} \tag{23}$$

for all $\boldsymbol{\varphi}$ in $\mathcal{C}^\infty(\overline{Q})$, and this relation holds for all $\boldsymbol{\varphi} \in \mathbb{H}^1(Q)$ by a density argument. Inequality (10) follows from (17), and Theorem 1 is now completely proved. □

2.2 Convergence of the approximate solutions

To pass to the limit as $\varepsilon \to 0$, we need the first estimate (17) and the following lemma.

Lemma 3 *If* $\mathbf{m}^{\varepsilon}$ *satisfies* (23) *then* $|\mathbf{m}^{\varepsilon}| \leq 1$ *a.e. in* Q.

Proof Noting that

$$\int_\Omega \mathbf{g} \cdot \boldsymbol{\varphi} \,\mathrm{d}x = \int_{\{|\mathbf{m}^{\varepsilon}|^2 > 1\}} \mathbf{g} \cdot \boldsymbol{\varphi} \,\mathrm{d}x + \int_{\{|\mathbf{m}^{\varepsilon}|^2 \leq 1\}} \mathbf{g} \cdot \boldsymbol{\varphi} \,\mathrm{d}x$$

for all $\mathbf{g}$, $\boldsymbol{\varphi}$ in $\mathbb{L}^2(\Omega)$.

So if we choose

$$\boldsymbol{\varphi} = \left(\max\left(\left|\mathbf{m}^{\varepsilon}\right|^2, 1\right) - 1\right) \mathbf{m}^{\varepsilon}$$

we have

$$\begin{cases} \boldsymbol{\varphi} = 0, & \text{if } |\mathbf{m}^{\varepsilon}|^2 \leq 1, \\ \boldsymbol{\varphi} = (|\mathbf{m}^{\varepsilon}|^2 - 1)\mathbf{m}^{\varepsilon}, & \text{if } |\mathbf{m}^{\varepsilon}|^2 > 1. \end{cases}$$

Let $A = \{|\mathbf{m}^\varepsilon|^2 > 1\}$, then (23) becomes

$$-\beta \int_0^T \int_A \nabla \mathbf{m}^\varepsilon \cdot \nabla\big(\big(|\mathbf{m}^\varepsilon|^2 - 1\big)\mathbf{m}^\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}t - \alpha \int_0^T \int_A \big(\partial_t \mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon\big)\big(|\mathbf{m}^\varepsilon|^2 - 1\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{1}{\varepsilon}\int_0^T \int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2 |\mathbf{m}^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

That is,

$$-\frac{\beta}{2}\int_0^T \int_A \big|\nabla\big(|\mathbf{m}^\varepsilon|^2 - 1\big)\big|^2\,\mathrm{d}x\,\mathrm{d}t - \beta \int_0^T \int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)|\nabla \mathbf{m}^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{\alpha}{4}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon}\int_0^T \int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2 |\mathbf{m}^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Then

$$\frac{\alpha}{4}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Therefore

$$\int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2(T)\,\mathrm{d}x \le \int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2(0)\,\mathrm{d}x,$$

and as $|\mathbf{m}^\varepsilon(0)| = 1$, we get

$$\int_A \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2(T)\,\mathrm{d}x \le 0,$$

which implies that $|\mathbf{m}^\varepsilon| \le 1$ a.e. on Q. □

Now we will look for an estimate of the term $\mathbf{m}^\varepsilon \times \Delta \mathbf{m}^\varepsilon$. Going back to (17) and taking into account the previous convergences in N, we get

$$\frac{\beta + \alpha\lambda}{2}\int_\Omega |\nabla \mathbf{m}^\varepsilon|^2(t)\,\mathrm{d}x + \alpha \int_0^t \int_\Omega |\partial_t \mathbf{m}^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{4\varepsilon}\left(1 + \frac{\alpha\lambda}{\beta}\right)\int_\Omega \big(|\mathbf{m}^\varepsilon|^2 - 1\big)^2(t)\,\mathrm{d}x \le \frac{\beta + \alpha\lambda}{2}\int_\Omega |\nabla \mathbf{m}_0|^2\,\mathrm{d}x \tag{24}$$

for all $t \in (0, T)$.

Thus

$$\begin{aligned}
&\big(\partial_t \mathbf{m}^\varepsilon\big)_\varepsilon \quad \text{is bounded in } \mathbb{L}^2(Q),\\
&\big(|\mathbf{m}^\varepsilon|^2 - 1\big)_\varepsilon \quad \text{is bounded in } L^\infty\big(0, T, L^2(\Omega)\big),\\
&\big(\mathbf{m}^\varepsilon\big)_\varepsilon \quad \text{is bounded in } L^\infty\big(0, T, \mathbb{H}^1(\Omega)\big).
\end{aligned}$$

Multiplying equation (8) by $\mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon$ and integrating over Ω, we get

$$-\int_\Omega |\mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon|^2\,\mathrm{d}x + \beta \int_\Omega \Delta \mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon\,\mathrm{d}x$$
$$+ \lambda \int_\Omega \mathbf{m}^\varepsilon \times \Delta \mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon\,\mathrm{d}x = 0. \tag{25}$$

Multiply this time equation (8) by $\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon$ and integrating over Ω we obtain

$$
\begin{aligned}
&-\alpha \int_\Omega \mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon \cdot \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x - \int_\Omega \mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x \\
&\quad + \lambda \int_\Omega \left|\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right|^2 \mathrm{d}x = 0.
\end{aligned} \tag{26}
$$

Multiplying equation (26) by λ and taking the sum with (25), we get

$$
-\int_\Omega \left|\mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + (\beta + \alpha\lambda) \int_\Omega \Delta\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x + \lambda^2 \int_\Omega \left|\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right|^2 \mathrm{d}x = 0.
$$

Then

$$
\lambda^2 \int_\Omega \left|\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right|^2 \mathrm{d}x = \int_\Omega \left|\mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x - (\beta + \alpha\lambda) \int_\Omega \Delta\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x. \tag{27}
$$

Multiplying (8) by $\partial_t \mathbf{m}^\varepsilon$, integrating over Ω, replacing $\int_\Omega \Delta\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x$ by its value in (27) and using Lemma 3, we have

$$
\begin{aligned}
&\lambda^2 \int_\Omega \left|\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right|^2 \mathrm{d}x \\
&\quad = \int_\Omega \left|\mathbf{m}^\varepsilon \times \partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + \frac{\alpha(\beta + \alpha\lambda)}{\lambda} \int_\Omega \left|\partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x \\
&\qquad + \frac{\beta(\beta + \alpha\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left|\nabla \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + \frac{(\beta + \alpha\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right)^2 \mathrm{d}x \\
&\quad \le \int_\Omega \left|\mathbf{m}^\varepsilon\right|^2 \left|\partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + \frac{\alpha(\beta + \alpha\lambda)}{\lambda} \int_\Omega \left|\partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + \frac{\beta(\beta + \alpha\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left|\nabla \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x \\
&\qquad + \frac{(\beta + \alpha\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right)^2 \mathrm{d}x \\
&\quad \le \left(1 + \frac{\alpha(\beta + \alpha\lambda)}{\lambda}\right) \int_\Omega \left|\partial_t \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x + \frac{\beta(\beta + \alpha\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left|\nabla \mathbf{m}^\varepsilon\right|^2 \mathrm{d}x \\
&\qquad + \frac{(\beta + \alpha\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\left|\mathbf{m}^\varepsilon\right|^2 - 1\right)^2 \mathrm{d}x.
\end{aligned}
$$

We integrate from 0 to t, and using (24), we get

$$
\lambda^2 \int_0^t \int_\Omega \left|\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right|^2 \mathrm{d}x \,\mathrm{d}t \le C, \tag{28}
$$

where C is a constant independent of ε. Hence

$$
\left(\mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon\right)_\varepsilon \quad \text{is bounded in } \mathbb{L}^2(Q). \tag{29}
$$

Up to a subsequence, we have the following convergences for $1 < p < \infty$:

$$
\begin{aligned}
&\mathbf{m}^\varepsilon \rightharpoonup \mathbf{m} \quad \text{weakly in } L^p\left(0, T, \mathbb{H}^1(\Omega)\right), \\
&\partial_t \mathbf{m}^\varepsilon \rightharpoonup \partial_t \mathbf{m} \quad \text{weakly in } \mathbb{L}^2(Q),
\end{aligned}
$$

$$\left|\mathbf{m}^{\varepsilon}\right|^{2}-1\longrightarrow 0\quad\text{strongly in } L^{2}\big(0,T,L^{2}(\Omega)\big)\text{ and }|\mathbf{m}|=1\text{ a.e.,}$$
$$\mathbf{m}^{\varepsilon}\times\Delta\mathbf{m}^{\varepsilon}\rightharpoonup\chi\quad\text{weakly in }\mathbb{L}^{2}(Q).\tag{30}$$

By the compactness embedding of $\mathbb{H}^1(Q)$ into $\mathbb{L}^q(Q)$ with $2 \leq q < 6$, we have

$$\mathbf{m}^{\varepsilon}\longrightarrow\mathbf{m}\quad\text{strongly in }\mathbb{L}^{2}(Q)\text{ and in }\mathbb{L}^{4}(Q).\tag{31}$$

In the following, we show

$$\chi=\mathbf{m}\times\Delta\mathbf{m}\in\mathbb{L}^{2}(Q).\tag{32}$$

Letting $\boldsymbol{\varphi} \in \mathbb{H}^1(Q)$, using the Green formula,

$$\int_{Q}\mathbf{m}^{\varepsilon}\times\Delta\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=-\int_{Q}\mathbf{m}^{\varepsilon}\times\nabla\mathbf{m}^{\varepsilon}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t.$$

By the previous convergences,

$$\begin{aligned}\int_{Q}\mathbf{m}^{\varepsilon}\times\nabla\mathbf{m}^{\varepsilon}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\longrightarrow{}&\int_{Q}\mathbf{m}\times\nabla\mathbf{m}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\\ ={}&-\int_{Q}\mathbf{m}\times\Delta\mathbf{m}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and therefore (32) is proved. In particular, we have

$$\mathbf{m}^{\varepsilon}\times\Delta\mathbf{m}^{\varepsilon}\rightharpoonup\mathbf{m}\times\Delta\mathbf{m}\quad\text{weakly in }\mathbb{L}^{2}(Q).$$

Now going back to (23) and taking $\boldsymbol{\varphi} = \mathbf{m}^{\varepsilon} \times \boldsymbol{\phi}$ with $\boldsymbol{\phi} \in \mathcal{C}^{\infty}(\overline{Q})$, we get

$$\begin{aligned}&\int_{Q}\partial_{t}\mathbf{m}^{\varepsilon}\times\mathbf{m}^{\varepsilon}\cdot\mathbf{m}^{\varepsilon}\times\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t+\beta\int_{Q}\mathbf{m}^{\varepsilon}\times\nabla\mathbf{m}^{\varepsilon}\cdot\nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\lambda\int_{Q}\mathbf{m}^{\varepsilon}\times\Delta\mathbf{m}^{\varepsilon}\cdot\mathbf{m}^{\varepsilon}\times\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t-\alpha\int_{Q}\partial_{t}\mathbf{m}^{\varepsilon}\cdot\mathbf{m}^{\varepsilon}\times\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}\tag{33}$$

For the first term of (33), we set $D_{\varepsilon} = \int_{Q} \partial_t \mathbf{m}^{\varepsilon} \times \mathbf{m}^{\varepsilon} \cdot \mathbf{m}^{\varepsilon} \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t$. We have

$$D_{\varepsilon}=\int_{Q}\big(\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\big)\mathbf{m}^{\varepsilon}\cdot\partial_{t}\mathbf{m}^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t-\int_{Q}\left|\mathbf{m}^{\varepsilon}\right|^{2}\partial_{t}\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

On the one hand

$$\begin{aligned}\int_{Q}\left|\mathbf{m}^{\varepsilon}\right|^{2}\partial_{t}\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t={}&\int_{Q}\big(\left|\mathbf{m}^{\varepsilon}\right|^{2}-1\big)\partial_{t}\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{Q}\partial_{t}\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\\&\longrightarrow\int_{Q}\partial_{t}\mathbf{m}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_Q (\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi})\mathbf{m}^\varepsilon \cdot \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{2}\int_Q \partial_t\big(|\mathbf{m}^\varepsilon|^2 - 1\big)\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{2}\bigg[\int_\Omega \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\,\mathrm{d}x\bigg]_0^T \\ &\quad - \frac{1}{2}\int_Q \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\partial_t\big(\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We choose $\boldsymbol{\phi}$ so that $\boldsymbol{\phi} = 0$ in $t = 0$ and $t = T$; then

$$\bigg[\int_\Omega \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\,\mathrm{d}x\bigg]_0^T = 0.$$

Therefore

$$\begin{aligned}\int_Q (\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi})\mathbf{m}^\varepsilon \cdot \partial_t \mathbf{m}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{2}\int_Q \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\partial_t\big(\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\big)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_Q \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\partial_t\mathbf{m}^\varepsilon \cdot \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\frac{1}{2}\int_Q \big(|\mathbf{m}^\varepsilon|^2 - 1\big)\mathbf{m}^\varepsilon \cdot \partial_t\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0.\end{aligned}$$

Hence

$$D_\varepsilon \longrightarrow -\int_Q \partial_t \mathbf{m} \cdot \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

For the second term of (33), we have

$$\beta\int_Q \mathbf{m}^\varepsilon \times \nabla\mathbf{m}^\varepsilon \cdot \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \beta\int_Q \mathbf{m} \times \nabla\mathbf{m} \cdot \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

For the third term of (33), we get

$$\lambda\int_Q \mathbf{m}^\varepsilon \times \Delta\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \lambda\int_Q \mathbf{m} \times \Delta\mathbf{m} \cdot \mathbf{m} \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

For the last term of (33), we have

$$\int_Q \partial_t\mathbf{m}^\varepsilon \cdot \mathbf{m}^\varepsilon \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_Q \partial_t\mathbf{m} \cdot \mathbf{m} \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

Letting ε tend to 0 in (33), we get

$$\begin{aligned}&-\int_Q \partial_t\mathbf{m} \cdot \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \beta\int_Q \mathbf{m} \times \nabla\mathbf{m} \cdot \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \lambda\int_Q \mathbf{m} \times \Delta\mathbf{m} \cdot \mathbf{m} \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t - \alpha\int_Q \partial_t\mathbf{m} \cdot \mathbf{m} \times \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t = 0\end{aligned}$$

for all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$. Inequality (7) follows from (24). We proved the following.

Theorem 2 *Let* $\mathbf{m}_0 \in \mathbb{H}^1(\Omega)$ *with* $|\mathbf{m}_0| = 1$ *a.e., then there exists a global weak solution of the problem* (1) *in the sense of Definition* 1.

3 The limit as $b \to 0$

The main purpose of this section is to reveal to relationships between the LLG equation we have studied in this paper, and the classical LLG equation (*i.e.*, without vertical spin stiffness field). We will prove the following theorem.

Theorem 3 *Let* $b \to 0$. *The weak solution* $\mathbf{m}^b$ *obtained in Section* 2 *weakly converges, up to a subsequence, to a solution of the classical LLG equation in the following sense.*

For all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$ *with* $\boldsymbol{\phi}(\cdot,0) = \boldsymbol{\phi}(\cdot,T) = 0$, *we have*

$$\int_Q \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t - \alpha \int_Q \mathbf{m} \times \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t = \beta \int_Q \mathbf{m} \times \nabla \mathbf{m} \cdot \nabla \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t.$$

Proof Using the fact that $|\mathbf{m}^b| = 1$ a.e. in Q and estimate (7), we deduce that

$$\left(\mathbf{m}^b\right)_b \quad \text{is bounded in } L^\infty\left(0,T,\mathbb{H}^1(\Omega)\right)$$

and

$$\left(\partial_t \mathbf{m}^b\right)_b \quad \text{is bounded in } \mathbb{L}^2(Q).$$

Hence, up to a subsequence, we have

$$\begin{aligned}
&\mathbf{m}^b \rightharpoonup \mathbf{m} \quad \text{weakly in } L^p\left(0,T,\mathbb{H}^1(\Omega)\right) \text{ for } 1 < p < \infty,\\
&\mathbf{m}^b \to \mathbf{m} \quad \text{strongly in } \mathbb{L}^2(Q),\\
&\partial_t \mathbf{m}^b \rightharpoonup \partial_t \mathbf{m} \quad \text{weakly in } \mathbb{L}^2(Q).
\end{aligned}$$

Then $|\mathbf{m}| = 1$ a.e. in Q. On the other hand, we have

$$\alpha \partial_t \mathbf{m}^b + \mathbf{m}^b \times \partial_t \mathbf{m}^b - \beta \Delta \mathbf{m}^b - \lambda \mathbf{m}^b \times \Delta \mathbf{m}^b - \beta \left|\nabla \mathbf{m}^b\right|^2 \mathbf{m}^b = 0 \quad \text{a.e. in } Q.$$

Multiplying this equation by $\partial_t \mathbf{m}^b$ and $\mathbf{m}^b \times \Delta \mathbf{m}^b$, respectively, and integrating over Ω, we get

$$\alpha \int_\Omega \left|\partial_t \mathbf{m}^b\right|^2 \mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla \mathbf{m}^b\right|^2 \mathrm{d}x - \lambda \int_\Omega \mathbf{m}^b \times \Delta \mathbf{m}^b \cdot \partial_t \mathbf{m}^b \,\mathrm{d}x = 0 \tag{34}$$

and

$$\lambda \int_\Omega \left|\mathbf{m}^b \times \Delta \mathbf{m}^b\right|^2 \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla \mathbf{m}^b\right|^2 \mathrm{d}x = \alpha \int_\Omega \mathbf{m}^b \times \Delta \mathbf{m}^b \cdot \partial_t \mathbf{m}^b \,\mathrm{d}x. \tag{35}$$

Combining (34) and (35), we obtain

$$\lambda^2 \int_\Omega \left|\mathbf{m}^b \times \Delta \mathbf{m}^b\right|^2 \mathrm{d}x = \alpha^2 \int_\Omega \left|\partial_t \mathbf{m}^b\right|^2 \mathrm{d}x + \left(\frac{\alpha\beta - \lambda}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla \mathbf{m}^b\right|^2 \mathrm{d}x.$$

We integrate from 0 to t to get

$$\lambda^2 \int_0^t \int_\Omega \left|\mathbf{m}^b \times \Delta \mathbf{m}^b\right|^2 \mathrm{d}x\,\mathrm{d}t + \left(\frac{\alpha\beta - \lambda}{2}\right) \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x$$
$$= \alpha^2 \int_0^t \int_\Omega \left|\partial_t \mathbf{m}^b\right|^2 \mathrm{d}x\,\mathrm{d}t + \left(\frac{\alpha\beta - \lambda}{2}\right) \int_\Omega \left|\nabla \mathbf{m}^b\right|^2 \mathrm{d}x \tag{36}$$

for all $t \in (0, T)$.

Recall that

$$\beta = a\left(1 + \alpha^2\right) \quad \text{and} \quad \lambda = b\left(1 + \alpha^2\right).$$

Since b is small enough, we assume that $b < a\alpha$ *i.e.*, $\lambda < \alpha\beta$. Using estimate (7), we have

$$\int_\Omega \left|\nabla \mathbf{m}^b\right|^2 \mathrm{d}x \leq \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x$$

and

$$\alpha^2 \int_0^t \int_\Omega \left|\partial_t \mathbf{m}^b\right|^2 \mathrm{d}x\,\mathrm{d}t \leq \frac{\alpha\beta(1+\alpha^2)}{2} \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x.$$

Then (36) implies that

$$b^2 \int_0^t \int_\Omega \left|\mathbf{m}^b \times \Delta \mathbf{m}^b\right|^2 \mathrm{d}x\,\mathrm{d}t \leq \frac{\alpha a}{2} \int_\Omega |\nabla \mathbf{m}_0|^2 \,\mathrm{d}x.$$

Hence

$$\left(b\mathbf{m}^b \times \Delta \mathbf{m}^b\right)_b \quad \text{is bounded in } \mathbb{L}^2(Q).$$

Therefore

$$b\mathbf{m}^b \times \Delta \mathbf{m}^b \rightharpoonup \delta \quad \text{weakly in } \mathbb{L}^2(Q).$$

Let $\boldsymbol{\psi} \in \mathbb{H}^1(Q)$. We have

$$\int_Q b\mathbf{m}^b \times \Delta \mathbf{m}^b \cdot \boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t = -b \int_Q \mathbf{m}^b \times \nabla \mathbf{m}^b \cdot \nabla \boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t,$$

which tends to zero as b goes to zero. We conclude that $\delta = 0$.

Now, we can pass to the limit as $b \to 0$ in the weak formulation,

$$\int_Q \partial_t \mathbf{m}^b \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t - \alpha \int_Q \mathbf{m}^b \times \partial_t \mathbf{m}^b \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \beta \int_Q \mathbf{m}^b \times \nabla \mathbf{m}^b \cdot \nabla \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t + \left(1 + \alpha^2\right) \int_Q b\mathbf{m}^b \times \Delta \mathbf{m}^b \cdot \mathbf{m}^b \times \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t.$$

We get

$$\int_Q \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t - \alpha \int_Q \mathbf{m} \times \partial_t \mathbf{m} \cdot \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t = \beta \int_Q \mathbf{m} \times \nabla \mathbf{m} \cdot \nabla \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t,$$

and Theorem 3 is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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