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Research article

On the approximate controllability for some impulsive fractional evolution hemivariational inequalities

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Abstract: In this paper, we study the approximate controllability for some impulsive fractional evolution hemivariational inequalities. We show the concept of mild solutions for these problems. The approximate controllability results are formulated and proved by utilizing fractional calculus, fixed points theorem of multivalued maps and properties of generalized Clarke subgradient under some certain conditions.

Keywords: Approximate controllability; hemivariational inequalities; fractional differential; mild solutions; generalized Clarke subdifferential
Mathematics Subject Classification:20M05, 49N25, 34G25, 65L05

1. Introduction

In this paper,we will study the approximate controllability results of the following impulsive fractional evolution hemivariational inequalities:

$$\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + Bu(t) + \partial F(t, x(t)), & t \in J,\ t \neq t_k,\ \frac{1}{2} < \alpha \leq 1, \\ \Delta x(t_k) \in I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ x(t) = x_0, & \end{cases} \tag{1}$$

where ${}^CD_t^\alpha$ denotes the Caputo fractional derivative of order α with the lower limit zero. $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ on a separable Hilbert space X. The notation ∂F stands for the generalized Clarke subgradient (cf. [5]) of a locally Lipschitz function $F(t, \cdot) : X \to R$. The control function $u(t)$ takes value in $L^2([0, b]; U)$ and U is a Hilbert space, B is a linear operator from U into X. The function $I_k : X \to X$ is continous, and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m = T, \Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k (k = 1, 2, \cdots, m)$.

Since the hemivariational inequality was introduced by Panagiotopoulos in [23] to solve the mechanical problems with nonconvex and nonsmooth superpotentials, an extensive attention has been paid to this field and the great progress has been made in the last two decades. As a natural generalization of variational inequality, the notion of hemivariational inequality plays an very important role in both the qualitative and numerical analysis of nonlinear boundary value problems arising in mechanics, physics, engineering sciences and so on. For more details, one can see, Carl and Motreanu [4], Liu [12,13], Migórski and Ochal [18,19], Panagiotopoulos [24,25]. The theory of the fractional derivatives and integrals is an expanding and vibrant branch of applied mathematics that has found numerous applications. Recently, both the ordinary and the partial differential equations of fractional order have been used within the last few decades for modeling of many physical and chemical processes and in engineering, see e.g. [8, 11, 14–16, 26, 28] and references therein.

It is well known that the controllability, introduced firstly by R.Kalman in 1960, plays an important role in control theory and engineering. It lies in the fact that they have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design, etc. For this reason, the controllability has become an active area of investigation by many researchers and an impressive progress has been made in recent years [1, 3, 11, 15–17, 27, 29]. However, to the best of our knowledge, the approximate controllability of some impulsive fractional evolution hemivariational inequalities is still an untreated topic, so it is more interesting and necessary to study it.

Motivated by the above mentioned works, the rest of this paper is organized as follows: In Section 2, we will show some definitions and preliminaries which will be used in the following parts. By applying the fixed point theorem of multivalued maps, the approximate controllability of the control system (1) is given in Section 3 under some appropriate conditions.

2. Preliminaries

In this section, we will give some definitions and preliminaries which will be used in the paper. For the uniformly bounded C_0-semigroup $T(t)(t \geq 0)$, we set $M := \sup_{t\in[0,\infty)} \|T(t)\|_{L_b(X)} < \infty$. The norm of the space X will be defined by $\|\cdot\|_X$. Let $C(J, X)$ denote the Banach space of all X-value continous functions from $J = [0, T]$ into X, the norm $\|\cdot\|_c$=sup$\|\cdot\|_X$. Let the another Banach space $PC(J, X)$=$\{x : J \to X, x \in C((t_k, t_{k+1}], X), k = 0, 1, 2, \cdots, n,$there exist$x(t_k^-), x(t_k^+), k = 1, 2, \cdots, n$, and $x(t_k^-) = x(t_k)\}$, the norm $\|x\|_{PC} = max\{sup\|x(t + 0)\|, sup\|x(t - 0)\|\}$. We can use $L^p(J, R)$ denote the Banach space of all Lebesgue measurable functions from J to R with $\|f\|_{L^p(J,R)} = (\int_J |f(t)|^p dt)^{\frac{1}{p}}$,$L^p(J, X)$ denote the Banach space of functions $f : J \to X$ which are Bochner integroble normed by $\|f\|_{L^p(J,X)}$, $u \in L^p(J, R)$.

Let us recall some known definitions, for more details, one can see [8] and [26].

Definition 2.1 For a given function $f : [0, +\infty) \to R$, the integral

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds, \qquad \alpha > 0,$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

The expression

$$^L D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} (\frac{d}{dt})^{(n)} \int_0^t (t - s)^{n-\alpha-1} f(s)dt,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order $\alpha > 0$.

Definition 2.2 Caputo's derivative for a function $f : [0, \infty) \to R$ can be defined as

$$^{c}D_t^{\alpha} f(t) = {}^{L}D_t^{\alpha}[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)], \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α.

Now, let us recall the definition of the generalized gradient of Clarke for a locally Lipschitz functional $h : E \to R$ (where E is a Banach space), cf. [5]. We denote by $h^0(y, z)$ the Clarke generalized directional derivative of h at y in the direction z, that is

$$h^0(y, z) := \limsup_{\lambda \to 0^+,\ \xi \to y} \frac{h(\xi + \lambda z) - h(\xi)}{\lambda}.$$

Recall also that the generalized Clarke subgradient of h at y, denote by $\partial h(y)$, is a subset of E^* is given by

$$\partial h(y) := \{y^* \in E^* : h^0(y, z) \geq \langle y^*, z \rangle_X,\ \forall z \in E\}.$$

The following basic properties of the generalized subgradient play important role in our main results.

Lemma 2.3 (see Proposition 2.1.2 of [5]). Let h be locally Lipschitz of rank K near y. Then

(a) $\partial h(y)$ is a nonempty, convex, weak*-compact subset of E^* and $\|y^*\|_{E^*} \leq K$ for every y^* in $\partial h(y)$;

(b) for every $z \in E$, one has $h^0(y, z) = \max\{\langle y^*, z \rangle : \text{ for all } y^* \in \partial h(y)\}$.

Lemma 2.4 (see Proposition 5.6.10 of [6]). If $h : E \to R$ is locally Lipschitz, then the multifunction $y \to \partial h(y)$ is upper semicontinuous (u.s.c. for short) from E into $E^*_{w^*}$ (where $E^*_{w^*}$ denotes the Banach space E^* furnished with the w^*-topology).

Next, we present a result on measurability of the multifunction of the subgradient type whose proofs can be found in Kulig [10].

Lemma 2.5 (Proposition 3.44 of [20], page 66). Let E be a separable reflexive Banach space, $0 < b < \infty$ and $h : (0, b) \times E \to R$ be a function such that $h(\cdot, x)$ is measurable for all $x \in E$ and $h(t, \cdot)$ is locally Lipschitz for all $t \in (0, b)$. Then the multifunction $(0, b) \times E \ni (t, x) \mapsto \partial h(t, x)$ is measurable, where ∂h denotes the Clarke generalized gradient of $h(t, \cdot)$.

Now, we also introduce some basic definitions and results from multivalued analysis. For more details ,one can see the book [7]:

• In a Banach space E, a multivalued map $F : E \to 2^E \setminus \{\emptyset\} := \mathcal{P}(E)$ is convex (closed) valued, if $F(x)$ is convex (closed) for all $x \in E$. F is bounded on bounded sets if $F(V) = \bigcup_{x \in V} f(x)$ is bounded in E, for any bounded set V of E (i.e., $\sup_{x \in V}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$).

• F is called u.s.c on E, if for each $x \in E$, the set $F(x)$ is a nonempty, closed subset of E, and if for each open set V of E containing $F(x)$, there exists an open neighborhood N of x such that $F(N) \subseteq V$.

• F is said to be completely continuous if $F(V)$ is relatively compact, for every bounded subset $V \subseteq E$.

• If the multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph (i.e., $x_n \to x,\ y_n \to y,\ y_n \in F(x_n)$ imply $y \in F(x)$).

• F has a fixed point if there is $x \in E$, such that $x \in F(x)$.

• A multivalued map $F : J \to \mathcal{P}(E)$ is measurable if $F^{-1}(C) = \{t \in J : F(t) \cap C \neq \emptyset\} \in \Sigma$ for every closed set $C \subseteq E$. If $F : J \times E \to \mathcal{P}(E)$, then measurability of F means that $F^{-1}(C) \in \Sigma \otimes \mathcal{B}_E$, where $\Sigma \otimes \mathcal{B}_E$ is the $\sigma-$algebra of subsets in $J \times E$ generated by the sets $A \times B$, $A \in \Sigma$, $B \in \mathcal{B}_E$, and $\mathcal{B}_E$ is the $\sigma-$algebra of the Borel sets in E.

Now, according to the paper [15, 16, 28, 30], we shall recall the following definitions:

Definition 2.6 For each $u \in L^2(J, U)$, a function $x \in C(J, X)$ is a solution (mild solution) of the system (1) if $x(0) = x_0$ and there exists $f \in L^p(J, X)$ $(p > \frac{1}{\alpha})$ such that $f(t) \in \partial F(t, x(t))$, $\mathcal{I}_i \in I_i(x(t_i^-))$, without loss of generality, let $t \in (t_k, t_{k+1}]$, $1 \leq k \leq m-1$.

$$x(t) = S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds. \quad (2.4)$$

where

$$S_\alpha(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \qquad T_\alpha(t) = \alpha \int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

and

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0, \infty),$$

ξ_α is a probability density function defined on $(0, \infty)$, that is

$$\xi_\alpha(\theta) \geq 0,\ \theta \in (0, \infty), \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

Lemma 2.7 (Lemma 3.2-3.4 in [28]) The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:

(i) For any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$,

$$\|S_\alpha(t)x\| \leq M\|x\|, \quad \text{and} \quad \|T_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) $\{S_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous.

(iii) For any $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators if $T(t)$ is compact.

The key tool in our main results is the following fixed point theorem stated in [2].

Theorem 2.8 (Bohnenblust-Karlin [2]). Let Ω be a nonempty subset of a Banach space E, which is bounded, closed and convex. Suppose that $F : \Omega \to 2^E \setminus \{\emptyset\}$ is u.s.c. with closed, convex values such that $F(\Omega) \subseteq \Omega$ and $F(\Omega)$ is compact. Then F has a fixed point.

3. Approximate controllability results

In this section, we investigate the approximate controllability of the control systems described by impulsive fractional evolution hemivariational inequalities.

Let $x(t; 0, x_0, u)$ be a solution of system (1) at time t corresponding to the control $u(\cdot) \in L^2(J, U)$ and the initial value $x_0 \in X$. The set $\Re(b, x_0) = \{x(b; 0, x_0, u) : u(\cdot) \in L^2(J, U)\}$ is called the reachable set of system (1) at terminal time b. Then, the following definition of the approximate controllability is standard.

Definition 3.1 The control system (1) is said to be approximately controllable on the interval J, if for every initial function $x_0 \in X$, we have $\overline{\Re(b, x_0)} = X$.

Now, we consider the following linear fractional differential system:

$$\begin{cases} {}^{C}D_t^{\alpha} x(t) = Ax(t) + Bu(t), \quad t \in J = [0, b], \\ x(0) = x_0. \end{cases} \tag{3.1}$$

It is convenient at this point to introduce the controllability operator associated with (3.1) as follows:

$$\Gamma_0^b = \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) BB^* T_\alpha^*(b-s) ds,$$

$$R(\varepsilon, \Gamma_0^b) = (\varepsilon I + \Gamma_0^b)^{-1}, \quad \varepsilon > 0,$$

respectively, where B^* denotes the adjoint of B and $T_\alpha^*(t)$ is the adjoint of $T_\alpha(t)$. It is straightforward to see that the operator Γ_0^b is a linear bounded operator.

The following Lemma is of great importance for our main results.

Lemma 3.2 [1,16]The linear fractional control system (3.1) is approximately controllable on J if and only if $\varepsilon R(\varepsilon, \Gamma_0^b) \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology.

To obtain the approximate controllability result, we impose the following hypotheses:

$H(1)$: The C_0-semigroup $T(t)$ is compact and $\sup_{t\in[0,\infty)} \|T(t)\|_{L_b(X)} \leq M$.

$H(2)$: $F : J \times X \to \mathbb{R}$ is a function such that:

(i) for all $x \in X$, the function $t \mapsto F(t, x)$ is measurable;

(ii) the function $x \mapsto F(t, x)$ is locally Lipschitz for a.e. $t \in J$;

(iii) there exists a function $a(t) \in L^p(J, R^+)(p > \frac{1}{\alpha})$ and a constant $c > 0$ such that

$$\|\partial F(t, x)\|_{X^*} = \sup\{\|f\|_{X^*} : f(t) \in \partial F(t; x)\} \leq a(t) + c\|x\|_X, \text{ for a.e. } t \in J, \text{ all } x \in X.$$

$H(3)$: $\mathcal{I}_i : X \to X (i = 1, 2, \cdots, m)$ satisfies:

(i) $\mathcal{I}_i$ maps a bounded set to a bounded set;

(ii) There exist constants $d_i > 0 (i = 1, 2, \cdots, m)$ such that

$$\|\mathcal{I}_i(x) - \mathcal{I}_i(y)\| \leq d_i \|x - y\|, \quad x, y \in X.$$

(iii)$\|\mathcal{I}(0)\| = max(\|\mathcal{I}_1(0)\|, \|\mathcal{I}_2(0)\|, \cdots, \|\mathcal{I}_m(0)\|)$.

Next, we define an operator $\mathcal{N} : L^{\frac{p}{p-1}}(J, X) \to 2^{L^p(J,X)}$ as follows

$$\mathcal{N}(x) = \{w \in L^p(J, X) : w(t) \in \partial F(t; x(t)) \text{ a.e. } t \in J\}, \ x \in L^{\frac{p}{p-1}}(J, X).$$

The following Lemma due to Migórski and Ochal [20] is crucial in our main results.

Lemma 3.3 If the assumption $H(2)$ holds, then the set $\mathcal{N}(x)$ has nonempty, convex and weakly compact values for $x \in L^{\frac{p}{p-1}}(J, X)$, that is the multifunction $t \mapsto \partial F(t, x(t))$ has a measurable X^* selection.

Proof. Our main idea comes from Lemma 5.3 of [20]. Firstly, it is easy to see that $\mathcal{N}(x)$ has convex and weakly compact values from Lemma 2.3. Now, we only show that its values are nonempty. Let

$x \in L^{\frac{p}{p-1}}(J, X)$. Then, by Theorem 2.35 (ii) of [20], there exists a sequence $\{\varphi_n\} \in L^{\frac{p}{p-1}}(J, X)$ of simple functions such that

$$\varphi_n(t) \to x(t), \quad \text{in } L^{\frac{p}{p-1}}(J, X). \tag{3.2}$$

From Lemma 2.5 and hypotheses $H(2)$(i), (ii), the multifunction $t \mapsto \partial F(t, x)$ is measurable from J into $\mathcal{P}_{fc}(X^*)$ (where $\mathcal{P}_{fc}(X^*) = \{\Omega \subseteq X^* : \Omega$ is nonempty, convex and closed $\}$) (since the weak and weak*-topologies on the dual space of a reflexive Banach space coincide (cf. e.g. p7 of [9]), the multifunction ∂F is $P_{fc}(X^*)$-valued). Applying Theorem 3.18 of [20], for every $n \geq 1$, there exists a measurable function $\zeta_n : J \to X^*$ such that $\zeta_n(t) \in \partial F_n(t, \varphi_n(t))$ a.e. $t \in J$. Next, from hypothesis $H(F)(iii)$, we have

$$\|\zeta_n\|_{X^*} \leq a(t) + c\|\varphi_n\|_X.$$

Hence, $\{\zeta_n\}$ remains in a bounded subset of X^*. Thus, by passing to a subsequence, if necessary, we may suppose, by Theorem 1.36 of [20], that $\zeta_n \to \zeta$ weakly in X^* with $\zeta \in X^*$. From Proposition 3.16 of [20], it follows that

$$\zeta(t) \in \overline{\text{conv}}\Big((w - X^*) - \limsup\{\zeta_n(t)\}_{n\geq 1}\Big) \ \text{ a.e. } t \in J, \tag{3.3}$$

where $\overline{\text{conv}}$ denotes the closed convex hull of a set. From hypothesis $H(2)$(ii) and Lemma 2.4, we know that the multifunction $x \mapsto \partial F(t, x(t))$ is u.s.c from X into $X^*_{w^*}$. Recalling that the graph of an u.s.c multifunction with closed values is closed (see Proposition 3.12 of [20]), we get for a.e. $t \in J$, if $f_n \in \partial F(t, \zeta_n)$, $f_n \in X^*$, $f_n \to f$ weakly in X^*, $\zeta_n \in L^{\frac{p}{p-1}}(J, X)$, $\zeta_n \to \zeta$ in $L^{\frac{p}{p-1}}(J, X)$, then $f \in \partial F(t, \zeta)$. Therefore, by (3.2), we have

$$(w - X^*) - \limsup \partial F(t, \zeta_n(t)) \subset \partial F(t, x(t)) \ \text{ a.e. } t \in J, \tag{3.4}$$

where the Kuratowski limit superior is given by

$$\begin{aligned} & (w - X^*) - \limsup \partial F(t, \varphi_n(t)) \\ = \ & \{\zeta^* \in X^* : \zeta^* = (w - X^*) - \limsup \partial\zeta^*_{n_k}\, \zeta^*_{n_k} \in \partial F(t, \varphi_n(t)),\ n_1 < n_2 < \cdots < n_k < \cdots\} \end{aligned}$$

(see Definition 3.14 of [20]). So, from (3.3) and (3.4), we deduce that

$$\begin{aligned} \zeta(t) \ &\subset \ \overline{\text{conv}}((w - X^*) - \limsup\{\zeta_n(t)\}_{n\geq 1}) \\ &\subset \ \overline{\text{conv}}((w - X^*) - \limsup \partial F(t, \varphi_n(t)) \\ &\subset \ \partial F(t, x(t)), \quad \text{a.e. } t \in J. \end{aligned}$$

Since $\zeta \in X^*$ and $\zeta(t) \in \partial F(t, x(t))$ a.e. $t \in J$, it is clear that $\zeta \in \mathcal{N}(x)$. This proves that $\mathcal{N}(x)$ has nonempty values and completes the proof .

we prove that there exists $f \in L^p(J, X)$ ($p > \frac{1}{\alpha}$) such that $f(t) \in \partial F(t, x(t))$, so the $\mathcal{I}_i \in I_i(x(t_i^-))$, we omit the same kind of arguement.

The following Lemma is of great importance in our main results.

Lemma 3.4 (see Lemma 11 in [19]). If $H(2)$ holds, the operator $\mathcal{N}$ satisfies: if $z_n \to z$ in $L^{\frac{p}{p-1}}(J, X)$, $w_n \rightharpoonup w$ in $L^p(J, X)$ and $w_n \in \mathcal{N}(z_n)$, then we have $w \in \mathcal{N}(z)$. (Where $\rightharpoonup$ means weak convergence).

Now, we are in the position to prove the existence results of this paper.

Theorem 3.5 Suppose that the hypotheses $H(1)$ and $H(2)$ are satisfied, then the system (1.1) has a mild solution on J provided that

$$\Big[1+\frac{M^2M_B^2b^\alpha}{\varepsilon\alpha[\Gamma(\alpha)]^2}\Big]\Big[\sum_{i=1}^{k}Md_i+\frac{Mcb^\alpha}{\Gamma(1+\alpha)}\Big]<\frac{1}{2},\quad \text{where} \quad M_B:=\|B\|.$$

Proof. For any $\varepsilon>0$, we consider the multivalued map $F_\varepsilon: C(J,X)\to 2^{C(J,X)}$ as follows

$$\begin{aligned} F_\varepsilon(x) \;=\; & \Big\{h\in C(J,X):\; h(t)=S_\alpha(t)x_0+\sum_{i=1}^{k}S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds \\ & +\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu_\varepsilon(s)ds,\;\text{ with }\; f\in\mathcal{N}(x)\Big\},\quad \text{for }\; x\in C(J,X), \end{aligned}$$

where

$$u_\varepsilon(t)=B^*T_\alpha^*(b-t)R(\varepsilon,\Gamma_0^b)\Big(x_1-S_\alpha(b)x_0-\sum_{i=1}^{k}S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-))-\int_0^b(b-\tau)^{\alpha-1}T_\alpha(b-\tau)f(\tau)d\tau\Big).$$

It is clear that the problem of finding mild solutions of (1) is reduced to find the fixed point of F_ε. We prove the operator F_ε satisfies all the conditions of the Theorem 2.8 and we divide the proof into several steps.

Step 1: F_ε is convex for each $x\in C(J,X)$.

In fact, for any $\rho_1,\ \rho_2$ belong to F_ε, then there exist $f_1,f_2\in\mathcal{N}(x)$ such that

$$\begin{aligned} \rho_i(t) \;=\; & S_\alpha(t)x_0+\sum_{i=1}^{k}S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f_i(s)ds \\ & +\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-s) \\ & \times R(\varepsilon,\Gamma_0^b)\Big(x_1-S_\alpha(b)x_0-\sum_{i=1}^{k}S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-))-\int_0^b(b-\tau)^{\alpha-1}T_\alpha(b-\tau)f(\tau)d\tau\Big)ds, \\ & i=1,2,\; t\in J. \end{aligned} \tag{3.1}$$

Let $\lambda\in[0,1]$, then for each $t\in J$, we have

$$\begin{aligned} & [\lambda\rho_1+(1-\lambda)\rho_2](t) \\ = \; & S_\alpha(t)x_0+\sum_{i=1}^{k}S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[\lambda f_1+(1-\lambda)f_2](s)ds \\ & +\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-s)R(\varepsilon,\Gamma_0^b)\Big(x_1-S_\alpha(b)x_0 \\ & -\sum_{i=1}^{k}S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-))-\int_0^b(b-\tau)^{\alpha-1}T_\alpha(b-\tau)[\lambda f_1+(1-\lambda)f_2](\tau)d\tau\Big)ds. \end{aligned} \tag{3.2}$$

From Lemma 2.3, we know that $\partial F(t, x(t))$ is convex, hence for $\lambda \in [0,1]$, $\lambda f_1 + (1-\lambda) f_2 \in \mathcal{N}(x)$, then $\lambda\rho_1(t) + (1-\lambda)\rho_2(t) \in F_\varepsilon$, which implies that F_ε is convex for each $x \in C(J, X)$.

Step 2: There exists a nonempty, bounded, closed and convex subset $B_r \subseteq C(J, X)$ such that $F_\varepsilon(B_r) \subseteq B_r$.

Take

$$\begin{aligned} r &= 2\Big[M\|x_0\| + \sum_{i=1}^{k} M\|\mathcal{I}(0)\| + (1 + \frac{M^2 M_B^2 b^\alpha}{\varepsilon\alpha[\Gamma(\alpha)]^2})\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}} b^{\alpha-\frac{1}{p}}\|a\|_{L^p} \\ &\quad + \frac{M^2 M_B^2 b^\alpha}{\varepsilon\alpha[\Gamma(\alpha)]^2}(\|x_1\| + M\|x_0\| + \sum_{i=1}^{k} M\|\mathcal{I}(0)\|)\Big], \end{aligned}$$

and denote $B_r = \{x \in C(J, X) : \|x(t)\|_X \le r\}$. Obviously, B_r is a bounded, closed and convex subset of $C(J, X)$. In fact, for any $x \in B_r$, $\varphi \in F_\varepsilon(x)$, there exists $f \in \mathcal{N}(x)$ such that

$$\begin{aligned} \varphi(t) &= S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s) ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) B B^* T_\alpha^*(b-s) \times R(\varepsilon, \Gamma_0^b)\Big(x_1 - S_\alpha(b)x_0 \\ &\quad - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1} T_\alpha(b-\tau) f(\tau) d\tau\Big) ds, \qquad t \in J. \end{aligned}$$

Taking the assumptions $H(1)$ and Hölder inequality into account, we obtain

$$\begin{aligned} \|\varphi(t)\| &\le \|S_\alpha(t)x_0\| + \|\sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-))\| + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s) f(s)\| ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s) B \times B^* T_\alpha^*(b-s) R(\varepsilon, \Gamma_0^b)(x_1 - S_\alpha(b)x_0 \\ &\quad - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1} T_\alpha(b-\tau) f(\tau) d\tau)\| ds \\ &\le M\|x_0\| + \sum_{i=1}^{k} M(d_i\|x(t_i^-)\| + \|\mathcal{I}_i(0)\|) + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[a(s) + c\|x(s)\|_X] ds \\ &\quad + \frac{M^2 M_B^2 b^\alpha}{\varepsilon\alpha[\Gamma(\alpha)]^2}\Big[\|x_1\| + M\|x_0\| + \sum_{i=1}^{k} M(d_i\|x(t_i^-)\| + \|\mathcal{I}_i(0)\|) \\ &\quad + \frac{M}{\Gamma(\alpha)}\int_0^b (b-\tau)^{\alpha-1}[a(\tau) + c\|x(\tau)\|_X\Big] d\tau \\ &\le M\|x_0\| + \sum_{i=1}^{k} M(d_i r + \|\mathcal{I}_i(0)\|) + \frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}} b^{\alpha-\frac{1}{p}}\|a\|_{L^p} + \frac{Mcb^\alpha}{\Gamma(1+\alpha)} r \\ &\quad + \frac{M^2 M_B^2 b^\alpha}{\varepsilon\alpha[\Gamma(\alpha)]^2}\Big[\|x_1\| + M\|x_0\| + \sum_{i=1}^{k} M(d_i r + \|\mathcal{I}_i(0)\|) \end{aligned}$$

$$+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}b^{\alpha-\frac{1}{p}}\|a\|_{L^p}+\frac{Mcb^{\alpha}}{\Gamma(1+\alpha)}r\Big]$$
$$\leq \quad r.$$

Thus, we obtain that $F_{\varepsilon}(B_r)\subseteq B_r$.

Step 3. F_{ε} is equicontinuous on B_r.

Firstly, for any $x\in B_r,\ \varphi\in F_{\varepsilon}(x)$, there exists $f\in\mathcal{N}(x)$ such that

$$\varphi(t) \quad = \quad S_{\alpha}(t)x_0+\sum_{i=1}^{k}S_{\alpha}(t-t_i)\mathcal{I}_i(x(t_i^-))+\int_0^t(t-s)^{\alpha-1}T_{\alpha}(t-s)[f(s)+Bu_{\varepsilon}(s)]ds, \quad t\in J.$$

For any $\epsilon>0$, when $\tau_1=0,\ 0<\tau_2\leq\delta_0$, we obtain

$$\begin{aligned}
&\|\varphi(\tau_2)-\varphi(\tau_1)\|=\|\varphi(\tau_2)-x_0\|\\
\leq\ &\|S_{\alpha}(\tau_2)x_0-x_0\|+\|\sum_{i=1}^{k}S_{\alpha}(t-t_i)\mathcal{I}_i(x(t_i^-))\|+\|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}T_{\alpha}(\tau_2-s)f(s)ds\|\\
&+\|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}T_{\alpha}(\tau_2-s)Bu_{\varepsilon}(s)ds\|\\
\leq\ &\|S_{\alpha}(\tau_2)x_0-x_0\|+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}\tau_2^{\alpha-\frac{1}{p}}+\frac{Mc\tau_2^{\alpha}}{\Gamma(1+\alpha)}r+\frac{MM_B}{\sqrt{2\alpha-1}\Gamma(\alpha)}\|u_{\varepsilon}\|_{L^2}\tau_2^{\alpha-\frac{1}{2}}.
\end{aligned}$$

Hence, we can choose $\delta_0>0$ is small enough so that for all $0<\tau_2\leq\delta_0$, the impulsive term is 0, $\|\varphi(\tau_2)-\varphi(\tau_1)\|<\frac{\epsilon}{2}$. Thus, for $\forall\epsilon>0,\ \forall\tau_1,\ \tau_2\in[0,\delta_0],\ \forall\varphi\in F_{\varepsilon}(x)$, we have $\|\varphi(\tau_2)-\varphi(\tau_1)\|<\epsilon$ independently of $x\in B_r$.

Next, for any $x\in B_r$ and $\frac{\delta_0}{2}\leq\tau_1<\tau_2\leq b$, we obtain

$$\begin{aligned}
&\|\varphi(\tau_2)-\varphi(\tau_1)\|\\
\leq\ &\|S_{\alpha}(\tau_2)x_0-S_{\alpha}(\tau_1)x_0\|+\|\int_0^{\tau_1}[(\tau_2-s)^{\alpha-1}-\tau_1-s)^{\alpha-1}]T_{\alpha}(\tau_2-s)f(s)ds\\
&+\|\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}[T_{\alpha}(\tau_2-s)-T_{\alpha}(\tau_1-s)]f(s)ds\|\\
&+\|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T_{\alpha}(\tau_2-s)f(s)ds\|\\
&+\|\int_0^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}]T_{\alpha}(\tau_1-s)Bu_{\varepsilon}(s)ds\|\\
&+\|\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}[T_{\alpha}(\tau_2-s)-T_{\alpha}(\tau_1-s)]Bu_{\varepsilon}(s)ds\|\\
&+\|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T_{\alpha}(\tau_2-s)Bu_{\varepsilon}(s)ds\|\\
&\leq Q_1+Q_2+Q_3+Q_4+Q_5+Q_6+Q_7.
\end{aligned}$$

By the assumptions and Hölder's inequality, we have

$$Q_2 \quad \leq \quad \frac{M}{\Gamma(\alpha)}\int_0^{\tau_1}[(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}]\|f(s)\|ds$$

$$
\begin{aligned}
&\leq \frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}[\tau_2^{\alpha-\frac{1}{p}}-\tau_1^{\alpha-\frac{1}{p}}+(\tau_2-\tau_1)^{\alpha-\frac{1}{p}}] \\
&\quad +\frac{Mcr}{\Gamma(1+\alpha)}[\tau_2^{\alpha}-\tau_1^{\alpha}+(\tau_2-\tau_1)^{\alpha}] \\
&\leq \frac{2M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}(\tau_2-\tau_1)^{\alpha-\frac{1}{p}}+\frac{2Mcr}{\Gamma(1+\alpha)}(\tau_2-\tau_1)^{\alpha},
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
Q_4 &\leq \frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}(\tau_2-\tau_1)^{\alpha-\frac{1}{p}}+\frac{Mcr}{\Gamma(1+\alpha)}(\tau_2-\tau_1)^{\alpha}, \\
Q_5 &\leq \frac{MM_B}{\sqrt{2\alpha-1}\Gamma(\alpha)}\|u_\varepsilon\|_{L^2}[\tau_2^{\alpha-\frac{1}{2}}-\tau_1^{\alpha-\frac{1}{2}}+(\tau_2-\tau_1)^{\alpha-\frac{1}{2}}], \\
Q_7 &\leq \frac{MM_B}{\sqrt{2\alpha-1}\Gamma(\alpha)}\|u_\varepsilon\|_{L^2}(\tau_2-\tau_1)^{\alpha-\frac{1}{2}}.
\end{aligned}
$$

For $\tau_1 \geq \frac{\delta_0}{2} > 0$ and $\delta_1 > 0$ small enough, we obtain

$$
\begin{aligned}
Q_3 &\leq \Big[\|\int_0^{\tau_1-\delta_1}(\tau_1-s)^{\alpha-1}[T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)]f(s)ds\| \\
&\quad +\|\int_{\tau_1-\delta_1}^{\tau_1}(\tau_1-s)^{\alpha-1}[T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)]f(s)ds\|\Big] \\
&\leq \sup_{s\in[0,\tau_1-\delta_1]}\|T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)\|\Big[(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}(\tau_1^{\alpha-\frac{1}{p}}-\delta_1^{\alpha-\frac{1}{p}}) \\
&\quad +\frac{cr}{\alpha}(\tau_1^{\alpha}-\delta_1^{\alpha})\Big]+\frac{2M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}\delta_1^{\alpha-\frac{1}{p}}+\frac{2Mcr}{\Gamma(1+\alpha)}\delta_1^{\alpha}, \\
Q_6 &\leq \sup_{s\in[0,\tau_1-\delta_1]}\|T_\alpha(\tau_2-s)-T_\alpha(\tau_1-s)\| \\
&\quad \times\sqrt{\frac{1}{2\alpha-1}}\|u_\varepsilon\|_{L^2}(\tau_1^{\alpha-\frac{1}{2}}-\delta_1^{\alpha-\frac{1}{2}})+\frac{2MM_B}{\sqrt{2\alpha-1}\Gamma(\alpha)}\|u_\varepsilon\|_{L^2}\delta_1^{\alpha-\frac{1}{2}}.
\end{aligned}
$$

Since the compactness of $T(t)(t > 0)$ and Lemma 2.7 imply the continuity of $T_\alpha(t)(t > 0)$ in t in the uniform operator topology, it can be easily seen that Q_3 and Q_6 tend to zero independently of $x \in B_r$ as $\tau_2 \to \tau_1,\ \delta_1 \to 0$. And it is clear that $Q_i(i = 1, 2, 4, 5, 7)$ tend to zero as $\tau_2 \to \tau_1$ does not depend on particular choice of x. Thus, one can choose $\delta = \min\{\delta_0, \delta_1\}$, then it is easy to get that $\|\varphi(\tau_2) - \varphi(\tau_1)\|$ tends to zero independently of $x \in B_r$ as $\delta \to 0$ which implies $\{(F_\varepsilon x)(t) : x \in B_r\}$ is an equicontinuous set in $C(J, X)$.

Step 4: F_ε is a compact multivalued map.

Let $t \in J$ be fixed, we show that the set $\Pi(t) = \{(F_\varepsilon x)(t) : x \in B_r\}$ is relatively compact in X.

Clearly, $\Pi(0) = \{x_0\}$ is compact, so it is only necessary to consider $t > 0$. Let $0 < t \leq b$ be fixed. For any $x \in B_r,\ \varphi \in F_\varepsilon(x)$, there exists $f \in \mathcal{N}(x)$ such that

$$
\varphi(t) = S_\alpha(t)x_0+\sum_{i=1}^{k}S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)[f(s)+Bu_\varepsilon(s)]ds, \quad t\in J.
$$

For each $\epsilon \in (0,t),\ t \in (0,b],\ x \in B_r$ and any $\delta > 0$, we define

$$\begin{aligned}
\varphi^{\epsilon,\delta}(t) &= S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) \\
&\quad +\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)[f(s)+Bu_\varepsilon(s)]d\theta ds. \\
&= S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) \\
&\quad +\alpha T(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta-\epsilon^\alpha\delta)[f(s)+Bu_\varepsilon(s)]d\theta ds.
\end{aligned}$$

From the compactness of $S_\alpha(t)(t>0)$ and $T(\epsilon^\alpha\delta)$ $(\epsilon^\alpha\delta>0)$, we obtain that the set

$$\Pi_{\epsilon,\delta}(t) = \{F_\varepsilon^{\epsilon,\delta}(x)(t) : x \in B_r\},$$

is relatively compact set in X for each $\epsilon \in (0,t)$ and $\delta > 0$. Moreover, we have

$$\begin{aligned}
&\|\varphi(t) - \varphi^{\epsilon,\delta}(x)(t)\| \\
&= \|\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)[f(s)+Bu_\varepsilon(s)]d\theta ds \\
&\quad -\alpha\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)[f(s)+Bu_\varepsilon(s)]d\theta ds\| \\
&\le \alpha M(\frac{p-1}{p\alpha-1})^{1-\frac{1}{p}}\|a\|_{L^p}\Big[b^{\alpha-\frac{1}{p}}\int_0^\delta \theta\xi_\alpha(\theta)d\theta + \frac{1}{\Gamma(1+\alpha)}\epsilon^{\alpha-\frac{1}{p}}\Big] + Mcr\Big[\frac{1}{\Gamma(1+\alpha)}\epsilon^\alpha \\
&\quad +b^\alpha\int_0^\delta \theta\xi_\alpha(\theta)d\theta\Big] + \alpha M\sqrt{\frac{1}{2\alpha-1}}\|u_\varepsilon\|_{L^2}\Big[b^{\alpha-\frac{1}{2}}\int_0^\delta \theta\xi_\alpha(\theta)d\theta + \frac{b^{\frac{1}{2}}}{\Gamma(1+\alpha)}\epsilon^{\alpha-\frac{1}{2}}\Big].
\end{aligned}$$

Since $\int_0^\infty \xi_\alpha(\theta)d\theta = 1$, the last inequality tends to zero when $\epsilon \to 0$ and $\delta \to 0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$ $(t > 0)$. Hence the set $\Pi(t)$ $(t > 0)$ is also relatively compact in X.

Step5: F_ε has a closed graph.

Let $x_n \to x_*$ in $C(J,X)$, $\varphi_n \in F_\varepsilon(x_n)$ and $\varphi_n \to \varphi_*$ in $C(J,X)$. we will show that $\varphi_* \in F_\varepsilon(x_*)$. Indeed, $\varphi_n \in F_\varepsilon(x_n)$ means that there exists $f_n \in \mathcal{N}(x_n)$ such that

$$\begin{aligned}
\varphi_n(t) &= S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f_n(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\times BB^*T_\alpha^*(b-s)R(\varepsilon,\Gamma_0^b)\Big(x_1 - S_\alpha(b)x_0 \\
&\quad -\sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1}T_\alpha(b-\tau)f_n(\tau)d\tau\Big)ds. \quad (3.5)
\end{aligned}$$

From Step 2, we know that $\{f_n\}_{n\ge1} \subseteq L^p(J,X)$ is bounded. Hence we may assume, passing to a subsequence if necessary, that

$$f_n \rightharpoonup f_*, \quad \text{for some } f_* \in L^p(J,X), \tag{3.6}$$

It follows from (3.5), (3.6) and Lemma 3.4 that

$$\begin{aligned}\varphi_*(t) &= S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f_*(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\times BB^*T_\alpha^*(b-s)R(\varepsilon,\Gamma_0^b)\Big(x_1 - S_\alpha(b)x_0 \\ &\quad - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1}T_\alpha(b-\tau)f_*(\tau)d\tau\Big)ds. \quad (3.7)\end{aligned}$$

Note that $x_n \to x_*$ in $C(J,X)$ and $f_n \in \mathcal{N}(x_n)$. From Lemma 3.4 and (3.6), we obtain $f_* \in \mathcal{N}(x_*)$ Hence, we prove that $\varphi_* \in F_\varepsilon(x_*)$, which implies that F_ε has a closed graph.

Hence by Steps 1-5 and Arzelà-Ascoli theorem, we obtain that F_ε is a completely continuous multivalued map, u.s.c. with convex closed values and satisfies all the assumptions of Theorem 2.8. Thus F_ε has a fixed point which is a mild solution of problem (1). This is the end of the proof.

The following result concerns the approximately controllable of the problem (1). We need the following assumption.

$H(2)$(iii)': There exists a function $\eta \in L^\infty(J,R^+)$ such that

$$\|\partial F(t,x)\|_{X^*} = \sup\{\|f\|_{X^*} : f(t) \in \partial F(t,x)\} \le \eta(t), \quad \text{for a.e. } t \in J, \text{ all } x \in X.$$

Now, we are now in a position to prove the main result of this paper.

Theorem 3.6 Assume that assumptions of Theorem 3.5 and $H(2)$(iii)' are satisfied, and the linear system (3.1) is approximately controllable on J, then system (1) is approximately controllable on J.

Proof. By employing the technique used in Theorem 3.5, we can easily show that, for all $\varepsilon > 0$, the operator F_ε has a fixed point in B_{r_0}, where $r_0 = r(\varepsilon)$. Let $x^\varepsilon(\cdot)$ be a fixed point of F_ε in B_{r_0}. Any fixed point of F_ε is a mild solution of (1.1), this means that there exists $f^\varepsilon \in \mathcal{N}(x^\varepsilon)$ such that for each $t \in J$,

$$\begin{aligned}x^\varepsilon(t) &\in S_\alpha(t)x_0 + \sum_{i=1}^{k} S_\alpha(t-t_i)\mathcal{I}_i(x(t_i^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f^\varepsilon(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-s)\times R(\varepsilon,\Gamma_0^b)\Big(x_1 - S_\alpha(b)x_0 \\ &\quad - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1}T_\alpha(b-\tau)f^\varepsilon(\tau)d\tau\Big)ds.\end{aligned}$$

Define $G(f^\varepsilon) = x_1 - S_\alpha(b)x_0 - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1}T_\alpha(b-\tau)f^\varepsilon(\tau)d\tau$.
Noting that $I - \Gamma_0^b R(\varepsilon,\Gamma_0^b) = \varepsilon R(\varepsilon,\Gamma_0^b)$, we get $x(b) = x_1 - \varepsilon R(\varepsilon,\Gamma_0^b)G(f^\varepsilon)$.
By assumption $H(2)$(iii)', we have $\int_0^b \|f^\varepsilon(s)\|^2 ds \le \|\eta\|_{L^2(J,R)}\sqrt{b}$.

Consequently the sequence $\{f^\varepsilon\}$ is uniformly bounded in $L^2(J,X)$. Thus, there is a subsequence, still denoted by $\{f^\varepsilon\}$, that converges weakly to say f in $L^2(J,X)$. Denoting

$$h = x_1 - S_\alpha(b)x_0 - \sum_{i=1}^{k} S_\alpha(b-t_i)\mathcal{I}_i(x(t_i^-)) - \int_0^b (b-\tau)^{\alpha-1}T_\alpha(b-\tau)f(\tau)d\tau.$$

we see that

$$
\begin{aligned}
\|G(f^{\varepsilon}) - h\| &= \| \int_0^b (b-\tau)^{\alpha-1} T_\alpha(b-\tau)[f^{\varepsilon}(\tau) - f(\tau)]d\tau \\
&\leq \sup_{0\leq t\leq b} \| \int_0^t (t-\tau)^{\alpha-1} T_\alpha(t-\tau)[f^{\varepsilon}(\tau) - f(\tau)]d\tau. \qquad (3.8)
\end{aligned}
$$

Using the Ascoli-Arzela theorem one can show that the linear operator $g \mapsto \int_0^{\cdot}(\cdot-\tau)^{\alpha-1}T_\alpha(\cdot-\tau)g(\tau)d\tau$: $L^2(J,X) \to C(J,X)$ is compact, consequently the right-hand side of (3.8) tends to zero as $\varepsilon \to 0^+$. This implies

$$
\begin{aligned}
\|x^{\varepsilon}(b) - x_1\| &= \|\varepsilon R(\varepsilon, \Gamma_0^b)G(f^{\varepsilon})\| \\
&\leq \|\varepsilon R(\varepsilon, \Gamma_0^b)(h)\| + \|\varepsilon R(\varepsilon, \Gamma_0^b)[G(f^{\varepsilon}) - h]\| \\
&\leq \|\varepsilon R(\varepsilon, \Gamma_0^b)(h)\| + \|G(f^{\varepsilon}) - h\| \to 0, \quad \text{as } \varepsilon \to 0^+.
\end{aligned}
$$

This proves the approximate controllability of system (1).

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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