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On the global existence of 3-D magneto-hydrodynamic system in the critical spaces

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Abstract

In this article, we prove the global existence of the three-dimensional inhomogeneous incompressible magneto-hydrodynamic system under the assumptions that the initial velocity field and the initial conductivity are small in the critical space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$.

MSC: 35Q35; 76D03

Keywords: magneto-hydrodynamic system; Littlewood-Paley theory; critical spaces

1 Introduction

In this paper, we consider the three-dimensional inhomogeneous incompressible magneto-hydrodynamics (MHD) which describes the coupling between the inhomogeneous Navier-Stokes system and the Maxwell equation [1]. We have

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \mathcal{M}) + \nabla \Pi = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = 0, \qquad \operatorname{div} B = 0, \\ \rho|_{t=0} = \rho_0, \qquad \rho u|_{t=0} = m_0, \qquad B|_{t=0} = B_0, \end{cases} \tag{1.1}$$

where the unknowns are the density ρ, the velocity $u = (u_1, u_2, u_3)$, the magnetic field $B = (B_1, B_2, B_3)$, and the pressure function Π. $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the deformation tensor and, in general, the viscosity coefficient μ is a smooth, positive function of the density ρ.

For the inhomogeneous MHD system (1.1), many results have been obtained. Gerbeau and Le Bris [2, 3], obtained global existence of weak solutions in whole space $\mathbb{R}^3$ or in the torus T^3. Abidi and Hmidi [4] proved the global existence of strong solutions with small initial density in Besov spaces. Moreover, Abidi and Hmidi [4] allowed for variable viscosity and conductivity coefficients but made the essential assumption that the initial data are closed to a constant state. Chen *et al.* [5] established the local strong solution in the presence of vacuum under the assumptions that both conductivity and viscosity are constants. The global existence of strong solutions was obtained by Huang and Wang [6]. Recently, Gui [7] showed global well-posedness of the two-dimensional inhomogeneous

MHD system with a constant viscosity and variable conductivity coefficients, but without the small density assumption.

If there is no magnetic field ($B = 0$), the MHD system turns out to be the inhomogeneous Navier-Stokes equations. In fact, due to the similarity of the second equation and the third equation in (1.1), the study of the MHD system has been along with that for Navier-Stokes one. There are a lot of studies of incompressible inhomogeneous Navier-Stokes equations. We should mention that Abidi *et al.* [8] proved the local well-posedness of the three-dimensional inhomogeneous incompressible isentropic Navier-Stokes equations in critical spaces but without the small density assumption. Motivated by [8], we shall investigate the global well-posedness of the 3-D incompressible inhomogeneous MHD (1.1) with constant viscosity coefficient in the critical spaces.

If the density ρ is away from zero, we denote $a \stackrel{\text{def}}{=} \rho^{-1} - 1$ and $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\rho)$, then the system (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \operatorname{div}(\tilde{\mu}(a)\mathcal{M})) = (1+a) B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = 0, \qquad \operatorname{div} B = 0, \\ (a,u,B)|_{t=0} = (a_0, u_0, B_0). \end{cases} \tag{MHD}$$

For simplicity, we just take $\mu(\rho) = 1$ and the space dimension $N = 3$. In this case, (MHD) becomes

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \Delta u) = (1+a) B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = 0, \qquad \operatorname{div} B = 0, \\ (a,u,B)|_{t=0} = (a_0, u_0, B_0), \end{cases} \tag{1.2}$$

or equivalently

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla \Pi = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = 0, \qquad \operatorname{div} B = 0, \\ (a,u,B)|_{t=0} = (a_0, u_0, B_0). \end{cases} \tag{1.3}$$

Our main result in this paper is as follows.

Theorem 1.1 *Let* $a_0 \in B^{3/2}_{2,1}(\mathbb{R}^3)$, $u_0 \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$, *and* $B_0 \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$, $\operatorname{div} B_0 = 0$, *and*

$$1 + a_0 \geq \underline{b} \tag{1.4}$$

for some positive constant $\underline{b}$. *There exists a small constant* c *depending on* $\|a_0\|_{B^{3/2}_{2,1}}$ *so that if*

$$\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|B_0\|_{\dot{B}^{1/2}_{2,1}} \leq c,$$

then (1.2) *has a unique global solution* (a,u,B) *satisfying*

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(B^{3/2}_{2,1})}+\big\|(u,B)\big\|_{\widetilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})}+\big\|(u,B)\big\|_{L^1_t(\dot{B}^{5/2}_{2,1})}+\|\nabla\Pi\|_{L^1_t(\dot{B}^{1/2}_{2,1})}\\&\quad\le C\big(\|a_0\|_{B^{3/2}_{2,1}}+\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}+1\big)\exp\{C\sqrt{t}\}\quad \textit{for any } t>0.\end{aligned}\tag{1.5}$$

We now make some comments on the analysis of this paper. Motivated by [8], in order to prove the global well-posedness of Theorem 1.1, we note that as long as $\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}$ is sufficiently small, the lifespan of the local solution thus obtained should be greater than 1, moreover, there exists $t_1\in(0,1)$ so that

$$\big\|u(t_1)\big\|_{\dot{B}^{1/2}_{2,1}\cap\dot{B}^{7/2}_{2,1}}+\big\|B(t_1)\big\|_{\dot{B}^{1/2}_{2,1}\cap\dot{B}^{7/2}_{2,1}}\le C\big(\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}\big).\tag{1.6}$$

We shall first solve v via the classical Navier-Stokes system:

$$\begin{cases}\partial_t v+v\cdot\nabla v-\Delta v+\nabla\Pi_v=0,\\ \operatorname{div} v=0,\\ v|_{t=t_1}=u(t_1),\end{cases}\tag{1.7}$$

and then solve $w=u-v$ via

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho(v+w))=0,\\ \rho\partial_t w+\rho(v+w)\cdot\nabla w-\Delta w+\nabla\Pi_w=(1-\rho)(\partial_t v+v\cdot\nabla v)-\rho w\cdot\nabla v+B\cdot\nabla B,\\ \partial_t B+(v+w)\cdot\nabla B-\Delta B=B\cdot\nabla(v+w),\\ \operatorname{div} w=0,\qquad \operatorname{div} B=0,\\ \rho|_{t=t_1}=\rho(t_1),\qquad w|_{t=t_1}=0,\qquad B|_{t=t_1}=B(t_1).\end{cases}\tag{1.8}$$

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts on Littlewood-Paley analysis that will be used later; then in Section 3, based on the local existence of the solutions and the priori estimates, we prove Theorem 1.1 by a standard continuity argument.

2 Preliminaries

Let us briefly explain how we may proceed in the case $x\in\mathbb{R}^3$ (see *e.g.* [9]). Let φ be a smooth function supported in the ring $\mathcal{C}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^3,\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and such that

$$\sum_{q\in\mathbb{Z}}\varphi\big(2^{-q}\xi\big)=1\quad\text{for }\xi\neq 0.$$

Then, for $u\in\mathcal{S}'(\mathbb{R}^3)$, we set

$$\forall q\in\mathbb{Z},\quad \dot{\Delta}_q u=\varphi\big(2^{-q}\mathrm{D}\big)u\quad\text{and}\quad \dot{S}_q u=\sum_{j\le q-1}\dot{\Delta}_j u.\tag{2.1}$$

We have the formal decomposition

$$u=\sum_{q\in\mathbb{Z}}\dot{\Delta}_q u,\quad \forall u\in\mathcal{S}'\big(\mathbb{R}^3\big)/\mathcal{P}\big[\mathbb{R}^3\big],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials (see [10]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_k \dot{\Delta}_q u \equiv 0 \quad \text{if } |k-q| \geq 2 \quad \text{and} \quad \dot{\Delta}_k(\dot{S}_{q-1} u \dot{\Delta}_q u) \equiv 0 \quad \text{if } |k-q| \geq 5. \tag{2.2}$$

We recall now the definition of the homogeneous Besov spaces from [11].

Definition 2.1 Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$, and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\right)_{\ell^r}.$$

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^3) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'(\mathbb{R}^3) \mid \|u\|_{\dot{B}^s_{p,r}} < \infty\}$.
- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \leq s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u \in \mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta u \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.

Lemma 2.1 (Bernstein-type lemma [11–13]) *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^3$. A constant C exists so that for any positive real number λ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (a,b) with $b \geq a \geq 1$, we have*

$$\begin{aligned} &\operatorname{Supp}\hat{u} \subset \lambda\mathcal{B} \quad \Rightarrow \quad \sup_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{L^b} \leq C^{k+1}\lambda^{k+3(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}, \\ &\operatorname{Supp}\hat{u} \subset \lambda\mathcal{C} \quad \Rightarrow \quad C^{-1-k}\lambda^k\|u\|_{L^a} \leq \sup_{|\alpha|=k}\left\|\partial^\alpha u\right\|_{L^a} \leq C^{1+k}\lambda^k\|u\|_{L^a}. \end{aligned} \tag{2.3}$$

Definition 2.2 (see [14, 15]) Let $s \leq \frac{3}{p}$ (respectively, $s \in \mathbb{R}$), $(r,\lambda,p) \in [1,+\infty]^3$, and $T \in]0,+\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{pr}(\mathbb{R}^3))$ as the completion of $C([0,T],\mathcal{S}(\mathbb{R}^3))$ by the norm

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \left(\sum_{q\in\mathbb{Z}} 2^{qrs}\left(\int_0^T \left\|\dot{\Delta}_q f(t)\right\|^\lambda_{L^p}\,dt\right)^{\frac{r}{\lambda}}\right)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For brevity, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$. In the particular case when $p = r = 2$, we denote this space by $\widetilde{L}^\lambda_T(\dot{H}^s)$.

3 Global existence of (1.2)

We start with the local existence of strong solutions whose proof can be found in Theorem 1.1 of [16].

Lemma 3.1 *Under the assumptions of Theorem* 1.1, *a positive time T exists so that* (1.2) *has a unique local solution $a \in \mathcal{C}_b([0,T]; B^{3/2}_{2,1}(\mathbb{R}^3))$; $u, B \in \mathcal{C}_b([0,T]; \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)) \cap L^1([0,T]; \dot{B}^{5/2}_{2,1}(\mathbb{R}^3))$ and, for $T \geq 1$, we have*

$$\left\|(u,B)\right\|_{\widetilde{L}^\infty([0,T];\dot{B}^{1/2}_{2,1})} + \int_0^T \left\|(u,B)(\tau)\right\|_{\dot{B}^{5/2}_{2,1}}\delta\tau \leq C\left(\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|B_0\|_{\dot{B}^{1/2}_{2,1}}\right). \tag{3.1}$$

Similar to Theorem 2 in [8], we can obtain a higher-order regularity of the local solution to (1.2) as follows.

Proposition 3.1 *Under the assumptions of Theorem* 1.1, *for any* $t_0 > 0$, *we have*

$$\begin{aligned}&\big\|(u,B)\big\|_{\widetilde{L}^\infty([t_0,T];\dot{B}^{3/2}_{2,1})}+\big\|(u,B)\big\|_{L^1([t_0,T];\dot{B}^{7/2}_{2,1})}+\|\nabla\Pi\|_{L^1([t_0,T];\dot{B}^{3/2}_{2,1})}\\&\quad\le C\big(\|a_0\|_{B^{3/2}_{2,1}}\big)\big(\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}\big)(1+1/\sqrt{t_0})\exp\big\{C\big(\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}\big)\big\}.\end{aligned}\tag{3.2}$$

Remark 3.1 Thanks to (3.1) and (3.2), there exists $t_1 \in (0,1)$ so that $u(t_1), B(t_1) \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\cap \dot{B}^{7/2}_{2,1}(\mathbb{R}^3)$ and it satisfies (1.6).

We are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Thanks to Lemma 3.1, we conclude that: given $a_0 \in B^{3/2}_{2,1}(\mathbb{R}^3)$, $u_0, B_0 \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ with $\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|B_0\|_{\dot{B}^{1/2}_{2,1}}$ sufficiently small, (1.2) has a unique local solution (a,u) satisfying $a \in \mathcal{C}([0,T^*); B^{3/2}_{2,1}(\mathbb{R}^3))$, and $u, B \in \mathcal{C}([0,T^*); \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}((0,T^*); \dot{B}^{5/2}_{2,1}(\mathbb{R}^3))$ for some $T^* > 1$. Our aim o is to prove that $T^* = \infty$.

As $\|u(t_1)\|_{\dot{B}^{1/2}_{2,1}\cap \dot{B}^{7/2}_{2,1}}$, $\|B(t_1)\|_{\dot{B}^{1/2}_{2,1}\cap \dot{B}^{7/2}_{2,1}}$ is very small provided $\|u_0\|_{\dot{B}^{1/2}_{2,1}}$, $\|B_0\|_{\dot{B}^{1/2}_{2,1}}$ is sufficiently small. Let v solve the classical Navier-Stokes system (1.7). As $u(t_1)$ is sufficient small in $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$, it follows from the classical theory of Navier-Stokes equations [15] that (1.7) has a unique global solution $v \in \mathcal{C}([t_1,+\infty); \dot{B}^{1/2}_{2,1}) \cap L^1([t_1,+\infty); \dot{B}^{5/2}_{2,1})$ satisfying

$$\begin{aligned}&\|v\|_{\widetilde{L}^\infty([t_1,+\infty);\dot{B}^{1/2}_{2,1})}+\|v\|_{L^1([t_1,+\infty);\dot{B}^{5/2}_{2,1})}+\|\nabla\Pi_v\|_{L^1([t_1,+\infty);\dot{B}^{1/2}_{2,1})}\le C\big\|u(t_1)\big\|_{\dot{B}^{1/2}_{2,1}},\\&\|\partial_t v\|_{L^1([t_1,+\infty);\dot{B}^{1/2}_{2,1})}\le C\big\|u(t_1)\big\|_{\dot{B}^{1/2}_{2,1}}+\big\|\mathrm{div}(v\otimes v)\big\|_{L^1([t_1,+\infty);\dot{B}^{1/2}_{2,1})}\le C\big\|u(t_1)\big\|_{\dot{B}^{1/2}_{2,1}}.\end{aligned}\tag{3.3}$$

With v thus obtained, we denote $w \stackrel{\mathrm{def}}{=} u - v$. Then, thanks to (1.3) and (1.7), w, B solves (1.8). Then the proof of Theorem 1.1 reduces to proving the global well-posedness of (1.8). For simplicity, we just present the *a priori* estimates for smooth enough solutions of (1.8) on $[0,T^*)$.

3.1 The higher regularities of *v*

Proposition 3.2 ([8]) *Let* (v,Π_v) *be the unique global solution of* (1.7) *which satisfies* (3.3). *Then, for* $s_1 \in [\frac{3}{2},\frac{7}{2}]$ *and* $s_2 \in [\frac{1}{2},\frac{3}{2}]$, *we have*

$$\|v\|_{\widetilde{L}^\infty([t_1,+\infty);\dot{B}^{s_1}_{2,1})}+\big\|(\Delta v,\nabla\Pi_v)\big\|_{L^1([t_1,+\infty);\dot{B}^{s_1}_{2,1})}\le C\|u_0\|_{\dot{B}^{1/2}_{2,1}},\tag{3.4}$$

$$\|\partial_t v\|_{\widetilde{L}^\infty([t_1,+\infty);\dot{B}^{s_2}_{2,1})}+\big\|(\partial_t\Delta v,\partial_t\nabla\Pi_v)\big\|_{L^1([t_1,+\infty);\dot{B}^{s_2}_{2,1})}\le C\|u_0\|_{\dot{B}^{1/2}_{2,1}}.\tag{3.5}$$

Corollary 3.1 ([8]) *Under the assumptions of Proposition* 3.2, *one has*

$$\|\nabla v\|_{L^2([t_1,+\infty);L^\infty)}+\|\Delta v-\nabla\Pi_v\|_{L^2([t_1,+\infty);L^\infty)}\le C\|u_0\|_{\dot{B}^{1/2}_{2,1}}.\tag{3.6}$$

3.2 The estimate of (w,B)

Lemma 3.2 (L^2 estimate of (w,B)) *We have for* $t_1 < t < T^*$,

$$\begin{aligned}&\|w\|_{L^\infty([t_1,t];L^2)}+\|\nabla w\|_{L^2([t_1,t];L^2)}+\|B\|_{L^\infty([t_1,t];L^2)}+\|\nabla B\|_{L^2([t_1,t];L^2)}\\&\quad\le C\big(\|u_0\|_{\dot{B}^{1/2}_{2,1}}+\|B_0\|_{\dot{B}^{1/2}_{2,1}}\big)\end{aligned}\tag{3.7}$$

with C being independent of t.

Proof First of all, thanks to (1.4), one deduces from the transport equation of (1.8) that

$$\big(1+\|a_0\|_{\dot B^{3/2}_{2,1}}\big)^{-1}\le \rho(t,x)\le \underline{b}^{-1}, \tag{3.8}$$

from which, with $1-\rho=\rho a$, we get by using the standard energy estimate to the w, B equation of (1.8)

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\sqrt{\rho}w\|_{L^2}^2+\|\nabla w\|_{L^2}^2=\int_{\mathbb{R}^3}\big((1-\rho)(\partial_t v+v\cdot\nabla v)+\rho w\cdot\nabla v+B\cdot\nabla B\big)\cdot w\,dx,\\
&\frac12\frac{d}{dt}\|B\|_{L^2}^2+\|\nabla B\|_{L^2}^2=\int_{\mathbb{R}^3}(B\cdot\nabla v\cdot B+B\cdot\nabla w\cdot B)\,dx;
\end{aligned}$$

thanks to $\operatorname{div} B=0$, one has

$$\int_{\mathbb{R}^3}B\cdot\nabla w\cdot B\,dx+\int_{\mathbb{R}^3}B\cdot\nabla B\cdot w\,dx=0.$$

Therefore, we get

$$\begin{aligned}
&\frac12\frac{d}{dt}\big(\|\sqrt{\rho}w\|_{L^2}^2+\|B\|_{L^2}^2\big)+\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\\
&\quad=\int_{\mathbb{R}^3}\big((1-\rho)(\partial_t v+v\cdot\nabla v)\cdot w+\rho w\cdot\nabla v\cdot w+B\cdot\nabla v\cdot B\big)\,dx\\
&\quad\le C\big(\|\sqrt{\rho}w\|_{L^2}\|a\|_{L^2}\|\partial_t v+v\cdot\nabla v\|_{L^\infty}+\|\nabla v\|_{L^\infty}\big(\|\sqrt{\rho}w\|_{L^2}^2+\|B\|_{L^2}^2\big)\big),
\end{aligned}\tag{3.9}$$

from which we get

$$\begin{aligned}
&\frac12\frac{d}{dt}\big(\|\sqrt{\rho}w\|_{L^2}^2+\|B\|_{L^2}^2\big)\\
&\quad\le C\big(\|\sqrt{\rho}w\|_{L^2}\|a\|_{L^2}\|\partial_t v+v\cdot\nabla v\|_{L^\infty}+\|\nabla v\|_{L^\infty}\big(\|\sqrt{\rho}w\|_{L^2}^2+\|B\|_{L^2}^2\big)\big)\\
&\quad\le C\big(\big(\|\sqrt{\rho}w\|_{L^2}+\|B\|_{L^2}\big)\|a\|_{L^2}\|\partial_t v+v\cdot\nabla v\|_{L^\infty}+\|\nabla v\|_{L^\infty}\big(\|\sqrt{\rho}w\|_{L^2}^2+\|B\|_{L^2}^2\big)\big),
\end{aligned}$$

from which we infer for $t\in(t_1,T^*)$ that

$$\begin{aligned}
&\frac{d}{dt}\big(e^{-2\int_{t_1}^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau}\big(\big\|\sqrt{\rho}w(t)\big\|_{L^2}^2+\big\|B(t)\big\|_{L^2}^2\big)\big)\\
&\quad\le C\|a_0\|_{L^2}e^{-2\int_{t_1}^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau}\big(\|\sqrt{\rho}w\|_{L^2}+\big\|B(t)\big\|_{L^2}\big)\|\Delta v-\nabla\Pi_v\|_{\dot B^{3/2}_{2,1}}.
\end{aligned}$$

This along with (1.6) and (3.4) implies

$$\begin{aligned}
\|\sqrt{\rho}w\|_{L^\infty([t_1,t];L^2)}^2+\big\|B(t)\big\|_{L^\infty([t_1,t];L^2)}^2&\le Ce^{\int_{t_1}^t\|\nabla v(\tau)\|_{L^\infty}\,d\tau}\|\Delta v-\nabla\Pi_v\|_{L^1([t_1,t],\dot B^{3/2}_{2,1})}\\
&\le C\big\|u(t_1)\big\|_{\dot B^{3/2}_{2,1}}\exp\big\{C\big\|u(t_0)\big\|_{\dot B^{1/2}_{2,1}}\big\}\\
&\le C\big(\|u_0\|_{\dot B^{1/2}_{2,1}}+\|B_0\|_{\dot B^{1/2}_{2,1}}\big).
\end{aligned}$$

Plugging the above estimate into (3.9) gives rise to

$$\|\nabla w\|_{L^2([t_1,t];L^2)}+\|\nabla B\|_{L^2([t_1,t];L^2)}\le C\big(\|u_0\|_{\dot B^{1/2}_{2,1}}+\|B_0\|_{\dot B^{1/2}_{2,1}}\big).$$

This completes the proof of Lemma 3.2. □

Lemma 3.3 (H^1 estimate of (w,B)) *There exist two positive constants c_1 and c_2 such that for $t \in [t_1, T^*)$,*

$$\begin{aligned}
&\|(\nabla w, \nabla B)\|^2_{L^\infty([t_1,t],L^2)} + \int_{t_1}^t \left(c_1\|\partial_t w\|^2_{L^2} + c_2\|(\nabla^2 B, \nabla^2 w)\|^2_{L^2} + \|\nabla \Pi_w\|^2_{L^2}\right) dt' \\
&\quad \le C\|u_0\|^2_{\dot{B}^{1/2}_{2,1}}
\end{aligned} \tag{3.10}$$

with C being independent of t.

Proof Taking the L^2 inner product of the w equation of (1.8) with $\frac{1}{\rho}\Delta w$ and using (3.8), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2} + \left\|\frac{1}{\sqrt{\rho}}\Delta w\right\|^2_{L^2} \le C\left\|\frac{1}{\sqrt{\rho}}\Delta w\right\|_{L^2} \big\{&\|\nabla w\|_{L^2}\|v\|_{L^\infty} + \|\nabla w\|_{L^6}\|w\|_{L^3} \\
&+ \|w\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla \Pi_w\|_{L^2} \\
&+ \|a\|_{L^2}\|\partial_t v + v\cdot\nabla v\|_{L^\infty} + \|\nabla B\|_{L^6}\|B\|_{L^3}\big\},
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}\|\nabla w\|^2_{L^2} + \left\|\frac{1}{\sqrt{\rho}}\Delta w\right\|^2_{L^2} \le{}& C\|\nabla w\|^2_{L^2}\|v\|^2_{L^\infty} + \|\Delta w\|^2_{L^2}\|w\|^2_{L^3} + \|w\|^2_{L^2}\|\nabla v\|^2_{L^\infty} \\
&+ \|\nabla \Pi_w\|^2_{L^2} + \|\Delta v - \nabla \Pi_v\|^2_{L^\infty} + \|\Delta B\|^2_{L^2}\|B\|^2_{L^3}.
\end{aligned}$$

Again thanks to the w equation of (1.8) and $\operatorname{div} w = 0$, one has

$$\begin{aligned}
\|\Delta w\|^2_{L^2} + \|\nabla \Pi_w\|^2_{L^2} ={}& \|\Delta w - \nabla \Pi_w\|^2_{L^2} \\
\le{}& C\big\{\|\sqrt{\rho}\partial_t w(t)\|^2_{L^2} + \|w(t)\|^2_{L^3}\|\Delta w(t)\|^2_{L^2} + \|v\|^2_{L^\infty}\|\nabla w\|^2_{L^2} \\
&+ \|\Delta v - \nabla \Pi_v\|^2_{L^\infty} + \|w\|^2_{L^2}\|\nabla v\|^2_{L^\infty} + \|B(t)\|^2_{L^3}\|\Delta B(t)\|^2_{L^2}\big\}.
\end{aligned}$$

As a consequence, we obtain for some positive constant c_1,

$$\begin{aligned}
\frac{d}{dt}\|\nabla w\|^2_{L^2} + c_1\|\nabla^2 w\|^2_{L^2} \le C\big\{&\|w\|^2_{L^3}\|\nabla^2 w\|^2_{L^2} + \|\sqrt{\rho}\partial_t w\|^2_{L^2} \\
&+ \|\nabla w\|^2_{L^2}\|v\|^2_{L^\infty} + \|w\|^2_{L^2}\|\nabla v\|^2_{L^\infty} \\
&+ \|\Delta v - \nabla \Pi_v\|^2_{L^\infty} + \|\Delta B\|^2_{L^2}\|B\|^2_{L^3}\big\}.
\end{aligned} \tag{3.11}$$

Taking the L^2 inner product of the B equation of (1.8) with ΔB,

$$\begin{aligned}
\frac{d}{dt}\|\nabla B\|^2_{L^2} + \|\nabla^2 B\|^2_{L^2} \le C\big\{&\|B\|^2_{L^3}\|\nabla^2 w\|^2_{L^2} + \|w\|^2_{L^3}\|\nabla^2 B\|^2_{L^2} \\
&+ \|\nabla B\|^2_{L^2}\|v\|^2_{L^\infty} + \|B\|^2_{L^2}\|\nabla v\|^2_{L^\infty}\big\}.
\end{aligned} \tag{3.12}$$

Along the same lines, we get by taking the L^2 inner product of the w equation of (1.8) with $\partial_t w$,

$$\begin{aligned}
\frac{d}{dt}\|\nabla w\|^2_{L^2} + \|\sqrt{\rho}\partial_t w\|^2_{L^2} \le C\big(&\|w\|^2_{L^3}\|\nabla^2 w\|^2_{L^2} + \|\nabla w\|^2_{L^2}\|v\|^2_{L^\infty} + \|w\|^2_{L^2}\|\nabla v\|^2_{L^\infty} \\
&+ \|\Delta v - \nabla \Pi_v\|^2_{L^\infty} + \|\Delta B\|^2_{L^2}\|B\|^2_{L^3}\big).
\end{aligned} \tag{3.13}$$

Thanks to (3.11), (3.12), and (3.13), there is a positive constant c_2 so that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\big)+c_2\|\partial_t w\|_{L^2}^2\\
&\qquad+\big(C_3-C_2\big(\|w\|_{L^3}^2+\|B\|_{L^3}^2\big)\big)\big(\|\nabla^2 w\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\big)\\
&\quad\le C_4\big(\|\nabla w\|_{L^2}^2\|v\|_{L^\infty}^2+\|w\|_{L^2}^2\|\nabla v\|_{L^\infty}^2+\|\Delta v-\nabla\Pi_v\|_{L^\infty}^2\\
&\qquad+\|\nabla B\|_{L^2}^2\|v\|_{L^\infty}^2+\|B\|_{L^2}^2\|\nabla v\|_{L^\infty}^2\big)\\
&\quad\le C_4\big(\big(\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\big)\|v\|_{L^\infty}^2\\
&\qquad+\big(\|w\|_{L^2}^2+\|B\|_{L^2}^2\big)\|\nabla v\|_{L^\infty}^2+\|\Delta v-\nabla\Pi_v\|_{L^\infty}^2\big).
\end{aligned}\tag{3.14}$$

Now let τ^* be determined by

$$\tau^*\stackrel{\text{def}}{=}\sup\left\{t\ge t_1,\ \|w(t)\|_{L^3}^2+\|B(t)\|_{L^3}^2\le\frac{C_3}{2C_2}\right\}.\tag{3.15}$$

We claim that $\tau^*=T^*$ provided that $\|u_0\|_{\dot B^{1/2}_{2,1}}+\|B_0\|_{\dot B^{1/2}_{2,1}}$ is sufficiently small. Otherwise for $t\in[t_1,\tau^*)$, it follows from (3.14) that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\big)+c_2\|\partial_t w\|_{L^2}^2+\frac{C_3}{2}\big(\|\nabla^2 w\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2\big)\\
&\quad\le C_4\big(\big(\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\big)\|v\|_{L^\infty}^2\\
&\qquad+\big(\|w\|_{L^2}^2+\|B\|_{L^2}^2\big)\|\nabla v\|_{L^\infty}^2+\|\Delta v-\nabla\Pi_v\|_{L^\infty}^2\big).
\end{aligned}\tag{3.16}$$

Applying Gronwall's inequality to (3.16) and using (3.7), (3.6), (1.6), and (3.3) together with the interpolation inequality yield

$$\begin{aligned}
&\|\nabla w\|_{L^2}^2+\|\nabla B\|_{L^2}^2\\
&\quad\le C_4\exp\left\{C_4\int_{t_1}^t\|v\|_{L^\infty}^2\,dt'\right\}\left[\|\nabla B(t_1)\|_{L^2}^2+\int_{t_1}^t\big(\|\nabla v\|_{L^\infty}^2+\|\Delta v-\nabla\Pi_v\|_{L^\infty}^2\big)\,dt'\right]\\
&\quad\le C_5\big(\|u_0\|_{\dot B^{1/2}_{2,1}}^2+\|B_0\|_{\dot B^{1/2}_{2,1}}^2\big).
\end{aligned}\tag{3.17}$$

However, notice from (3.7) and (3.17) that

$$\begin{aligned}
\|w(t)\|_{L^3}^2+\|B(t)\|_{L^3}^2&\le C\big(\|w(t)\|_{L^2}\|\nabla w(t)\|_{L^2}+\|B(t)\|_{L^2}\|\nabla B(t)\|_{L^2}\big)\\
&\le C_6\big(\|u_0\|_{\dot B^{1/2}_{2,1}}^2+\|B_0\|_{\dot B^{1/2}_{2,1}}^2\big)\le\frac{C_3}{4C_2}\quad\text{for }t\in[t_1,\tau^*)
\end{aligned}$$

provided that $\|u_0\|_{\dot B^{1/2}_{2,1}}^2+\|B_0\|_{\dot B^{1/2}_{2,1}}^2\le\frac{C_3}{4C_2C_6}$, which contradicts with (3.15). This in turn shows that $\tau^*=T^*$. Then integrating (3.16) and using (3.6) leads to (3.10). This completes the proof of the lemma. □

Lemma 3.4 (H^2 estimate of (w,B)) *There exists a time independent constant C so that for $t\in[t_1,T^*)$,*

$$\big\|\big(\nabla^2 w,\nabla^2 B\big)\big\|_{L^\infty([t_1,t],L^2)}+\big\|(\nabla w_t,\nabla B_t)\big\|_{L^2([t_1,t],L^2)}+\big\|\big(\nabla^2 w,\nabla^2 B\big)\big\|_{L^2([t_1,t],L^6)}\le C.\tag{3.18}$$

Proof Step 1. L^2 estimate of $(\sqrt{\rho}w_t, B_t)$.

We get by first acting ∂_t to the w equation of (1.8) and then taking the L^2 inner product of the resulting equation with $\partial_t w$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}w_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^3}(1-\rho)w_t\cdot\partial_t(\Delta v - \nabla\Pi_v)\,dx \\
&\qquad - \int_{\mathbb{R}^3}\rho_t w_t\cdot\big(w_t + (w+v)\cdot\nabla w + w\cdot\nabla v + (\Delta v - \nabla\Pi_v)\big)\,dx \\
&\qquad - \int_{\mathbb{R}^3}\rho w_t\cdot\big((v+w)_t\cdot\nabla w + w_t\cdot\nabla v + w\cdot\nabla v_t\big)\,dx \\
&\qquad + \int_{\mathbb{R}^3}\partial_t(B\cdot\nabla B)w_t\,dx \\
&\quad \stackrel{\text{def}}{=} I + II + III + IV.
\end{aligned}\tag{3.19}$$

The estimates of I, II, and III are similar to [8],

$$|I| \le C\|a_0\|_{\dot{B}^{3/2}_{2,1}}\|\sqrt{\rho}w_t\|_{L^2}\big\|\partial_t(\Delta v - \nabla\Pi_v)\big\|_{L^2},\tag{3.20}$$

$$\begin{aligned}
|II| \le{}& \frac{1}{4}\|\nabla w_t\|_{L^2}^2 + C\big\{\|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\Delta v - \nabla\Pi_v\|_{L^4}^2 \\
&+ \big\|\nabla^2 v\big\|_{L^6}^2 + \|\sqrt{\rho}w_t\|_{L^2}^2\big(\|v\|_{L^\infty}^2 + \|\nabla w\|_{L^2}\big\|\nabla^2 w\big\|_{L^2}\big) \\
&+ \|\sqrt{\rho}w_t\|_{L^2}\big\|\nabla(\Delta v - \nabla\Pi_v)\big\|_{L^4}\big\},
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
|III| \le{}& \frac{1}{16}\|\nabla w_t\|_{L^2}^2 + C\big\{\|\sqrt{\rho}w_t\|_{L^2}^2\big(\|\nabla w\|_{L^2}\big\|\nabla^2 w\big\|_{L^2} + \|\nabla v\|_{L^\infty}\big) \\
&+ \|\sqrt{\rho}w_t\|_{L^2}\big(\|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4}\big)\big\},
\end{aligned}\tag{3.22}$$

which, thanks to (3.10), yields

$$\begin{aligned}
|IV| &= \left|\int_{\mathbb{R}^3}B_t\cdot\nabla B\cdot w_t\,dx + \int_{\mathbb{R}^3}B\cdot\nabla B_t\cdot w_t\,dx\right| \\
&\le \varepsilon\|\nabla w_t\|_{L^2}^2 + C_\varepsilon\|B_t\|_{L^2}^2\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}.
\end{aligned}$$

This together with (3.19), (3.20), (3.21), and (3.22) yields

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}w_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2 \\
&\quad \le C\|\sqrt{\rho}w_t\|_{L^2}\big[\big\|\nabla(\Delta v - \nabla\Pi_v)\big\|_{L^4} \\
&\qquad + \big\|\partial_t(\Delta v - \nabla\Pi_v)\big\|_{L^2} + \|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4}\big] \\
&\qquad + C\|\sqrt{\rho}w_t\|_{L^2}^2\big[\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^2}\big\|\nabla^2 w\big\|_{L^2}\big] \\
&\qquad + C\big[\|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\Delta v - \nabla\Pi_v\|_{L^4}^2 + \big\|\nabla^2 v\big\|_{L^6}^2\big] \\
&\qquad + C_\varepsilon\|B_t\|_{L^2}^2\big\|\nabla^2 B\big\|_{L^2}\|\nabla B\|_{L^2}.
\end{aligned}\tag{3.23}$$

On the other hand, acting by ∂_t to the B equation of (1.8) and then taking the L^2 inner product of the resulting equation with $\partial_t B$ we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|B_t\|_{L^2}^2 + \|\nabla B_t\|_{L^2}^2 \\
&\quad= \int_{\mathbb{R}^3} B_t\cdot\nabla(v+w)\cdot B_t + B\cdot\nabla(v+w)_t\cdot B_t - (v+w)_t\cdot\nabla B\cdot B_t\,dx \\
&\quad\le \varepsilon\big(\|\nabla w_t\|_{L^2}^2 + \|\nabla B_t\|_{L^2}^2\big) + C\|B_t\|_{L^2}^2\big(\|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2} \\
&\qquad + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2} + \|\nabla v\|_{L^\infty}\big) + C\|B_t\|_{L^2}\big(\|\nabla v_t\|_{L^4} + \|v_t\|_{L^\infty}\big).
\end{aligned}\tag{3.24}$$

Thanks to (3.23) and (3.24), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|B_t\|_{L^2}^2\big) + \|\nabla B_t\|_{L^2}^2 + \|\nabla w_t\|_{L^2}^2 \\
&\quad\le C\big(\|\sqrt{\rho}w_t\|_{L^2} + \|B_t\|_{L^2}\big) \\
&\qquad\times\big[\|\nabla(\Delta v - \nabla\Pi_v)\|_{L^4} + \|\partial_t(\Delta v - \nabla\Pi_v)\|_{L^4} + \|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4}\big] \\
&\qquad + C\big(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|B_t\|_{L^2}^2\big) \\
&\qquad\times\big[\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2} + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}\big] \\
&\qquad + C\big[\|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\Delta v - \nabla\Pi_v\|_{L^4}^2 + \|\nabla^2 v\|_{L^6}^2\big] \\
&\quad\stackrel{\text{def}}{=} C\big(\|\sqrt{\rho}w_t\|_{L^2} + \|B_t\|_{L^2}\big)f_3(t) + C\big(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|B_t\|_{L^2}^2\big)f_1(t) + f_2(t) \\
&\quad\le C\big(f_3(t) + f_1(t)\big)\big(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|B_t\|_{L^2}^2\big) + f_2(t) + f_3(t).
\end{aligned}$$

We use

$$\begin{aligned}
f_1(t) &\stackrel{\text{def}}{=} \|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2} + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}, \\
f_2(t) &\stackrel{\text{def}}{=} \|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|\Delta v - \nabla\Pi_v\|_{L^4}^2 + \|\nabla^2 v\|_{L^6}^2, \\
f_3(t) &\stackrel{\text{def}}{=} \|\nabla(\Delta v - \nabla\Pi_v)\|_{L^4} + \|\partial_t(\Delta v - \nabla\Pi_v)\|_{L^2} + \|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4}.
\end{aligned}$$

Applying Gronwall's inequality to (3.23) yields for $t \in (t_1, T^*)$,

$$\begin{aligned}
&\|\sqrt{\rho}w_t(t)\|_{L^2}^2 + \|B_t(t)\|_{L^2}^2 + \int_{t_1}^t \|\nabla w_t(t')\|_{L^2}^2 + \|\nabla B_t(t')\|_{L^2}^2\,dt' \\
&\quad\le C\exp\Big\{C\int_{t_1}^t \big(f_1(t') + f_3(t')\big)\,dt'\Big\} \\
&\qquad\times\Big(\|(\sqrt{\rho}w_t)(t_1)\|_{L^2}^2 + \|B_t(t_1)\|_{L^2}^2 + \int_{t_1}^t \big(f_2(t') + f_3(t')\big)\,dt'\Big).
\end{aligned}\tag{3.25}$$

However, notice that $\dot{B}_{2,1}^{3/2} \hookrightarrow L^\infty$, $\dot{B}_{2,1}^1 \hookrightarrow L^6$, $\dot{B}_{2,1}^{3/4} \hookrightarrow L^4$, and we deduce from (3.3), (3.4), (3.5), (3.7), and (3.10) that

$$\int_{t_1}^t \big(f_1(t') + f_2(t') + f_3(t')\big)\,dt' \le C$$

with C being independent of t. Taking the L^2 inner product of the w, B equation of (1.8) with w_t, B_t at $t = t_1$ and using (3.2) give rise to

$$\begin{aligned}
\big\|(\sqrt{\rho}w_t)(t_1)\big\|_{L^2} &\le C\big(\big\|a(t_1)\big\|_{L^2}\big\|(\partial_t v + v\cdot\nabla v)(t_1)\big\|_{\dot B^{3/2}_{2,1}} + \big\|B\cdot\nabla B(t_1)\big\|_{L^2}\big)\\
&\le C\big(\big\|v(t_1)\big\|_{\dot B^{7/2}_{2,1}} + \big\|v(t_1)\big\|_{\dot B^{3/2}_{2,1}}\big\|v(t_1)\big\|_{\dot B^{5/2}_{2,1}} + \big\|B(t_1)\big\|^{\frac{5}{4}}_{\dot B^{1/2}_{2,1}}\big\|B(t_1)\big\|^{\frac{3}{4}}_{\dot B^{5/2}_{2,1}}\big)\\
&\le C,\\
\big\|B_t(t_1)\big\|_{L^2} \le \big(C\big\|v\cdot\nabla B(t_1)\big\|_{L^2} &+ \big\|\Delta B(t_1)\big\|_{L^2} + \big\|B\cdot\nabla v(t_1)\big\|_{L^2}\big)\\
&\le C\big(\big\|v(t_1)\big\|_{\dot B^{1/2}_{2,1}}\big\|B(t_1)\big\|^{\frac{1}{4}}_{\dot B^{1/2}_{2,1}}\big\|B(t_1)\big\|^{\frac{3}{4}}_{\dot B^{5/2}_{2,1}}\\
&\quad + \big\|B(t_1)\big\|_{\dot B^{1/2}_{2,1}}\big\|v(t_1)\big\|^{\frac{1}{4}}_{\dot B^{1/2}_{2,1}}\big\|v(t_1)\big\|^{\frac{3}{4}}_{\dot B^{5/2}_{2,1}} + \big\|B(t_1)\big\|_{\dot B^{2}_{2,1}}\big)\\
&\le C.
\end{aligned}$$

As a consequence, we deduce from (3.25) that

$$\sup_{t\in[t_1,T^*)}\big(\big\|\sqrt{\rho}w_t(t)\big\|^2_{L^2} + \big\|B_t(t)\big\|^2_{L^2}\big) + \int_{t_1}^t \big\|\nabla w_t(t')\big\|^2_{L^2} + \big\|\nabla B_t(t')\big\|^2_{L^2}\,dt' \le C. \tag{3.26}$$

Step 2. The estimate of $(\nabla^2 w, \nabla^2 B)$.

We first observe from the w and B equation of (1.8) that

$$\begin{aligned}
\big\|\nabla^2 w\big\|_{L^2} + \|\nabla\Pi_w\|_{L^2} &\le \varepsilon\big(\big\|\nabla^2 w\big\|_{L^2} + \big\|\nabla^2 B\big\|_{L^2}\big) + C\big\{\|\sqrt{\rho}w_t\|_{L^2} + \|\nabla w\|^3_{L^2} + \|\nabla B\|^3_{L^2}\\
&\quad + \|v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}\|w\|_{L^2} + \|v\|_{\dot B^{7/2}_{2,1}} + \|v\|_{\dot B^{3/2}_{2,1}}\|v\|_{\dot B^{5/2}_{2,1}}\big\},\\
\big\|\nabla^2 B\big\|_{L^2} &\le \|B_t\|_{L^2} + \big\|(v+w)\cdot\nabla B\big\|_{L^2} + \big\|\nabla(v+w)\cdot B\big\|_{L^2}\\
&\le \varepsilon\big(\big\|\nabla^2 w\big\|_{L^2} + \big\|\nabla^2 w\big\|_{L^2}\big) + C\big\{\|\nabla w\|^2_{L^2}\|\nabla B\|_{L^2} + \|\nabla B\|^2_{L^2}\|\nabla w\|_{L^2}\\
&\quad + \|B_t\|_{L^2} + \|v\|_{L^\infty}\|\nabla B\|_{L^2} + \|\nabla v\|_{L^\infty}\|B\|_{L^2}\big\},
\end{aligned}$$

which along with (3.4), (3.5), (3.7), (3.10), and (3.26) ensures that

$$\sup_{t\in[t_1,T^*)}\big(\big\|\nabla^2 w(t)\big\|_{L^2} + \big\|\nabla^2 B(t)\big\|_{L^2} + \big\|\nabla\Pi_w(t)\big\|_{L^2}\big) \le C. \tag{3.27}$$

On the other hand, let (v,q) solve

$$-\Delta v + \nabla q = f, \qquad \operatorname{div} v = 0.$$

Then one has $\nabla q = -\nabla(-\Delta)^{-1}\operatorname{div} f$ and, for any $r\in(1,\infty)$,

$$\|\nabla q\|_{L^r} \le C\|f\|_{L^r} \quad \text{and} \quad \|\Delta v\|_{L^r} \le C\|f\|_{L^r};$$

from this and the w equation of (1.8), we infer

$$\begin{aligned}
\big\|\nabla^2 w\big\|_{L^6} + \|\nabla\Pi_w\|_{L^6} &\le C\big(\|w_t\|_{L^6} + \|w\cdot\nabla w\|_{L^6} + \|v\cdot\nabla w\|_{L^6} + \|w\cdot\nabla v\|_{L^6}\\
&\quad + \big\|(1-\rho)\big\|_{L^6}\|\partial_t v + v\cdot\nabla v\|_{L^\infty} + \|B\cdot\nabla B\|_{L^6}\big),
\end{aligned}$$

which along with (3.10) implies

$$\begin{aligned}
&\|\nabla^2 w\|_{L^6} + \|\nabla\Pi_w\|_{L^6} \\
&\quad\le C\bigl(\|\nabla w_t\|_{L^2} + \|\nabla w\|_{L^2}\|\nabla^2 w\|_{L^2}^{\frac12}\|\nabla^2 w\|_{L^6}^{\frac12} + \|\nabla^2 w\|_{L^2}\|v\|_{L^\infty} + \|\nabla w\|_{L^2}\|\nabla v\|_{L^\infty} \\
&\qquad + \|v\|_{\dot B^{7/2}_{2,1}} + \|v\|_{\dot B^{3/2}_{2,1}}\|v\|_{\dot B^{5/2}_{2,1}} + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^{\frac12}\|\nabla^2 B\|_{L^6}^{\frac12}\bigr) \\
&\quad\le \varepsilon\bigl(\|\nabla^2 w\|_{L^6} + \|\nabla^2 B\|_{L^6}\bigr) + C\bigl(\|\nabla w_t\|_{L^2} + \|\nabla^2 w\|_{L^2} + \|\nabla^2 B\|_{L^2} + \|v\|_{\dot B^{7/2}_{2,1}} + \|v\|_{\dot B^{5/2}_{2,1}}\bigr).
\end{aligned}$$

On the other hand, following the B equation of (1.8), we infer

$$\|\nabla^2 B\|_{L^6} \le \varepsilon\bigl(\|\nabla^2 w\|_{L^6} + \|\nabla^2 B\|_{L^6}\bigr) + C\bigl(\|\nabla B_t\|_{L^2} + \|\nabla^2 w\|_{L^2} + \|\nabla^2 B\|_{L^2} + \|v\|_{\dot B^{5/2}_{2,1}}\bigr).$$

Therefore, we get

$$\begin{aligned}
&\|\nabla^2 w\|_{L^6} + \|\nabla^2 B\|_{L^6} + \|\nabla\Pi_w\|_{L^6} \\
&\quad\le C\bigl(\|\nabla w_t\|_{L^2} + \|\nabla B_t\|_{L^2} + \|\nabla^2 w\|_{L^2} + \|\nabla^2 B\|_{L^2} + \|v\|_{\dot B^{7/2}_{2,1}} + \|v\|_{\dot B^{5/2}_{2,1}}\bigr).
\end{aligned}$$

Therefore thanks to (3.4), (3.26), and (3.10), we obtain

$$\begin{aligned}
&\|\nabla^2 w\|^2_{L^2([t_1,t];L^6)} + \|\nabla^2 B\|^2_{L^2([t_1,t];L^6)} + \|\nabla\Pi_w\|^2_{L^2([t_1,t];L^6)} \\
&\quad\le C\bigl\{\|\nabla w_t\|^2_{L^2([t_1,t];L^2)} + \|\nabla B_t\|^2_{L^2([t_1,t];L^2)} + \|\nabla^2 w\|^2_{L^2([t_1,t];L^2)} + \|\nabla^2 B\|^2_{L^2([t_1,t];L^2)} \\
&\qquad + \|v\|^2_{L^2([t_1,t];\dot B^{7/2}_{2,1})} + \|v\|^2_{L^2([t_1,t];\dot B^{5/2}_{2,1})}\bigr\} \le C.
\end{aligned}$$

This completes the proof of the lemma. □

3.3 Proof of Theorem 1.1

We first rewrite the momentum equation in (1.3) as

$$\partial_t u + u\cdot\nabla u - \Delta u + \nabla\Pi = (1-\rho)(\partial_t u + u\cdot\nabla u) + B\cdot\nabla B.$$

Then it follows from the classical theory of the homogeneous Navier-Stokes equations (see [15] for instance) that with $t \in [t_1, T^*)$,

$$\begin{aligned}
&\|u\|_{\widetilde L^\infty([t_1,t];\dot B^{1/2}_{2,1})} + \|u\|_{L^1([t_1,t];\dot B^{5/2}_{2,1})} + \|\nabla\Pi\|_{L^1([t_1,t];\dot B^{1/2}_{2,1})} \\
&\quad\le C\|u(t_1)\|_{\dot B^{1/2}_{2,1}} + \int_{t_1}^t \|\nabla u\|_{L^\infty}\|u\|_{\dot B^{1/2}_{2,1}}\,dt' + \|(1-\rho)(\partial_t u + u\cdot\nabla u)\|_{L^1([t_1,t];\dot B^{1/2}_{2,1})} \\
&\qquad + \|B\cdot\nabla B\|_{L^1([t_1,t];\dot B^{1/2}_{2,1})}.
\end{aligned} \tag{3.28}$$

Applying the product law in Besov spaces gives

$$\|(1-\rho)(\partial_t u + u\cdot\nabla u)\|_{L^1([t_1,t];\dot B^{1/2}_{2,1})} \le C\|1-\rho\|_{L^\infty([t_1,t];\dot B^{3/2}_{2,1})}\|\partial_t u + u\cdot\nabla u\|_{L^1([t_1,t];\dot B^{1/2}_{2,1})}.$$

Yet thanks to Lemma 2.1 and (3.18), one has

$$\|\partial_t u\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})} \le Ct^{1/2}\|\partial_t w\|_{L^2([t_1,t];H^1)} + \|u(t_1)\|_{\dot{B}^{1/2}_{2,1}} \le C(t^{1/2}+1),$$

$$\|u\cdot\nabla u\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})} \le C\int_{t_1}^t \big(\|\nabla w\|_{L^2}\|\Delta w\|_{L^2} + \|v\|^2_{\dot{B}^{3/2}_{2,1}}\big)\,dt' \le C,$$

$$\|B\cdot\nabla B\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})} \le C\int_{t_1}^t \|B\|_{\dot{H}^1}\|\nabla B\|_{\dot{H}^1}\,dt' \le C\int_{t_1}^t \|\nabla B\|_{L^2}\|\Delta B\|_{L^2}\,dt' \le C.$$

Thanks to Theorem 2.87 in [13] and Proposition 1 in [8], we have

$$\begin{aligned}&\|1-\rho\|_{\widetilde{L}^\infty([t_1,t];\dot{B}^{3/2}_{2,1})}\\ &\quad\le C\|a\|_{\widetilde{L}^\infty([t_1,t];\dot{B}^{3/2}_{2,1})} \le C\|a(t_1)\|_{\dot{B}^{3/2}_{2,1}}\exp\left\{C\int_{t_1}^t \|u(t')\|_{\dot{B}^{3/2}_{6,1}}\,dt'\right\}.\end{aligned}\tag{3.29}$$

However, applying Lemma 2.1 leads to

$$\|u\|_{\dot{B}^{3/2}_{6,1}} \le C\|\nabla w\|^{1/2}_{L^6}\|\nabla^2 w\|^{1/2}_{L^6} + \|v\|_{\dot{B}^{5/2}_{2,1}},$$

which along with (3.10) and (3.18) implies

$$\begin{aligned}&\|\nabla u\|_{L^1([t_1,t];L^\infty)} + \|u\|_{L^1([t_1,t];\dot{B}^{3/2}_{6,1})}\\ &\quad\le C\|u\|_{L^1([t_1,t];\dot{B}^{3/2}_{6,1})} \le C\big\{\|v\|_{L^1([t_1,t];\dot{B}^{5/2}_{2,1})} + t^{1/2}\|\Delta w\|^{1/2}_{L^2([t_1,t];L^2)}\|\Delta w\|^{1/2}_{L^2([t_1,t];L^6)}\big\}\\ &\quad\le C(1+t^{1/2}).\end{aligned}$$

Therefore, we obtain

$$\|(1-\rho)(\partial_t u + u\cdot\nabla u)\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})} \le C\|a(t_1)\|_{\dot{B}^{3/2}_{2,1}}\exp\{Ct^{1/2}\}.$$

Then applying Gronwall's inequality to (3.28) gives rise to

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty([t_1,t];\dot{B}^{1/2}_{2,1})} + \|u\|_{L^1([t_1,t];\dot{B}^{5/2}_{2,1})} + \|\nabla\Pi\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})}\\ &\quad\le C\exp\{C\sqrt{t}\}\big(\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|a_0\|_{B^{3/2}_{2,1}} + C\big).\end{aligned}\tag{3.30}$$

We first observe from the B equation of (1.8) that

$$\partial_t B - \Delta B = u\cdot\nabla B + B\cdot\nabla u.$$

Similarly, we get

$$\begin{aligned}&\|B\|_{\widetilde{L}^\infty([t_1,t];\dot{B}^{1/2}_{2,1})} + \|B\|_{L^1([t_1,t];\dot{B}^{5/2}_{2,1})}\\ &\quad\le C\|B(t_1)\|_{\dot{B}^{1/2}_{2,1}} + \|u\cdot\nabla B\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})} + \|B\cdot\nabla u\|_{L^1([t_1,t];\dot{B}^{1/2}_{2,1})}\\ &\quad\le C\|B_0\|_{\dot{B}^{1/2}_{2,1}} + C.\end{aligned}\tag{3.31}$$

From (3.29), (3.30), and (3.31), we infer by a standard argument that $T^* = \infty$. Moreover, the global solution thus obtained, (a, u, B, Π), satisfies (1.5). This completes the proof of Theorem 1.1. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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