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# Existence of solutions for functional boundary value problems of second-order nonlinear differential equations system at resonance

Weihua Jiang* and Bingzhi Sun

*Correspondence:
weihuajiang@hebust.edu.cn
College of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei, P.R. China

**Abstract**

In this paper, by using the coincidence degree theory due to Mawhin and constructing suitable operators, we study the solvability for functional boundary value problems of second-order nonlinear differential equations system at resonance with $\dim \operatorname{Ker} L = 3$ and 4, respectively.

**Keywords:** coincidence degree theory; functional boundary condition; resonance; Fredholm operator

## 1 Introduction

The existence of solutions for integer order differential equations with specific boundary conditions and resonance scenarios have been studied by many authors (see [1–14] and the references therein). Recently, attention has shifted to problems with linear functional conditions. The differential operator $L : C^1[0,1] \to L^1[0,1]$, $Lx = x''$ known to us is done in [15] for a resonant problem, where the authors studied the existence of solutions to the problem of second-order nonlinear differential equation

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), & 0 < t < 1, \\ \Gamma_1(x) = 0, \qquad \Gamma_2(x) = 0, \end{cases}$$

which generalizes recent work on multi-point and integral boundary value problems. Although it excellently generalizes and extends many results for nonlocal second-order problems at resonance, it does not contain a complete analysis for this problem. For example, in [16], to see this, set $B_1(t) = \alpha b, B_1(1) = \alpha a, B_2(t) = b, B_2(1) = a$, where $a, b, \alpha \in R$ and $a, b \neq 0$, then $B_1(t)B_2(1) = B_1(1)B_2(t)$ with $\operatorname{Ker} L = \{c(at - b) : c \in R\}$, $\dim \operatorname{Ker} L = 1$. This case cannot be derived from the results of [15] pertaining to the cases of resonance. And in [15], the authors also make the unnecessary artificial assumptions $\Gamma_1(t^2) \neq 0, \Gamma_1(t^3) \neq 0$, notably, for these assumptions, some interesting results have been obtained in [17] for a resonant problem that allow us to bypass above minor technical difficulty (see Lemma 1.1 below). Thus, we improve the results of [1–13] and [14] in that respect as well. In addition, it clearly can also be used for higher order problems with functional conditions see [18, 19]. Inspired by the above literature, we will study the existence of solutions to functional boundary value problems of differential equations system. To the best of our knowledge, this subject has not been studied. In the present paper, we investigate the following equations:

$$\begin{cases} x''(t) = f(t, x(t), y(t), x'(t), y'(t)), & t \in [0,1], \\ y''(t) = g(t, x(t), y(t), x'(t), y'(t)), & t \in [0,1], \\ \Gamma_1(x) = 0, \qquad \Gamma_2(x) = 0, \\ \Gamma_3(y) = 0, \qquad \Gamma_4(y) = 0, \end{cases} \tag{1.1}$$

where $\Gamma_i : C^1[0,1] \to \mathbb{R}$, $i = 1, 2, 3, 4$, are continuous linear functionals. we will always suppose that the following condition holds:

$(H)$ Let $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy Carathéodory conditions, *i.e.*, $f(\cdot, u)$ and $g(\cdot, u)$ are measurable for each fixed $u \in \mathbb{R}^4$, $f(t, \cdot)$ and $g(t, \cdot)$ are continuous for a.e. $t \in [0,1]$ and $\sup\{|f(t,x)| : x \in D_0\}, \sup\{|g(t,x)| : x \in D_0\} \in L^1([0,1])$ for any compact set $D_0 \in \mathbb{R}^4$.

**Lemma 1.1** ([17]) *There must exist* $h_1 \in L^1[0,1]$ *such that* $(\Gamma_1 - \alpha_1\Gamma_2)(\int_0^t (t-s)h_1(s)\,ds) = 1$.

**Definition 1.1** We say $(x, y) \in C^1[0,1] \times C^1[0,1]$ is a solution of functional boundary value problems (FBVPs) (1.1) which means that $(x, y)$ satisfies (1.1).

## 2 Preliminaries

We present some necessary definitions and lemmas. Consider the following conditions:

$(A_1)$ $\frac{\Gamma_1(t)}{\Gamma_2(t)} = \frac{\Gamma_1(1)}{\Gamma_2(1)}$, $\Gamma_3(1) = 0$, $\Gamma_3(t) = 0$, $\Gamma_4(1) = 0$, $\Gamma_4(t) = 0$,
$(A_2)$ $\frac{\Gamma_3(t)}{\Gamma_4(t)} = \frac{\Gamma_3(1)}{\Gamma_4(1)}$, $\Gamma_1(1) = 0$, $\Gamma_1(t) = 0$, $\Gamma_2(1) = 0$, $\Gamma_2(t) = 0$,
$(A_3)$ $\Gamma_1(1) = \Gamma_1(t) = \Gamma_2(1) = \Gamma_2(t) = 0$, $\Gamma_3(1) = \Gamma_3(t) = \Gamma_4(1) = \Gamma_4(t) = 0$.

We should prove the following. If $(A_1)$ or $(A_2)$ holds, then

$$\operatorname{Ker} L = \{(c_1(at-b), ct+d) | c_1, c, d \in \mathbb{R}, a^2 + b^2 \neq 0\} \quad \text{or}$$
$$\operatorname{Ker} L = \{at+b, c_2(ct-d) | a, b, c_2 \in \mathbb{R}, c^2 + d^2 \neq 0\}.$$

If $(A_3)$ holds, then $\operatorname{Ker} L = \{(at+b, ct+d) | a, b, c, d \in \mathbb{R}\}$. In fact, if exchange the places of $\Gamma_1$ and $\Gamma_3$, $\Gamma_2$ and $\Gamma_4$ in the boundary value conditions, respectively, condition $(A_1)$ just becomes $(A_2)$. So we only need to focus on the FBVPs (1.1) under conditions $(A_1)$, $(A_3)$.

As usual, we shall use the classical spaces $C^1[0,1]$ and $L^1[0,1]$. For $(x, y) \in C^1[0,1] \times C^1[0,1]$, we define the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}$, where $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. We denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. Similarly, for $(u, v) \in L^1[0,1] \times L^1[0,1]$, we denote the norm $\|(u,v)\|_1$ and define the norm $\|(u,v)\|_1 = \max\{\|u\|_1, \|v\|_1\}$, where $\|u\|_1 = \int_0^1 |u(t)|\,dt, u \in L^1[0,1]$. We also use the Sobolev space $W^{2,1}(0,1)$ defined by

$$W^{2,1}(0,1) = \{(x,y) \in C[0,1] \times C[0,1] | x, x', y, y' \text{ are absolutely continuous on } [0,1]\}.$$

Let $Y = C^1[0,1] \times C^1[0,1]$ with norm $\|(x,y)\|$, $Z = L^1[0,1] \times L^1[0,1]$ with norm $\|(x,y)\|_1$. Clearly, $Y, Z$ are Banach spaces.

Let the linear operator $L : \operatorname{dom} L \subset Y \to Z$ be defined by $L(x,y) = (x'', y'')$, where

$$\operatorname{dom} L = \{(x,y) \in W^{2,1}(0,1) : \Gamma_1(x) = 0, \Gamma_2(x) = 0, \Gamma_3(y) = 0, \Gamma_4(y) = 0\}.$$

Let the nonlinear operator $N : Y \to Z$ be defined by

$$\big(N(x,y)\big)(t) = \big(f\big(t,x(t),y(t),x'(t),y'(t)\big), g\big(t,x(t),y(t),x'(t),y'(t)\big)\big).$$

Then FBVPs (1.1) can be written as $L(x,y) = N(x,y)$.

**Definition 2.1** Let $Y$, $Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator. $Y$ is said to be the Fredholm operator of index zero provided that:

(i) $\operatorname{Im} L$ is a closed subset of $Z$;
(ii) $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L < +\infty$.

Let $Y$, $Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator. $L$ is said to be the Fredholm operator of index zero. $P : Y \to Y$, $Q : Z \to Z$ are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is reversible. We denote the inverse of the mapping by $K_P$ (generalized inverse operator of $L$). If $\Omega$ is an open bounded subset of $Y$ such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the mapping $N : Y \to Z$ will be called $L$-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ and $K_P(I - Q)N : \overline{\Omega} \to Y$ are continuous and compact.

The following is the Kolmogorov-Riesz criterion (see, for example, [20])

**Lemma 2.1** *For* $1 \le p < \infty, E \subset L^P[0,1]$ *is compact if*

(a) *$E$ is bounded*;
(b) *the limit* $\lim_{\varepsilon\to 0} \int_0^1 |g(s+\varepsilon) - g(s)|^p \, ds = 0$ *is uniform in* $E$.

**Lemma 2.2** ([16]) *Let* $L : \operatorname{dom} L \subset Y \to Z$ *be a Fredholm operator of index zero and* $N : Y \to Z$ *is* $L$*-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied*:

(i) $Lu \neq \lambda Nu$ *for every* $(u,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
(ii) $Nu \notin \operatorname{Im} L$ *for every* $u \in \operatorname{Ker} L \cap \partial\Omega$;
(iii) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $Q : Z \to Z$ *is a continuous projector such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

*Then the equation* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Now, we give $\operatorname{Ker} L, \operatorname{Im} L$ and some necessary operators under conditions $(A_1)$ and $(A_3)$, respectively.

**Lemma 2.3** *There exist* $m_i, n_i \in \mathbb{N}^+, m_i, n_i > 1, m_i \neq n_i, i = 1,2$ *such that* $\Gamma_1(t^{n_1})\Gamma_2(t^{m_1}) - \Gamma_1(t^{m_1})\Gamma_2(t^{n_1}) \neq 0$, $\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2}) \neq 0$.

*Proof* For convenience, assume, by way of contradiction, that $\frac{\Gamma_1(t^{m_1})}{\Gamma_2(t^{m_1})} = \frac{\Gamma_1(t^{n_1})}{\Gamma_2(t^{n_1})} = k$ for all $m_1, n_1 \in \mathbb{N}^*$, so we have

$$\Gamma_1\big(t^{m_1}\big) = k\Gamma_2\big(t^{m_1}\big) \quad \text{or} \quad \Gamma_1\big(t^{n_1}\big) = k\Gamma_2\big(t^{n_1}\big).$$

By $(A_2)$, $(\Gamma_1 - k\Gamma_2)(1) = (\Gamma_1 - k\Gamma_2)(t) = 0$. Thus, $\Gamma_1(p(t)) = k\Gamma_2(p(t))$ for every polynomial $p$.

Since $\Gamma_1(x) - k\Gamma_2(x) \neq 0$ on all of $x \in C^1[0,1]$, there exists $v_0 \in C^1[0,1]$ such that $\Gamma_1(v_0) - k\Gamma_2(v_0) \neq 0$. Choose a sequence of polynomials $\{p_m\}$ such that $\|v_0 - p_m\| < \frac{1}{m}$. Then $0 \neq |(\Gamma_1 - k\Gamma_2)(v_0)| = |(\Gamma_1 - k\Gamma_2)(v_0 - p_m) + (\Gamma_1 - k\Gamma_2)(p_m)| = |(\Gamma_1 - k\Gamma_2)(v_0 - p_m)| \leq \|(\Gamma_1 - k\Gamma_2)\|\|v_0 - p_m\| < (\beta_1 + |\alpha|\beta_2)\frac{1}{m}$ for all $m \in \mathbb{N}$, which is a contradiction. Similarly, for $\Gamma_3$ and $\Gamma_4$, we omit the corresponding details as straightforward. □

For convenience, we denote

($B_1$) The linear functionals $\Gamma_1, \Gamma_2 : Y \to \mathbb{R}$ satisfy $\Gamma_2(t) = b, \Gamma_2(1) = a, \Gamma_1(t) = \alpha_1 b, \Gamma_1(1) = \alpha_1 a$, where $a^2 + b^2 \neq 0, \alpha_1, a, b \in \mathbb{R}$.

($B_2$) The functionals $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 : Y \to \mathbb{R}$ are linear continuous with respective norms $\beta_1, \beta_2, \beta_3, \beta_4$, that is, $|\Gamma_i(x)| \leq \beta_i\|x\|, |\Gamma_j(y)| \leq \beta_j\|y\|, i = 1,2, j = 3,4$.

**Lemma 2.4** *Assume* ($A_1$) *holds, then* $L : \operatorname{dom} L \subset Y \to Z$ *is a Fredholm mapping of index zero*, $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L = 3$.

*Proof* If $(x,y) \in \operatorname{Ker} L$ and $L(x,y) = (x'', y'') = (0,0)$, we have $(x(t), y(t)) = (k_1 t + k_2, k_3 t + k_4)$.
Based on the condition ($A_1$), we have

$$x = c_1(at - b), \qquad y = ct + d,$$

where $c_1, c, d \in \mathbb{R}$. So,

$$\operatorname{Ker} L = \{(c_1(at - b), ct + d) | a^2 + b^2 \neq 0, c_1, c, d \in \mathbb{R}\}, \qquad \dim \operatorname{Ker} L = 3.$$

Now, we verify

$$\begin{aligned} \operatorname{Im} L = \biggl\{ &(u,v) \in Z : (\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)u(s)\,ds\right) = 0, \\ &\Gamma_j\left(\int_0^t (t-s)v(s)\,ds\right) = 0, j = 3,4 \biggr\}. \end{aligned} \tag{2.1}$$

Let $(u,v) \in \operatorname{Im} L$, then there exists $(x,y) \in \operatorname{dom} L$ such that $L(x,y) = (u,v)$, that is,

$$\begin{cases} x(t) = \int_0^t (t-s)u(s)\,ds + x(0) + x'(0)t, \\ y(t) = \int_0^t (t-s)v(s)\,ds + y(0) + y'(0)t, \end{cases}$$

and $\Gamma_i(x) = 0, \Gamma_j(y) = 0, i = 1,2, j = 3,4$. Hence,

$$\begin{cases} \Gamma_i(x) = \Gamma_i(\int_0^t (t-s)u(s)\,ds) + \Gamma_i(1)x(0) + x'(0)\Gamma_i(t) = 0, \quad i = 1,2, \\ \Gamma_j(y) = \Gamma_j(\int_0^t (t-s)v(s)\,ds) + \Gamma_j(1)y(0) + y'(0)\Gamma_j(t) = 0, \quad j = 3,4. \end{cases}$$

Considering the resonance condition ($B_1$), we have

$$(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)u(s)\,ds\right) = 0, \qquad \Gamma_j\left(\int_0^t (t-s)v(s)\,ds\right) = 0, \quad j = 3,4.$$

That is,

$$\begin{aligned}\operatorname{Im} L \subseteq \biggl\{&(u,v) \in Z : (\Gamma_1 - \alpha_1\Gamma_2)\biggl(\int_0^t (t-s)u(s)\,ds\biggr) = 0,\\ &\Gamma_j\biggl(\int_0^t (t-s)v(s)\,ds\biggr) = 0, j = 3,4\biggr\}.\end{aligned}$$

If

$$\begin{aligned}(u,v) \in \biggl\{&(u,v) \in Z : (\Gamma_1 - \alpha_1\Gamma_2)\biggl(\int_0^t (t-s)u(s)\,ds\biggr) = 0,\\ &\Gamma_j\biggl(\int_0^t (t-s)v(s)\,ds\biggr) = 0, j = 3,4\biggr\},\end{aligned}$$

take

$$\bigl(x(t),y(t)\bigr) = \biggl(-\frac{bt+a}{a^2+b^2}\Gamma_2\biggl(\int_0^t (t-s)u(s)\,ds\biggr) + \int_0^t (t-s)u(s)\,ds, \int_0^t (t-s)v(s)\,ds\biggr).$$

It is clear that $L(x,y) = (x'',y'') = (u,v)$ and $\Gamma_i(x) = 0, \Gamma_j(y) = 0, i = 1,2, j = 3,4$.
That is, $(u,v) \in \operatorname{Im} L$, *i.e.*

$$\begin{aligned}\biggl\{&(u,v) \in Z : (\Gamma_1 - \alpha_1\Gamma_2)\biggl(\int_0^t (t-s)u(s)\,ds\biggr) = 0,\\ &\Gamma_j\biggl(\int_0^t (t-s)v(s)\,ds\biggr) = 0, j = 3,4\biggr\} \subseteq \operatorname{Im} L.\end{aligned}$$

Combining the above we obtain (2.1).
Define $Q : Z \to Z$ as follows: $Q(u,v) = (Q_1 u, (T_1 v)t^{n_2-2} + (T_2 v)t^{m_2-2})$, where

$$\begin{aligned}Q_1 u &= (\Gamma_1 - \alpha_1\Gamma_2)\biggl(\int_0^t (t-s)u(s)\,ds\biggr)h_1(t),\\ T_1 v &= \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\int_0^t (t-s)v(s)\,ds) - \Gamma_3(t^{m_2})\Gamma_4(\int_0^t (t-s)v(s)\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})},\\ T_2 v &= -\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\int_0^t (t-s)v(s)\,ds) - \Gamma_3(t^{n_2})\Gamma_4(\int_0^t (t-s)v(s)\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})},\end{aligned}$$

and $h_1$ is introduced in Lemma 1.1, $m_2$ and $n_2$ are the same as in Lemma 2.3.
By Lemma 2.3, $(B_1)$, and the property of $h_1$ in Lemma 1.1, we have

$$\begin{aligned}Q_1^2 u &= (\Gamma_1 - \alpha_1\Gamma_2)\biggl(\int_0^t (t-s)u(s)\,ds\biggr)h_1(t) = Q_1 u,\\ T_1&\bigl((T_1 v)t^{n_2-2}\bigr)\\ &= \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\int_0^t (t-s)s^{n_2-2}\,ds) - \Gamma_3(t^{m_2})\Gamma_4(\int_0^t (t-s)s^{n_2-2}\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_1 v\\ &= \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\frac{t^{n_2}}{n_2(n_2-1)}) - \Gamma_3(t^{m_2})\Gamma_4(\frac{t^{n_2}}{n_2(n_2-1)})]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_1 v\\ &= T_1 v,\end{aligned}$$

$$\begin{aligned}
&T_1\bigl((T_2v)t^{m_2-2}\bigr)\\
&\quad= \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\int_0^t(t-s)s^{m_2-2}\,ds)-\Gamma_3(t^{m_2})\Gamma_4(\int_0^t(t-s)s^{m_2-2}\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_2v\\
&\quad= \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\frac{t^{m_2}}{m_2(m_2-1)})-\Gamma_3(t^{m_2})\Gamma_4(\frac{t^{m_2}}{m_2(m_2-1)})]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_2v\\
&\quad= 0,\\
&T_2\bigl((T_1v)t^{n_2-2}\bigr)\\
&\quad= -\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\int_0^t(t-s)s^{n_2-2}\,ds)-\Gamma_3(t^{n_2})\Gamma_4(\int_0^t(t-s)s^{n_2-2}\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_1v\\
&\quad= -\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\frac{t^{n_2}}{n_2(n_2-1)})-\Gamma_3(t^{n_2})\Gamma_4(\frac{t^{n_2}}{n_2(n_2-1)})]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_1v\\
&\quad= 0,\\
&T_2\bigl((T_2v)t^{m_2-2}\bigr)\\
&\quad= -\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\int_0^t(t-s)s^{m_2-2}\,ds)-\Gamma_3(t^{n_2})\Gamma_4(\int_0^t(t-s)s^{m_2-2}\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_2v\\
&\quad= -\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\frac{t^{m_2}}{m_2(m_2-1)})-\Gamma_3(t^{n_2})\Gamma_4(\frac{t^{m_2}}{m_2(m_2-1)})]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}T_2v\\
&\quad= T_2v.
\end{aligned}$$

We have, for each $(u,v)\in Z$,

$$\begin{aligned}
Q^2(u,v) &= \bigl(Q_1^2u, T_1\bigl[(T_1v)t^{n_2-2}+(T_2v)t^{m_2-2}\bigr]t^{n_2-2}+T_2\bigl[(T_1v)t^{n_2-2}+(T_2v)t^{m_2-2}\bigr]t^{m_2-2}\bigr)\\
&= \bigl(Q_1u,(T_1v)t^{n_2-2}+(T_2v)t^{m_2-2}\bigr)\\
&= Q(u,v).
\end{aligned}$$

So $Q:Z\to Z$ is a continuous linear projector such that $\operatorname{Im}L=\operatorname{Ker}Q$ and $\operatorname{Im}Q=\{(c_1h_1(t),ct^{n_2-2}+dt^{m_2-2})|c_1,c,d\in\mathbb{R}\}$. It is clear that $Z=\operatorname{Im}L\oplus\operatorname{Im}Q$ and $\dim\operatorname{Ker}L=\operatorname{co}\dim\operatorname{Im}L=3$, that is, $L$ is a Fredholm mapping of index zero. □

Define an operator $P:Y\to Y$ as follows:

$$P(x,y)(t)=\left(\frac{1}{a^2+b^2}\bigl(ax'(0)-bx(0)\bigr)(at-b),y'(0)t+y(0)\right),\quad t\in[0,1].$$

It is easy to check that $P^2(x,y)=P(x,y)$, $(x,y)\in Y$, it is also elementary to confirm the identity $\operatorname{Im}P=\operatorname{Ker}L$. So, $Y=\operatorname{Ker}L\oplus\operatorname{Ker}P$.

The mapping $K_P:\operatorname{Im}L\to\operatorname{dom}L\cap\operatorname{Ker}P$ defined by

$$K_P(u,v)(t)=\left(-\frac{bt+a}{a^2+b^2}\Gamma_2\left(\int_0^t(t-s)u(s)\,ds\right)+\int_0^t(t-s)u(s)\,ds,\int_0^t(t-s)v(s)\,ds\right)$$

is the inverse of $L$. In fact, $LK_P(u,v) = (u,v)$ for all $(u,v) \in \operatorname{Im} L$. For $(x,y) \in \operatorname{dom} L \cap \operatorname{Ker} P$,

$$\begin{aligned}
K_P L(x,y)(t) &= \bigl(K_P\bigl(x'',y''\bigr)\bigr)(t) \\
&= \left(-\frac{bt+a}{a^2+b^2}\Gamma_2\left(\int_0^t (t-s)x''(s)\,ds\right) + \int_0^t (t-s)x''(s)\,ds, \int_0^t (t-s)y''(s)\,ds\right) \\
&= \biggl(-\frac{bt+a}{a^2+b^2}\Gamma_2\bigl(x(t)-x'(0)t-x(0)\bigr) + x(t) - x'(0)t \\
&\quad - x(0), y(t) - y'(0)t - y(0)\biggr) \\
&= \left(-\frac{bt+a}{a^2+b^2}\bigl(bx'(0)+ax(0)\bigr) + x(t) - x'(0)t - x(0), y(t) - y'(0)t - y(0)\right) \\
&= \left(x(t) - \frac{at-b}{a^2+b^2}\bigl(ax'(0)-bx(0)\bigr), y(t) - y'(0)t - y(0)\right) \\
&= \bigl(x(t), y(t)\bigr) - P(x,y)(t) \\
&= \bigl(x(t), y(t)\bigr).
\end{aligned}$$

Thus, $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$.

**Lemma 2.5** *If* $(A_3)$ *holds, Then* $L : \operatorname{dom} L \subset Y \to Z$ *is a Fredholm mapping of index zero,* $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L = 4$.

*Proof* Considering $(A_3)$, for every $a,b,c,d \in \mathbb{R}$, $\Gamma_i(at+b) = a\Gamma_i(t) + b\Gamma_i(1) = 0$, $\Gamma_j(ct+d) = c\Gamma_j(t) + d\Gamma_j(1) = 0$, $i = 1,2, j = 3,4$.

So it is easy to obtain

$$\operatorname{Ker} L = \bigl\{(at+b, ct+d) | a,b,c,d \in \mathbb{R}\bigr\}, \qquad \dim \operatorname{Ker} L = 4.$$

For each $(u,v) \in \operatorname{Im} L$, there exists $(x,y) \in \operatorname{dom} L$ such that $L(x,y) = (x'',y'') = (u,v)$. Hence,

$$\begin{cases} x(t) = \int_0^t (t-s)u(s)\,ds + x(0) + x'(0)t, \\ y(t) = \int_0^t (t-s)v(s)\,ds + y(0) + y'(0)t. \end{cases}$$

From the above equations, we have

$$\begin{aligned}
&\Gamma_i(x) = \Gamma_i\left(\int_0^t (t-s)u(s)\,ds\right) = 0, \\
&\Gamma_j(y) = \Gamma_j\left(\int_0^t (t-s)v(s)\,ds\right) = 0, \quad i = 1,2, j = 3,4.
\end{aligned}$$

Therefore,

$$\operatorname{Im} L \subseteq \left\{(u,v) \in Z : \Gamma_i\left(\int_0^t (t-s)u(s)\,ds\right) = 0, \Gamma_j\left(\int_0^t (t-s)v(s)\,ds\right) = 0, i = 1,2, j = 3,4\right\}.$$

For each $(u,v)\in Z$ satisfying $\Gamma_i(\int_0^t(t-s)u(s)\,ds)=0, \Gamma_j(\int_0^t(t-s)v(s)\,ds)=0, i=1,2, j=3,4$, let

$$x(t)=\int_0^t(t-s)u(s)\,ds,\qquad y(t)=\int_0^t(t-s)v(s)\,ds.$$

We have $L(x,y)=(u(t),v(t)), t\in(0,1)$ and

$$\begin{aligned}
&\Gamma_i(x)=\Gamma_i\left(\int_0^t(t-s)u(s)\,ds\right)=0,\quad i=1,2,\\
&\Gamma_j(y)=\Gamma_j\left(\int_0^t(t-s)v(s)\,ds\right)=0,\quad j=3,4.
\end{aligned}$$

That is, $(u,v)\in\operatorname{Im}L$, *i.e.*,

$$\left\{(u,v)\in Z:\Gamma_i\left(\int_0^t(t-s)u(s)\,ds\right)=0,\Gamma_j\left(\int_0^t(t-s)v(s)\,ds\right)=0,i=1,2,j=3,4\right\}\subseteq\operatorname{Im}L.$$

From the above two aspects, we have

$$\operatorname{Im}L=\left\{(u,v)\in Z:\Gamma_i\left(\int_0^t(t-s)u(s)\,ds\right)=0,i=1,2,\Gamma_j\left(\int_0^t(t-s)v(s)\,ds\right)=0,j=3,4\right\}.$$

By Lemma 2.3, define $Q:Z\to Z$ as follows:

$$\begin{aligned}
&Q(u,v)\\
&\quad=\left(\frac{n_1(n_1-1)[\Gamma_2(t^{m_1})\Gamma_1(\int_0^t(t-s)u(s)\,ds)-\Gamma_1(t^{m_1})\Gamma_2(\int_0^t(t-s)u(s)\,ds)]}{\Gamma_1(t^{n_1})\Gamma_2(t^{m_1})-\Gamma_1(t^{m_1})\Gamma_2(t^{n_1})}t^{n_1-2}\right.\\
&\qquad-\frac{m_1(m_1-1)[\Gamma_4(t^{n_1})\Gamma_1(\int_0^t(t-s)u(s)\,ds)-\Gamma_1(t^{n_1})\Gamma_2(\int_0^t(t-s)u(s)\,ds)]}{\Gamma_1(t^{n_1})\Gamma_2(t^{m_1})-\Gamma_1(t^{m_1})\Gamma_2(t^{n_1})}t^{m_1-2},\\
&\qquad\frac{n_2(n_2-1)[\Gamma_4(t^{m_2})\Gamma_3(\int_0^t(t-s)v(s)\,ds)-\Gamma_3(t^{m_2})\Gamma_4(\int_0^t(t-s)v(s)\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}t^{n_2-2}\\
&\qquad\left.-\frac{m_2(m_2-1)[\Gamma_4(t^{n_2})\Gamma_3(\int_0^t(t-s)v(s)\,ds)-\Gamma_3(t^{n_2})\Gamma_4(\int_0^t(t-s)v(s)\,ds)]}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2})-\Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}t^{m_2-2}\right).
\end{aligned}$$

Similarly, we can get $Q^2(u,v)=Q(u,v)$, so $Q:Z\to Z$ is a well-defined projector. Now, it is obvious that $\operatorname{Im}L=\operatorname{Ker}Q$. Noting that $Q$ is a linear projector, we have $Z=\operatorname{Im}Q\oplus\operatorname{Ker}Q$. So, $Z=\operatorname{Im}L\oplus\operatorname{Im}Q$ and $\dim\operatorname{Ker}L=\dim\operatorname{Im}Q=\operatorname{co}\dim\operatorname{Im}L=4$. So, $L$ is a Fredholm mapping of index zero. □

Let the mapping $P:Y\to Y$ be defined by

$$P(x,y)(t)=\big(x(0)+x'(0)t,y(0)+y'(0)t\big),\quad t\in[0,1].$$

Noting that $P$ is a continuous linear projector and $\operatorname{Ker}P=\{(x,y)\in Y:x(0)=0,x'(0)=0,y(0)=0,y'(0)=0\}$, it is easy to know that $Y=\operatorname{Ker}L\oplus\operatorname{Ker}P$.

The generalized inverse operator of $L$, $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be defined by

$$K_P(u,v)(t) = \left( \int_0^t (t-s)u(s)\,ds, \int_0^t (t-s)v(s)\,ds \right)$$

is the inverse of $L$. In fact, if $(u,v) \in \operatorname{Im} L$, then

$$LK_P(u,v) = \left( \left[ \int_0^t (t-s)u(s)\,ds \right]'', \left[ \int_0^t (t-s)v(s)\,ds \right]'' \right) = (u,v).$$

If $(x,y) \in \operatorname{dom} L \cap \operatorname{Ker} P$, then $L(x,y) = (x'', y''), x(0) + x'(0)t = 0$ and $y(0) + y'(0)t = 0$. We have

$$\begin{aligned} K_P L(x,y) &= \left( \int_0^t (t-s)x''(s)\,ds, \int_0^t (t-s)y''(s)\,ds \right) \\ &= \big( x(t) - x'(0)t - x(0), y(t) - y'(0)t - y(0) \big) \\ &= \big( x(t), y(t) \big). \end{aligned}$$

Thus, $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$.

## 3 Main results

By making use of Lemmas 2.2, 2.3 and 2.4, we can obtain the following existence theorem for FBVPs (1.1) at $\dim \operatorname{Ker} L = 3$.

**Theorem 3.1** *Assume* $(A_1)$, $(H)$ *and the following conditions hold*:

$(D_1)$. *There exist constants* $M_1 > 0, M_2 > 0$ *such that, for* $(x,y) \in \operatorname{dom} L$, *if* $|x(t)| + |x'(t)| > M_1$, *for* $t \in [0,1]$, *then*

$$(\Gamma_1 - \alpha_1 \Gamma_2)\left( \int_0^t (t-s) f\big(s, x(s), y(s), x'(s), y'(s)\big)\,ds \right) \neq 0,$$

*if* $|y(t)| + |y'(t)| > M_2$, *for* $t \in [0,1]$,

$$\Gamma_3\left( \int_0^t (t-s) g\big(s, x(s), y(s), x'(s), y'(s)\big)\,ds \right) \neq 0,$$

*or*

$$\Gamma_4\left( \int_0^t (t-s) g\big(s, x(s), y(s), x'(s), y'(s)\big)\,ds \right) \neq 0.$$

$(D_2)$. *There exist nonnegative functions* $a_i, b_i, e_i, d_i, \rho_i \in L^1[0,1], i = 1,2$ *such that*

$$\begin{aligned} &\big| f(t, x_1, x_2, y_1, y_2) \big| < \rho_1(t) + a_1(t)|x_1| + b_1(t)|x_2| + e_1(t)|y_1| + d_1(t)|y_2|, \\ &\big| g(t, x_1, x_2, y_1, y_2) \big| \\ &\quad < \rho_2(t) + a_2(t)|x_1| + b_2(t)|x_2| + e_2(t)|y_1| + d_2(t)|y_2|, \quad t \in [0,1], x_i, y_i \in \mathbb{R}, i = 1,2. \end{aligned}$$

$(D_3)$. *There exist constants* $E_i > 0, i = 1,2,3$, *such that either for each* $(c_1, b_3, b_4) \in \mathbb{R}^3$:

$|c_1| > E_1$, *then*

$$c_1(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) > 0, \tag{3.1}$$

$|b_3| > E_2$, *then*

$$b_3\Gamma_3\left(\int_0^t (t-s)g\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) > 0, \tag{3.2}$$

$|b_4| > E_3$, *then*

$$b_4\Gamma_4\left(\int_0^t (t-s)g\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) > 0, \tag{3.3}$$

*or* $(c_1, b_3, b_4) \in \mathbb{R}^3 : |c_1| > E_1$, *then*

$$c_1(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) < 0, \tag{3.4}$$

$|b_3| > E_2$, *then*

$$b_3\Gamma_3\left(\int_0^t (t-s)g\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) < 0, \tag{3.5}$$

$|b_4| > E_3$, *then*

$$b_4\Gamma_4\left(\int_0^t (t-s)g\big(s, c_1(as-b), b_3 s + b_4, c_1 a, b_3\big)\,ds\right) < 0. \tag{3.6}$$

*Then FBVPs* (1.1) *has at least one solution in* $C^1[0,1] \times C^1[0,1]$ *provided that*

$$B_1 + \frac{C_1 B_2}{1 - C_2} < 1, \qquad C_2 + \frac{C_1 B_2}{1 - B_1} < 1,$$

*where* $B_1 = \|a_1\|_1 + \|e_1\|_1, B_2 = \|a_2\|_1 + \|e_2\|_1, C_1 = \|b_1\|_1 + \|d_1\|_1, C_2 = \|b_2\|_1 + \|d_2\|_1$.

The proof of Theorem 3.1 will be based on the next two lemmas.

**Lemma 3.1** *Assume that* $(A_1), (H), (D_1), (D_2)$ *and* $(D_3)$ *hold. Then*

$$\Omega_1 = \big\{(x,y) \in \operatorname{dom} L \setminus \operatorname{Ker} L : L(x,y) = \lambda N(x,y), \textit{ for some } \lambda \in [0,1]\big\},$$

*and*

$$\Omega_2 = \big\{(x,y) \in \operatorname{Ker} L : N(x,y) \in \operatorname{Im} L\big\}$$

*are bounded.*

*Proof* For $(x,y) \in \Omega_1$, we have $(x,y) \notin \operatorname{Ker} L, \lambda \neq 0$ and $N(x,y) \in \operatorname{Im} L$.

So

$$(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) = 0,$$
$$\Gamma_3\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) = 0,$$

and

$$\Gamma_4\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) = 0.$$

By $(D_1)$, there exist constants $t_i \in [0,1], i = 1,2$ such that $|x(t_1)| \le M_1, |x'(t_1)| \le M_1, |y(t_2)| \le M_2, |y'(t_2)| \le M_2$.

Since $x(t) = x(t_1) + \int_{t_1}^t x'(s)\,ds$, $y(t) = y(t_2) + \int_{t_2}^t y'(s)\,ds$, we get

$$\big|x(t)\big| \le \|x'\|_\infty + M_1, \qquad \big|y(t)\big| \le \|y'\|_\infty + M_2, \quad t \in [0,1]. \tag{3.7}$$

Thus,

$$\big\|(x,y)\big\| \le \max\big\{\|x'\|_\infty, \|y'\|_\infty\big\} + \max\{M_1,M_2\}. \tag{3.8}$$

By $L(x,y) = \lambda N(x,y)$, we obtain

$$\big(x'(t),y'(t)\big) = \left(\lambda\int_{t_1}^t f\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds + x'(t_1),\right.$$
$$\left.\lambda\int_{t_2}^t g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds + y'(t_2)\right),$$

thus, $|x'(t)| < \|N_1x\|_1 + M_1, |y'(t)| < \|N_2y\|_1 + M_2$, where $N(x,y) = (N_1x, N_2y)$,

$$N_1x = f\big(s,x(s),x'(s),y(s),y'(s)\big), \qquad N_2y = g\big(s,x(s),x'(s),y(s),y'(s)\big).$$

That is, $\max\{\|x'\|_\infty, \|y'\|_\infty\} < \|N(x,y)\|_1 + \max\{M_1,M_2\}$.

By $(D_2)$ and (3.7), we have

$$\begin{aligned}\big|x'(t)\big| &< \|\rho_1\|_1 + \|a_1\|_1\|x\|_\infty + \|b_1\|_1\|y\|_\infty + \|e_1\|_1\|x'\|_\infty + \|d_1\|_1\|y'\|_\infty + M_1\\ &< \|\rho_1\|_1 + \|a_1\|_1M_1 + \|b_1\|_1M_2 + \big(\|a_1\|_1 + \|e_1\|_1\big)\|x'\|_\infty\\ &\quad + \big(\|b_1\|_1 + \|d_1\|_1\big)\|y'\|_\infty + M_1,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\big|y'(t)\big| &< \|\rho_2\|_1 + \|a_2\|_1M_1 + \|b_2\|_1M_2 + \big(\|a_2\|_1 + \|e_2\|_1\big)\|x'\|_\infty\\ &\quad + \big(\|b_2\|_1 + \|d_2\|_1\big)\|y'\|_\infty + M_2,\end{aligned} \tag{3.10}$$

for the sake of brevity, let $A_1 = \|\rho_1\|_1 + \|a_1\|_1M_1 + \|b_1\|_1M_2 + M_1, A_2 = \|\rho_2\|_1 + \|a_2\|_1M_1 + \|b_2\|_1M_2 + M_2$, then by (3.10) and (3.9), we have $\|y'\|_\infty < \frac{A_2+B_2\|x'\|_\infty}{1-C_2}$, $\|x'\|_\infty < \frac{A_1+\frac{C_1A_2}{1-C_2}}{1-B_1-\frac{C_1B_2}{1-C_2}}$.

Similarly, $\|y'\|_\infty < \frac{A_2+\frac{B_2A_1}{1-B_1}}{1-C_2-\frac{C_1B_2}{1-B_1}}$.

By (3.8), $\|(x,y)\| < \infty$. Therefore $\Omega_1$ is bounded.

For $(x,y) \in \Omega_2$, $(x,y) = (c_1(at-b), b_3t+b_4), c_1, b_3, b_4 \in \mathbb{R}$ and $N(x,y) \in \operatorname{Im} L$. So,

$$(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right) = 0$$

and

$$b_j\Gamma_j\left(\int_0^t (t-s)g\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right) = 0, \quad j = 3,4.$$

Considering $(D_3)$, $|c_1| \le E_1, |b_3| \le E_2, |b_4| \le E_3$, we have $\|x\| \le E_1\|at-b\|$, $\|y\| \le E_2 + E_3$. Therefore $\Omega_2$ is bounded. □

**Lemma 3.2** *Assume that* $(A_1)$, $(H)$ *and* $(D_3)$ *hold. Then*

$$\Omega_3 = \bigl\{(x,y) \in \operatorname{Ker} L : \lambda J(x,y) + (1-\lambda)QN(x,y) = 0, \lambda \in [0,1]\bigr\}$$

*is bounded, where* $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is homeomorphous*: $(x,y) = (c_1(at-b), b_3+b_4t), c_1, b_3, b_4 \in \mathbb{R}$,

$$\begin{aligned} &J(x,y) \\ &\quad = \Biggl(c_1h_1, \\ &\qquad \frac{n_2(n_2-1)[\Gamma_4(t^{m_2})b_3 - \Gamma_3(t^{m_2})b_4]t^{n_2-2} - m_2(m_2-1)[\Gamma_4(t^{n_2})b_3 - \Gamma_3(t^{n_2})b_4]t^{m_2-2}}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}\Biggr). \end{aligned}$$

*Proof* For $(x,y) \in \Omega_3$, $\lambda J(x,y) + (1-\lambda)QN(x,y) = 0$. If $\lambda = 1$, then $c_1 = 0, b_3 = 0, b_4 = 0$. That is, $(x,y) = 0$. If $\lambda \ne 1$, we can have

$$\lambda c_1h_1 = -(1-\lambda)(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right)h_1, \tag{3.11}$$

$$\begin{aligned} &\Gamma_4\bigl(t^{m_2}\bigr)\left(\lambda b_3 + (1-\lambda)\Gamma_3\left(\int_0^t (t-s)g\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right)\right) \\ &\quad - \Gamma_3\bigl(t^{m_2}\bigr)\left(\lambda b_4 + (1-\lambda)\Gamma_4\left(\int_0^t (t-s)g\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right)\right) = 0, \end{aligned}$$

and

$$\begin{aligned} &\Gamma_4\bigl(t^{n_2}\bigr)\left(\lambda b_3 + (1-\lambda)\Gamma_3\left(\int_0^t (t-s)g\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right)\right) \\ &\quad - \Gamma_3\bigl(t^{n_2}\bigr)\left(\lambda b_4 + (1-\lambda)\Gamma_4\left(\int_0^t (t-s)g\bigl(s, c_1(as-b), b_3s+b_4, c_1a, b_3\bigr)\,ds\right)\right) = 0. \end{aligned}$$

From Lemma 2.3,

$$\begin{vmatrix} \Gamma_4\bigl(t^{m_2}\bigr) & \Gamma_3\bigl(t^{m_2}\bigr) \\ \Gamma_4\bigl(t^{n_2}\bigr) & \Gamma_3\bigl(t^{n_2}\bigr) \end{vmatrix} \ne 0,$$

it yields

$$\begin{cases} \lambda b_3 + (1-\lambda)\Gamma_3(\int_0^t (t-s)g(s,c_1(as-b),b_3s+b_4,c_1a,b_3)\,ds) = 0, \\ \lambda b_4 + (1-\lambda)\Gamma_4(\int_0^t (t-s)g(s,c_1(as-b),b_3s+b_4,c_1a,b_3)\,ds) = 0, \end{cases}$$

if $|c_1| > E_1, |b_3| > E_2, |b_4| > E_3$, considering above equalities, (3.11) and (3.1)-(3.3), we have

$$\lambda c_1^2 h_1 = -(1-\lambda)c_1(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\bigl(s,c_1(as-b),b_3s+b_4,c_1a,b_3\bigr)\,ds\right) < 0,$$

$$\lambda b_3^2 = -(1-\lambda)b_3\Gamma_3\left(\int_0^t (t-s)g\bigl(s,c_1(as-b),b_3s+b_4,c_1a,b_3\bigr)\,ds\right) < 0,$$

$$\lambda b_4^2 = -(1-\lambda)b_4\Gamma_4\left(\int_0^t (t-s)g\bigl(s,c_1(as-b),b_3s+b_4,c_1a,b_3\bigr)\,ds\right) < 0.$$

Thus $|c_1| \le E_1, |b_3| \le E_2, |b_4| \le E_3$. So, $\Omega_3$ is bounded.

If (3.4)-(3.6) hold, then let

$$\Omega_3 = \bigl\{(x,y) \in \operatorname{Ker} L : -\lambda J(x,y) + (1-\lambda)QN(x,y) = 0, \lambda \in [0,1]\bigr\}.$$

By the same method we can also see that $\Omega_3$ is bounded. □

*Proof of Theorem* 3.1 Let $\Omega$ be a bounded open subset of $Y$ such that $\bigcup_{j=1}^3 \overline{\Omega}_j \subset \Omega$. The compactness of $K_P(I-Q)N : \overline{\Omega} \to Y$ and $QN(\overline{\Omega})$ will follow from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. Thus $N$ is $L$-compact on $\overline{\Omega}$.

Then from above arguments, we have

(i) $L(x,y) \neq \lambda N(x,y)$, for every $((x,y),\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(ii) $N(x,y) \notin \operatorname{Im} L$, for every $(x,y) \in \operatorname{Ker} L \cap \partial\Omega$.

At last we will prove that (iii) of Lemma 2.2. is satisfied.

Let $H((x,y),\lambda) = \pm\lambda J(x,y) + (1-\lambda)QN(x,y) = 0$, noting that $\Omega_3 \subset \Omega$, we know $H((x,y),\lambda) \neq 0$ for every $((x,y),\lambda) \in \partial\Omega \cap \operatorname{Ker} L$. Thus, by the homotopic property of degree

$$\begin{aligned} \deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(x,y,0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg\bigl(H(x,y,1), \Omega \cap \operatorname{Ker} L, 0\bigr) = \deg(\pm J, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \end{aligned}$$

Then by Lemma 2.2, $L(x,y) = N(x,y)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. The proof of Theorem 3.1 is completed. □

**Theorem 3.2** *Assume* $(A_3), (D_2), (H)$ *and the following conditions hold*:

$(D_4)$. *There exist constants* $M_3 > 0, M_4 > 0$ *such that, for* $(x,y) \in \operatorname{dom} L$, *if* $|x(t)| + |x'(t)| > M_3$, *for* $t \in [0,1]$, *then*

$$\Gamma_1\left(\int_0^t (t-s)f\bigl(s,x(s),y(s),x'(s),y'(s)\bigr)\,ds\right) \neq 0,$$

*or*

$$\Gamma_2\left(\int_0^t (t-s)f\bigl(s,x(s),y(s),x'(s),y'(s)\bigr)\,ds\right) \neq 0,$$

*if* $|y(t)| + |y'(t)| > M_4$, *for* $t \in [0,1]$, *then*

$$\Gamma_3\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) \neq 0,$$

*or*

$$\Gamma_4\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) \neq 0.$$

$(D_5)$. *There exist constants* $E_i > 0, i = 4,5$, *such that either for each* $(a_1,a_2,b_3,b_4) \in \mathbb{R}^4$:
$|a_1| > E_4$, *then*

$$a_1\Gamma_1\left(\int_0^t (t-s)f(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) > 0, \tag{3.12}$$

$|a_2| > E_5$, *then*

$$a_2\Gamma_2\left(\int_0^t (t-s)f(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) > 0, \tag{3.13}$$

$|b_3| > E_6$, *then*

$$b_3\Gamma_3\left(\int_0^t (t-s)g(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) > 0, \tag{3.14}$$

$|b_4| > E_7$, *then*

$$b_4\Gamma_4\left(\int_0^t (t-s)g(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) > 0, \tag{3.15}$$

*or for each* $(a_1,a_2,b_3,b_4) \in \mathbb{R}^4$:
$|a_1| > E_4$, *then*

$$a_1\Gamma_1\left(\int_0^t (t-s)f(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) < 0, \tag{3.16}$$

$|a_2| > E_5$, *then*

$$a_2\Gamma_2\left(\int_0^t (t-s)f(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) < 0, \tag{3.17}$$

$|b_3| > E_6$, *then*

$$b_3\Gamma_3\left(\int_0^t (t-s)g(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) < 0, \tag{3.18}$$

$|b_4| > E_7$, *then*

$$b_4\Gamma_4\left(\int_0^t (t-s)g(s,a_1s+a_2,b_3s+b_4,a_1,b_3)\,ds\right) < 0. \tag{3.19}$$

*Then FBVP* (1.1) *has at least one solution in* $C^1[0,1] \times C^1[0,1]$ *provided that*

$$B_1 + \frac{C_1 B_2}{1 - C_2} < 1, \qquad C_2 + \frac{C_1 B_2}{1 - B_1} < 1,$$

*where* $B_1 = \|a_1\|_1 + \|e_1\|_1, B_2 = \|a_2\|_1 + \|e_2\|_1, C_1 = \|b_1\|_1 + \|d_1\|_1, C_2 = \|b_2\|_1 + \|d_2\|_1$.

The proof of Theorem 3.2 will also be based on the next two lemmas.

**Lemma 3.3** *Assume that* $(A_3), (B_2), (H), (D_2), (D_4)$ *and* $(D_5)$ *hold. Then*

$$\Omega_1 = \bigl\{(x,y) \in \operatorname{dom} L \setminus \operatorname{Ker} L : L(x,y) = \lambda N(x,y), \textit{ for some } \lambda \in [0,1]\bigr\}$$

*and*

$$\Omega_2 = \bigl\{(x,y) \in \operatorname{Ker} L : N(x,y) \in \operatorname{Im} L\bigr\}$$

*are bounded.*

*Proof* For $(x,y) \in \Omega_1$, we have $(x,y) \notin \operatorname{Ker} L, \lambda \neq 0$ and $N(x,y) \in \operatorname{Im} L$.
So

$$\Gamma_1\left(\int_0^t (t-s) f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\, ds\right) = 0,$$
$$\Gamma_2\left(\int_0^t (t-s) f\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\, ds\right) = 0,$$
$$\Gamma_3\left(\int_0^t (t-s) g\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\, ds\right) = 0,$$
$$\Gamma_4\left(\int_0^t (t-s) g\bigl(s, x(s), x'(s), y(s), y'(s)\bigr)\, ds\right) = 0.$$

By $(D_4)$, there exist constants $t_i \in [0,1], i = 3,4$ such that

$$\bigl|x(t_3)\bigr| \le M_3, \qquad \bigl|x'(t_3)\bigr| \le M_3, \qquad \bigl|y(t_4)\bigr| \le M_4, \qquad \bigl|y'(t_4)\bigr| \le M_4.$$

Since

$$x(t) = x(t_3) + \int_{t_3}^t x'(s)\, ds, \qquad y(t) = y(t_4) + \int_{t_4}^t y'(s)\, ds,$$

we get

$$\bigl|x(t)\bigr| \le \bigl\|x'\bigr\|_\infty + M_3, \qquad \bigl|y(t)\bigr| \le \bigl\|y'\bigr\|_\infty + M_4, \quad t \in [0,1]. \tag{3.20}$$

Thus,

$$\bigl\|(x,y)\bigr\| \le \max\bigl\{\bigl\|x'\bigr\|_\infty, \bigl\|y'\bigr\|_\infty\bigr\} + \max\{M_3, M_4\}. \tag{3.21}$$

By the proof of method in Lemma 3.1, we obtain $\|x'\|_\infty < \frac{A_3+\frac{C_1A_4}{1-C_2}}{1-B_1-\frac{C_1B_2}{1-C_2}}$, $\|y'\|_\infty < \frac{A_4+\frac{B_2A_3}{1-B_1}}{1-C_2-\frac{C_1B_2}{1-B_1}}$, where $A_3 = \|\rho_1\|_1 + \|a_1\|_1 M_3 + \|b_1\|_1 M_4 + M_3, A_4 = \|\rho_2\|_1 + \|a_2\|_1 M_3 + \|b_2\|_1 M_4 + M_4$, by (3.21), $\|(x,y)\| < \infty$. Therefore $\Omega_1$ is bounded.

For $(x,y) \in \Omega_2, (x,y)(t) = (a_1 + a_2 t, b_3 + b_4 t), a_i, b_j \in \mathbb{R}, i = 1,2, j = 3,4, t \in [0,1]$ and $N(x,y) \in \operatorname{Im} L$.

So

$$\Gamma_1\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right) = 0,$$
$$\Gamma_2\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right) = 0,$$
$$\Gamma_3\left(\int_0^t (t-s)g(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right) = 0,$$

and

$$\Gamma_4\left(\int_0^t (t-s)g(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right) = 0.$$

Considering $(D_5)$, $|a_1| \le E_4, |a_2| \le E_5, |b_3| \le E_6, |b_4| \le E_7$, so $\|x\| \le E_4 + E_5$, $\|y\| \le E_6 + E_7$.

Therefore, $\Omega_2$ is bounded. □

**Lemma 3.4** *Assume that* $(A_3), (B_2), (H)$ *and* $(D_5)$ *hold. Then*

$$\Omega_3 = \left\{(x,y) \in \operatorname{Ker} L : \lambda J(x,y) + (1-\lambda)QN(x,y) = 0, \lambda \in [0,1]\right\}$$

*is bounded, where* $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is homeomorphous*: $(x,y)(t) = (a_1 + a_2 t, b_3 + b_4 t)$, $a_1, a_2, b_3, b_4 \in \mathbb{R}$,

$$\begin{aligned} &J(x,y)(t) \\ &\quad= \left(\frac{n_1(n_1-1)[\Gamma_2(t^{m_1})a_1 - \Gamma_1(t^{m_1})a_2]t^{n_1-2} - m_1(m_1-1)[\Gamma_2(t^{n_1})a_1 - \Gamma_1(t^{n_1})a_2]t^{m_1-2}}{\Gamma_1(t^{n_1})\Gamma_2(t^{m_1}) - \Gamma_1(t^{m_1})\Gamma_2(t^{n_1})},\right. \\ &\qquad \left.\frac{n_2(n_2-1)[\Gamma_4(t^{m_2})b_3 - \Gamma_3(t^{m_2})b_4]t^{n_2-2} - m_2(m_2-1)[\Gamma_4(t^{n_2})b_3 - \Gamma_3(t^{n_2})b_4]t^{m_2-2}}{\Gamma_3(t^{n_2})\Gamma_4(t^{m_2}) - \Gamma_3(t^{m_2})\Gamma_4(t^{n_2})}\right). \end{aligned}$$

*Proof* For every $(x,y) \in \Omega_3$, $\lambda J(x,y) + (1-\lambda)QN(x,y) = 0$. If $\lambda = 1$, then $a_1 = 0, a_2 = 0, b_3 = 0, b_4 = 0$. That is, $(x,y) = 0$. If $\lambda \neq 1$, we can have

$$\begin{aligned} &\Gamma_2(t^{m_1})\left(\lambda a_1 + (1-\lambda)\Gamma_1\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right)\right) \\ &\quad - \Gamma_1(t^{m_1})\left(\lambda a_2 + (1-\lambda)\Gamma_2\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right)\right) = 0, \\ &\Gamma_2(t^{n_1})\left(\lambda a_1 + (1-\lambda)\Gamma_1\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right)\right) \\ &\quad - \Gamma_1(t^{n_1})\left(\lambda a_2 + (1-\lambda)\Gamma_2\left(\int_0^t (t-s)f(s, a_1 + a_2 s, b_3 + b_4 s, a_2, b_4)\,ds\right)\right) = 0, \end{aligned}$$

$$\begin{aligned}&\Gamma_4\bigl(t^{m_2}\bigr)\biggl(\lambda b_3+(1-\lambda)\Gamma_3\biggl(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds\biggr)\biggr)\\&\quad-\Gamma_3\bigl(t^{m_2}\bigr)\biggl(\lambda b_4+(1-\lambda)\Gamma_4\biggl(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds\biggr)\biggr)=0,\end{aligned}$$

and

$$\begin{aligned}&\Gamma_4\bigl(t^{n_2}\bigr)\biggl(\lambda b_3+(1-\lambda)\Gamma_3\biggl(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds\biggr)\biggr)\\&\quad-\Gamma_3\bigl(t^{n_2}\bigr)\biggl(\lambda b_4+(1-\lambda)\Gamma_4\biggl(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds\biggr)\biggr)=0.\end{aligned}$$

From Lemma 2.3,

$$\begin{vmatrix}\Gamma_2(t^{m_1}) & \Gamma_1(t^{m_1})\\ \Gamma_2(t^{n_1}) & \Gamma_1(t^{n_1})\end{vmatrix}\neq 0\quad\text{and}\quad\begin{vmatrix}\Gamma_4(t^{m_2}) & \Gamma_3(t^{m_2})\\ \Gamma_4(t^{n_2}) & \Gamma_3(t^{n_2})\end{vmatrix}\neq 0,$$

it yields

$$\begin{cases}\lambda a_1^2+(1-\lambda)a_1\Gamma_1(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds)=0,\\ \lambda a_2^2+(1-\lambda)a_2\Gamma_2(\int_0^t(t-s)f(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds)=0,\\ \lambda b_3^2+(1-\lambda)b_3\Gamma_3(\int_0^t(t-s)g(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds)=0,\\ \lambda b_4^2+(1-\lambda)b_4\Gamma_4(\int_0^t(t-s)g(s,a_1+a_2s,b_3+b_4s,a_2,b_4)\,ds)=0,\end{cases}$$

if $|a_1|>E_4,|a_2|>E_5,|b_3|>E_6,|b_4|>E_7$, considering the above equalities and (3.12)-(3.15), we have $\|x\|\le E_4+E_5$, $\|y\|\le E_6+E_7$. So, $\Omega_3$ is bounded.

If (3.16)-(3.19) hold, then let $\Omega_3=\{(x,y)\in\operatorname{Ker}L:-\lambda J(x,y)+(1-\lambda)QN(x,y)=0,\lambda\in[0,1]\}$. Similar to the above arguments, we can show that $\Omega_3$ is bounded, too. □

*Proof of Theorem* 3.2 Let $\Omega$ be a bounded open subset of $Y$ such that $\bigcup_{j=1}^3\overline{\Omega}_j\subset\Omega$. The compactness of $K_P(I-Q)N:\overline{\Omega}\to Y$ and $QN(\overline{\Omega})$ will follow from the Arzela-Ascoli theorem and the Kolmogorov-Riesz criterion, respectively. Thus $N$ is $L$-compact on $\overline{\Omega}$. Then from the above arguments, we have

(i) $L(x,y)\neq\lambda N(x,y)$, for every $((x,y),\lambda)\in[(\operatorname{dom}L\setminus\operatorname{Ker}L)\cap\partial\Omega]\times(0,1)$;

(ii) $N(x,y)\notin\operatorname{Im}L$, for every $(x,y)\in\operatorname{Ker}L\cap\partial\Omega$.

At last we will prove that (iii) of Lemma 2.2 is satisfied.

Let $H((x,y),\lambda)=\pm\lambda J(x,y)+(1-\lambda)QN(x,y)=0$, noting that $\Omega_3\subset\Omega$, we know $H((x,y),\lambda)\neq 0$ for every $((x,y),\lambda)\in\partial\Omega\cap\operatorname{Ker}L$. Thus, by the homotopic property of degree

$$\begin{aligned}\deg(QN|_{\operatorname{Ker}L},\Omega\cap\operatorname{Ker}L,0)&=\deg\bigl(H(x,y,0),\Omega\cap\operatorname{Ker}L,0\bigr)\\&=\deg\bigl(H(x,y,1),\Omega\cap\operatorname{Ker}L,0\bigr)=\deg(\pm J,\Omega\cap\operatorname{Ker}L,0)\neq 0.\end{aligned}$$

Then by Lemma 2.2, $L(x,y)=N(x,y)$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$. The proof of Theorem 3.2 is completed. □

The next lemma provides norm estimates needed for the following result.

**Lemma 3.5** *For* $(u,v) \in Z, K_P(u,v) = (K_{P_1}u, K_{P_2}v)$, *where* $K_{P_1}u = -\frac{bt+a}{a^2+b^2}\Gamma_2(\int_0^t (t-s)u(s)\,ds) + \int_0^t (t-s)u(s)\,ds, K_{P_2}v = \int_0^t (t-s)v(s)\,ds$, *then*

(1) $\|K_{P_1}u\| \le \|K_{P_1}\|\|u\|_1$,

(2) $\|K_{P_2}v\| \le \|v\|_1$,

*where* $\|K_{P_1}\| = (\frac{\|bt+a\|\beta_2}{a^2+b^2} + 1)$.

*Proof* Observe that due to $|\Gamma_2(x)| \le \beta_2\|x\|$,

$$\begin{aligned}
|K_{P_1}u| &= \left|-\frac{bt+a}{a^2+b^2}\Gamma_2\left(\int_0^t (t-s)u(s)\,ds\right) + \int_0^t (t-s)u(s)\,ds\right| \\
&\le \frac{|bt+a|}{a^2+b^2}\beta_2\left\|\left(\int_0^t (t-s)u(s)\,ds\right)\right\| + \left\|\left(\int_0^t (t-s)u(s)\,ds\right)\right\| \\
&\le \left(\frac{|bt+a|}{a^2+b^2}\beta_2 + 1\right)\|u\|_1 \le \left(\frac{\|bt+a\|}{a^2+b^2}\beta_2 + 1\right)\|u\|_1
\end{aligned}$$

and $|(K_{P_1}u)'(t)| \le (\frac{|b|}{a^2+b^2}\beta_2 + 1)\|u\|_1$; (1) follows from the above two inequalities. Similarly, we can obtain (2). □

**Theorem 3.3** *Assume* $(A_1)$ *with* $a \neq 0$, $(H)$, $(D_3)$ (*of Theorem* 3.1) *and the following conditions hold*:

$(D_5)$. *There exist constants* $M_1, M_5, M_6 > 0$ *such that, for* $(x,y) \in \operatorname{dom} L$, *if* $|x'(t)| > M_1$, *for* $t \in [0,1]$, *then*

$$(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)f\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) \neq 0,$$

*if* $|y'(t)| > M_5$,

$$\Gamma_3\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) \neq 0,$$

*or if* $|y(t)| > M_6$,

$$\Gamma_4\left(\int_0^t (t-s)g\big(s,x(s),y(s),x'(s),y'(s)\big)\,ds\right) \neq 0.$$

$(D_6)$. *There exist nonnegative functions* $a_i, b_i, e_i, d_i, \rho_i \in L^1[0,1], i = 1,2$ *such that*

$$\begin{aligned}
&\big|f(t,x_1,x_2,y_1,y_2)\big| < \rho_1(t) + a_1(t)|x_1| + b_1(t)|x_2| + e_1(t)|y_1| + d_1(t)|y_2|, \\
&\big|g(t,x_1,x_2,y_1,y_2)\big| < \rho_2(t) + a_2(t)|x_1| + b_2(t)|x_2| \\
&\qquad\qquad + e_2(t)|y_1| + d_2(t)|y_2|, \quad t \in [0,1], x_i, y_i \in \mathbb{R}, i = 1,2,
\end{aligned}$$

*where*

$$\begin{aligned}
&\big(\|t-b/a\| + \big(\|t-b/a\|+1\big)\|K_{P_1}\|\big)\big(\|a_1\|_1 + \|b_1\|_1\big) + \big(\|t-b/a\| \\
&\quad + \big(\|t-b/a\|+1\big)\|K_{P_1}\|\big)\frac{6(\|a_2\|_1 + \|b_2\|_1)(\|e_1\|_1 + \|d_1\|_1)}{1 - 6(\|e_2\|_1 + \|d_2\|_1)} < 1,
\end{aligned}$$

$$6\big(\|e_2\|_1+\|d_2\|_1\big)+\frac{6(\|a_2\|_1+\|b_2\|_1)(\|e_1\|_1+\|d_1\|_1)}{1-(\|t-b/a\|+(\|t-b/a\|+1)\|K_{P_1}\|)(\|a_1\|_1+\|b_1\|_1)}<1.$$

*Then FBVP* (1.1) *has at least one solution in* $C^1[0,1]\times C^1[0,1]$.

*Proof* As in the proof of Lemma 3.1, by $(D_5)$, there exist constants $M_i>0, t_i\in[0,1], i=5,6,7$ such that $|x'(t_5)|\le M_1, |y'(t_6)|\le M_5, |y(t_7)|\le M_6$. Since $x'(t)=x'(t_5)+\int_{t_5}^t x''(s)\,ds$, $y'(t)=y'(t_6)+\int_{t_6}^t y''(s)\,ds$, we get

$$\big|x'(t)\big|\le\|N_1x\|_1+M_1,\qquad\big|y'(t)\big|\le\|N_2y\|_1+M_5,\quad t\in[0,1],\tag{3.22}$$

where $N(x,y)=(N_1x,N_2y)$, $N_1x=f(s,x(s),y(s),x'(s),y'(s))$, and $N_2y=g(s,x(s),y(s),x'(s),y'(s))$. Write $(x,y)=(x_1,y_1)+(x_2,y_2)$, where $(x_1,y_1)=(I-P)(x,y)\in\operatorname{dom}L\cap\operatorname{Ker}P$ and $(x_2,y_2)=P(x,y)\in\operatorname{Im}P$.

Then since $(x_1,y_1)=(I-P)(x,y)\in\operatorname{dom}L\cap\operatorname{Ker}P$, $(x_1,y_1)=K_PL(x_1,y_1)=K_PL(I-P)(x,y)=\lambda K_PN(x,y)$.

As in the proof of Lemma 3.5,

$$\|x_1\|\le\|K_{P_1}\|\|N_1x\|_1,\qquad\|y_1\|\le\|N_2y\|_1.\tag{3.23}$$

Now, $(x_2,y_2)=(x,y)-(x_1,y_1)$, so $x_2'=x'-x_1', y_2'=y'-y_1'$ and $|x_2'(t)|\le|x'(t)|+|x_1'(t)|<M_1+(\|K_{P_1}\|+1)\|N_1x\|_1$, $|y_2'(t)|\le|y'(t)|+|y_1'(t)|<M_5+2\|N_2y\|_1$ by (3.23). Recall that $(x_2,y_2)(t)=P(x,y)(t)=(c(x)(at-b),y'(0)t+y(0))$, where

$$c(x)=\frac{1}{a^2+b^2}\big(ax'(0)-bx(0)\big)$$

is introduced for the sake of brevity. Hence

$$\big|x_2'(t)\big|=\big|c(x)a\big|<M_1+\big(\|K_{P_1}\|+1\big)\|N_1x\|_1.$$

That is,

$$\big|c(x)\big|\le\frac{1}{|a|}\big(M_1+\big(\|K_{P_1}\|+1\big)'\|N_1x\|_1\big).$$

Thus,

$$\|x_2\|=\big|c(x)\big|\|at-b\|<\|t-b/a\|\big(M_1+\big(\|K_{P_1}\|+1\big)\|N_1x\|_1\big).\tag{3.24}$$

Similarly, it is easy to obtain $|y'(0)|<M_5+2\|N_2y\|_1$. In addition, $|y_2(t_7)|\le|y(t_7)|+|y_1(t_7)|\le M_6+\|N_2y\|_1$, so, $|y_2(t_7)|=|y'(0)t_7+y(0)|\le M_6+\|N_2y\|_1$ and $|y(0)|\le M_5+M_6+3\|N_2y\|_1$, thus

$$\|y_2\|\le\big\|y'(0)t\big\|+\big|y(0)\big|\le 2M_5+M_6+5\|N_2y\|_1.\tag{3.25}$$

By (3.23) and (3.24), $\|x\|\le\|x_1\|+\|x_2\|\le C_3+C_4\|N_1x\|_1$, where

$$C_3=\|t-b/a\|M_1,\qquad C_4=\|t-b/a\|+\big(\|t-b/a\|+1\big)\|K_{P_1}\|.$$

$\|y\| \le \|y_1\| + \|y_2\| \le 2M_5 + M_6 + 6\|N_2 y\|_1$ by (3.23) and (3.25). Finally, it follows from $(D_6)$ that

$$\|x\| \le \frac{C_3 + C_4(\|\rho_1\| + (\|e_1\|_1 + \|d_1\|_1)\frac{2M_5+M_6+6\|\rho_2\|_1}{1-6(\|e_2\|_1+\|d_2\|_1)})}{1 - C_4(\|a_1\|_1 + \|b_1\|_1) - C_4\frac{6(\|a_2\|_1+\|b_2\|_1)}{1-6(\|e_2\|_1+\|d_2\|_1)}},$$

$$\|y\| \le \frac{2M_5 + M_6 + 6\|\rho_2\|_1 + \frac{6(\|a_2\|_1+\|b_2\|_1)(C_3+C_4\|\rho_1\|_1)}{1-C_4(\|a_1\|_1+\|b_1\|_1)}}{1 - 6(\|e_2\|_1 + \|d_2\|_1) - \frac{6(\|a_2\|_1+\|b_2\|_1)(\|e_1\|_1+\|d_1\|_1)}{1-C_4(\|a_1\|_1+\|b_1\|_1)}}.$$

Therefore $\Omega_1$ is bounded. The rest of the proof repeats that of Theorem 3.1. □

We now provide an example that satisfies the assumptions of Theorem 3.3. Consider the kind of equation system

$$\begin{cases} x''(t) = t - 1 + \frac{1}{32}\sin x(t) + \frac{1}{32}\sin y(t) + \frac{1}{32}x'(t) + \frac{1}{32}\sin y'(t), \\ y''(t) = g(t, x(t), y(t), x'(t), y'(t)), \\ \Gamma_1(x) = x'(0) + 2x(\frac{1}{2}) = 0, \qquad \Gamma_2(x) = x(0) - 2\int_0^1 x(s)\,ds = 0, \\ \Gamma_3(y) = 2\int_0^{\frac{1}{2}} y(s)\,ds - y(\frac{1}{2}) + \frac{1}{4}y'(\frac{1}{2}) = 0, \\ \Gamma_4(y) = y'(1) - y'(\frac{1}{2}) = 0, \end{cases}$$

where

$$\begin{aligned} &g\bigl(t, x(t), y(t), x'(t), y'(t)\bigr) \\ &\quad = \begin{cases} t + \frac{1}{32}\sin x(t) + \frac{1}{32}y(0) + \frac{1}{32}\sin x'(t) + \frac{1}{32}\sin y'(t), & t \in [0, \frac{1}{2}], \\ t + \frac{1}{32}\sin x(t) + \frac{1}{32}\sin y(t) + \frac{1}{32}\sin x'(t) + \frac{1}{32}y'(t), & t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

It is easy to see that $\Gamma_1(t) = 2, \Gamma_1(1) = 2, \Gamma_2(t) = -1, \Gamma_2(1) = -1, \Gamma_3(t) = \Gamma_3(1) = \Gamma_4(t) = \Gamma_1(1) = 0$, so that $\alpha_1 = -2, a = b = -1$ and $\operatorname{Ker} L = \{(c_1(t-1), b_3 t + b_4) | c_1, b_3, b_4 \in \mathbb{R}\}$. It is not difficult to verify that $h_1 \equiv -\frac{12}{5}$ satisfies Lemma 1.1.

Also,

$$\bigl|\Gamma_2(x)\bigr| \le \bigl|x(0)\bigr| + 2\int_0^1 \bigl|x(s)\bigr|\,ds \le 3\|x\|,$$

that is, $\beta_2 = 3, \rho_1 = 0, \rho_2 = 1, \|a_1\|_1 = \frac{1}{32}, \|b_1\|_1 = \frac{1}{32}, \|e_1\|_1 = \frac{1}{32}, \|d_1\|_1 = \frac{1}{32}, \|a_2\|_1 = \|b_2\|_1 = \|e_2\|_1 = \|d_2\|_1 = \frac{1}{32}, \|K_{P_1}\| = 4, \|K_{P_2}\| = 1, \|t - b/a\| = 1$,

$$\begin{aligned} &\bigl(\|t - b/a\| + \bigl(\|t - b/a\| + 1\bigr)\|K_{P_1}\|\bigr)\bigl(\|a_1\|_1 + \|b_1\|_1\bigr) + \bigl(\|t - b/a\| \\ &\quad + \bigl(\|t - b/a\| + 1\bigr)\|K_{P_1}\|\bigr)\frac{6(\|a_2\|_1 + \|b_2\|_1)(\|e_1\|_1 + \|d_1\|_1)}{1 - 6(\|e_2\|_1 + \|d_2\|_1)} = \frac{144}{160} < 1, \end{aligned}$$

and

$$6\bigl(\|e_2\|_1 + \|d_2\|_1\bigr) + \frac{6(\|a_2\|_1 + \|b_2\|_1)(\|e_1\|_1 + \|d_1\|_1)}{1 - (\|t - b/a\| + (\|t - b/a\| + 1)\|K_{P_1}\|)(\|a_1\|_1 + \|b_1\|_1)} = \frac{3}{7} < 1.$$

Let $M_1 = 36$. Since $N(x,y) = (N_1x, N_2y)$, if $x'(t) > 36$, then $N_1x(t) > -1 - \frac{3}{32} + \frac{1}{32}M_1 > 0$, and if $x'(t) < -36$, then $N_1x(t) < \frac{3}{32} - \frac{1}{32}M_1 < 0$. Taking $M_5 = 36, M_6 = 36$, if $y'(t) > 36$, then $N_2y(t) > 0$, and if $y'(t) < -36$, then $N_2y(t) < 0$ for $t \in [\frac{1}{2}, 1]$. And if $y(t) > 36$, then $N_2y(t) > 0$, and $y(t) < -36$, then $N_2y < 0$ for $t \in [0, \frac{1}{2}]$.

Observe that

$$
\begin{aligned}
&(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)N_1x(s)\,ds\right) = \int_0^1 \kappa(s)N_1(s)\,ds,\\
&\Gamma_3\left(\int_0^t (t-s)N_2y(s)\,ds\right) = \int_{\frac{1}{2}}^1 N_2y(s)\,ds,\\
&\Gamma_4\left(\int_0^t (t-s)N_2y(s)\,ds\right) = \int_0^{\frac{1}{2}} s^2N_2y(s)\,ds,
\end{aligned}
$$

where

$$
\kappa(s) = \begin{cases} -1 + 2s - 2s^2, & s \in [0, \frac{1}{2}],\\ -2 + 4s - 2s^2, & s \in [\frac{1}{2}, 1]. \end{cases}
$$

Obviously, $\kappa(s) < 0, s^2 \geq 0$ in$[0,1]$, therefore,

$$
\begin{aligned}
&(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)N_1x(s)\,ds\right) \neq 0, \qquad \Gamma_3\left(\int_0^t (t-s)N_2y(s)\,ds\right) \neq 0,\\
&\Gamma_4\left(\int_0^t (t-s)N_2y(s)\,ds\right) \neq 0
\end{aligned}
$$

provided $(x,y) \in \operatorname{dom} L \setminus \operatorname{Ker} L$ satisfies $|x'(t)| > M_1 = 36, |y'(t)| > M_5 = 36, |y(t)| > M_6 = 36$. Hence $(D_5)$ holds.

Finally, for$(x,y) \in \operatorname{Ker} L, x_{c_1}(t) = c_1(t-1), y_b(t) = b_3t + b_4$.

Consequently,

$$
c_1(\Gamma_1 - \alpha_1\Gamma_2)\left(\int_0^t (t-s)N_1x(s)\,ds\right) = \int_0^1 \kappa(s)c_1N_1x_{c_1}(s)\,ds > 0,
$$

since $\kappa(s) < 0$ in $[0,1]$ and

$$
c_1N_1x_{c_1}(t) \leq \frac{1}{32}|c_1| + \frac{1}{32}|c_1| + \frac{1}{32}|c_1| - \frac{1}{32}c_1^2 < 0
$$

provided $|c_1| > E_1 = 3$. When $|b_3| > E_2 = 35, |b_4| > E_3 = 35$,

$$
\begin{aligned}
&b_3\Gamma_3\left(\int_0^t (t-s)N_2y(s)\,ds\right) = \int_{\frac{1}{2}}^1 s^2b_3N_2y_b(s)\,ds > 0,\\
&b_4\Gamma_4\left(\int_0^t (t-s)N_2y_b(s)\,ds\right) = \int_0^{\frac{1}{2}} b_4N_2y_b(s)\,ds > 0,
\end{aligned}
$$

since $s^2 > 0$ in $[\frac{1}{2}, 1]$, and

$$
b_3N_2y_b(t) > -|b_3| - \frac{3}{32}|b_3| + \frac{1}{32}b_3^2 > 0, \quad t \in \left[\frac{1}{2}, 1\right],
$$

$$b_4 N_2 y_b(t) > -|b_4| - \frac{3}{32}|b_4| + \frac{1}{32}b_4^2 > 0, \quad t \in \left[0, \frac{1}{2}\right],$$

then condition $(D_3)$ is satisfied. It follows from Theorem 3.3 that there must be at least one solution in $C^1[0,1] \times C^1[0,1]$.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The main idea of this paper was proposed by SB and JW. SB prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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