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On weak convergence of an iterative algorithm for common solutions of inclusion problems and fixed point problems in Hilbert spaces

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Abstract
In this paper, a monotone inclusion problem and a fixed point problem of nonexpansive mappings are investigated based on a Mann-type iterative algorithm with mixed errors. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.
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1 Introduction

Variational inclusion has become rich of inspiration in pure and applied mathematics. In recent years, classical variational inclusion problems have been extended and generalized to study a large variety of problems arising in image recovery, economics, and signal processing; for more details, see [1–14]. Based on the projection technique, it has been shown that the variational inclusion problems are equivalent to the fixed point problems. This alternative formulation has played a fundamental and significant part in developing several numerical methods for solving variational inclusion problems and related optimization problems.

The purposes of this paper is to study the zero point problem of the sum of a maximal monotone mapping and an inverse-strongly monotone mapping, and the fixed point problem of a nonexpansive mapping. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a Mann-type iterative algorithm with mixed errors is investigated. A weak convergence theorem is established. Applications of the main results are also discussed in this section.

2 Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C.

Let $S: C \to C$ be a mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$.

Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty, closed, and convex; see [15].

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-strongly monotone. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone. It is not hard to see that inverse-strongly monotone mappings are monotone and Lipschitz continuous.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

In this paper, we use $\mathrm{VI}(C, A)$ to denote the solution set of (2.1). It is known that $\omega \in C$ is a solution to (2.1) iff ω is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant, and I stands for the identity mapping. If A is α-inverse-strongly monotone and $\lambda \in (0, 2\alpha]$, then the mapping $P_C(I - \lambda A)$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\big\|(I - \lambda A)x - (I - \lambda A)y\big\|^2 \\
&\quad = \big\|(x - y) - \lambda(Ax - Ay)\big\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2.
\end{aligned}$$

This shows that $P_C(I - \lambda A)$ is nonexpansive. It follows that $\mathrm{VI}(C, A)$ is closed and convex.

A multivalued operator $T : H \to 2^H$ with the domain $D(T) = \{x \in H : Tx \neq \emptyset\}$ and the range $R(T) = \{Tx : x \in D(T)\}$ is said to be monotone if for $x_1 \in D(T)$, $x_2 \in D(T)$, $y_1 \in Tx_1$, and $y_2 \in Tx_2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. A monotone operator T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and let $T : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : H \to H$ by $J_\lambda = (I + \lambda T)^{-1}$. It is called the resolvent of T. We know that $T^{-1}0 = F(J_\lambda)$ for all $\lambda > 0$ and J_λ is firmly nonexpansive, that is,

$$\|J_\lambda x - J_\lambda y\|^2 \leq \langle J_\lambda x - J_\lambda y, x - y \rangle, \quad \forall x, y \in H;$$

for more details, see [16–22] and the references therein.

In [19], Kamimura and Takahashi investigated the problem of finding zero points of a maximal monotone operator based on the following algorithm:

$$x_0 \in H, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{\lambda_n}x_n, \quad n = 0,1,2,\ldots,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is a positive sequence, $T : H \to 2^H$ is maximal monotone and $J_{\lambda_n} = (I + \lambda_n T)^{-1}$. They showed that the sequence $\{x_n\}$ converges weakly to some $z \in T^{-1}(0)$ provided that the control sequence satisfies some restrictions. Further, using this result, they also investigated the case that $T = \partial f$, where $f : H \to (-\infty,\infty]$ is a proper lower semicontinuous convex function. Convergence theorems are established in the framework of real Hilbert spaces.

In [16], Takahashi an Toyoda investigated the problem of finding a common solutions of the variational inequality problem (2.1) and a fixed point problem of nonexpansive mappings based on the following algorithm:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. They showed that the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{VI}(C,A) \cap F(S)$ provided that the control sequence satisfies some restrictions.

In [23], Tada and Takahashi investigated the problem of finding common solutions of an equilibrium problem and a fixed point problem of nonexpansive mappings based on the following algorithm: $x_0 \in H$ and

$$\begin{cases} u_n \in C \text{ such that } F(u_n,u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping and $F : C \times C \to R$ is a bifunction. They showed that the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{VI}(C,A) \cap F(S)$ provided that the control sequence satisfies some restrictions.

Recently, fixed point and zero point problems have been studied by many authors based on iterative methods; see, for example, [23–34] and the references therein. In this paper, motivated by the above results, we consider the problem of finding a common solution to the zero point problem and the fixed point problem based on Mann-type iterative methods with errors. Weak convergence theorems are established in the framework of Hilbert spaces.

To obtain our main results in this paper, we need the following lemmas.

Recall that a space is said to satisfy Opial's condition [35] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, where $\rightharpoonup$ denotes the weak convergence, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Indeed, the above inequality is equivalent to the following:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 2.1 [34] *Let C be a nonempty, closed, and convex subset of H, let $A : C \to H$ be a mapping, and let $B : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_\lambda(I - \lambda A)) = (A + B)^{-1}(0)$, where $J_\lambda(I - \lambda A)$ is the resolvent of B for $\lambda > 0$.*

Lemma 2.2 [36] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \le (1 + b_n)a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 [37] *Suppose that H is a real Hilbert space and $0 < p \le t_n \le q < 1$ for all $n \ge 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \qquad \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4 [15] *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $A : C \to H$ be an α-inverse-strongly monotone mapping, let $S : C \to C$ be a nonexpansive mapping and let B be a maximal monotone operator on H such that the domain of B is included in C. Assume that $\mathcal{F} = F(S) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive real number sequence and let $\{e_n\}$ be a bounded error sequence in C. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n Sx_n + \beta_n J_{\lambda_n}(x_n - \lambda_n Ax_n) + \gamma_n e_n \tag{3.1}$$

for all $n \in \mathbb{N}$, where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a \le \beta_n \le b < 1$,
(b) $0 < c \le \lambda_n \le d < 2\alpha$,
(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

where a, b, c, and d are some real numbers. Then the sequence $\{x_n\}$ generated in (3.1) *converges weakly to some point in $\mathcal{F}$.*

Proof Notice that $I - \lambda_n A$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\big\|(I - \lambda_n A)x - (I - \lambda_n A)y\big\|^2 \\
&\quad = \big\|(x - y) - \lambda_n (Ax - Ay)\big\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay\rangle + {\lambda_n}^2 \|Ax - Ay\|^2 \\
&\quad \le \|x - y\|^2 - \lambda_n (2\alpha - \lambda_n)\|Ax - Ay\|^2.
\end{aligned}$$

In view of the restriction (b), we find that $I - \lambda_n A$ is nonexpansive. Fixing $p \in \mathcal{F}$, we find from Lemma 2.1 that

$$p = Sp = J_{\lambda_n}(p - \lambda_n Ap).$$

Put $y_n = J_{\lambda_n}(x_n - \lambda_n Ax_n)$. Since J_{λ_n} and $I - \lambda_n A$ are nonexpansive, we have

$$\begin{aligned}
\|y_n - p\| &\le \big\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\big\| \\
&\le \|x_n - p\|.
\end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|e_n - p\| \\
&\le \|x_n - p\| + \gamma_n \|e_n - p\|.
\end{aligned} \tag{3.3}$$

We find that $\lim_{n\to\infty} \|x_n - p\|$ exists with the aid of Lemma 2.2. This in turn implies that $\{x_n\}$ and $\{y_n\}$ are bounded. Put $\lim_{n\to\infty} \|x_n - p\| = L > 0$. Notice that

$$\begin{aligned}
\big\|Sx_n - p + \gamma_n(e_n - Sx_n)\big\| &\le \|Sx_n - p\| + \gamma_n \|e_n - Sx_n\| \\
&\le \|x_n - p\| + \gamma_n \|e_n - Sx_n\|.
\end{aligned}$$

This implies from the restriction (c) that

$$\limsup_{n\to\infty} \big\|Sx_n - p + \gamma_n(e_n - Sx_n)\big\| \le L.$$

We also have

$$\begin{aligned}
\big\|y_n - p + \gamma_n(e_n - Sx_n)\big\| &\le \|y_n - p\| + \gamma_n \|e_n - Sx_n\| \\
&\le \|x_n - p\| + \gamma_n \|e_n - Sx_n\|.
\end{aligned}$$

This implies from the restriction (c) that

$$\limsup_{n\to\infty} \big\|y_n - p + \gamma_n(e_n - Sx_n)\big\| \le L.$$

On the other hand, we have

$$\|x_{n+1} - p\| = \big\|(1 - \beta_n)\big(Sx_n - p + \gamma_n(e_n - Sx_n)\big) + \beta_n\big(y_n - p + \gamma_n(e_n - Sx_n)\big)\big\|.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|Sx_n - y_n\| = 0. \tag{3.4}$$

Notice that

$$\begin{aligned}\|y_n - p\|^2 &\le \big\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\big\|^2 \\ &\le \|x - p\|^2 - 2\alpha\lambda_n\|Ax - Ap\|^2 + \lambda_n{}^2\|Ax - Ap\|^2 \\ &= \|x - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax - Ap\|^2. \end{aligned} \tag{3.5}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 + \gamma_n\|e_n - p\|^2 \\ &\le \|x_n - p\|^2 - \beta_n\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 + \gamma_n\|e_n - p\|^2. \end{aligned} \tag{3.6}$$

It follows that

$$\beta_n\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|e_n - p\|^2.$$

In view of the restrictions (a), (b), and (c), we obtain that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.7}$$

Notice that

$$\begin{aligned}\|y_n - p\|^2 &= \big\|J_{\lambda_n}(x_n - \lambda_n Ax_n) - J_{\lambda_n}(p - \lambda_n Ap)\big\|^2 \\ &\le \big\langle (x_n - \lambda_n Ax_n) - (p - \lambda_n Ap), y_n - p\big\rangle \\ &= \frac{1}{2}\Big(\big\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\big\|^2 + \|y_n - p\|^2 \\ &\quad - \big\|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap) - (y_n - p)\big\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|y_n - p\|^2 - \big\|x_n - y_n - \lambda_n(Ax_n - Ap)\big\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 \\ &\quad + 2\lambda_n\|x_n - y_n\|\|Ax_n - Ap\|\Big) \\ &\le \frac{1}{2}\Big(\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Ax_n - Ap\|\Big). \end{aligned}$$

It follows that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\|\|Ax_n - Ap\|. \tag{3.8}$$

On the other hand, we have

$$\|x_{n+1}-p\|^2 \le \alpha_n\|x_n-p\|^2 + \beta_n\|y_n-p\|^2 + \gamma_n\|e_n-p\|^2. \tag{3.9}$$

Substituting (3.8) into (3.9), we arrive at

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2 + \beta_n\|x_n-p\|^2 - \beta_n\|x_n-y_n\|^2 \\ &\quad + 2\beta_n\lambda_n\|x_n-y_n\|\|Ax_n-Ap\| + \gamma_n\|e_n-p\|^2 \\ &\le \|x_n-p\|^2 - \beta_n\|x_n-y_n\|^2 + 2\beta_n\lambda_n\|x_n-y_n\|\|Ax_n-Ap\| + \gamma_n\|e_n-p\|^2.\end{aligned}$$

It derives that

$$\beta_n\|x_n-y_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\beta_n\lambda_n\|x_n-y_n\|\|Ax_n-Ap\| + \gamma_n\|e_n-p\|^2.$$

In view of the restrictions (a) and (c), we find from (3.7) that

$$\lim_{n\to\infty}\|x_n-y_n\| = 0. \tag{3.10}$$

Notice that

$$\|Sx_n-x_n\| \le \|Sx_n-y_n\| + \|y_n-x_n\|.$$

It follows from (3.4) and (3.10) that

$$\lim_{n\to\infty}\|Sx_n-x_n\| = 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \omega \in C$, where $\rightharpoonup$ denotes the weak convergence. From Lemma 2.4, we find that $\omega \in F(S)$. In view of (3.10), we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup \omega$. Notice that

$$y_n = J_{\lambda_n}(x_n - \lambda_n Ax_n).$$

This implies that

$$x_n - \lambda_n Ax_n \in (I+\lambda_n B)y_n.$$

That is,

$$\frac{x_n-y_n}{\lambda_n} - Ax_n \in By_n.$$

Since B is monotone, we get for any $(u,v)\in B$ that

$$\left\langle y_n-u, \frac{x_n-y_n}{\lambda_n} - Ax_n - v\right\rangle \ge 0. \tag{3.12}$$

Replacing n by n_i and letting $i \to \infty$, we obtain from (3.10) that

$$\langle \omega - u, -A\omega - v \rangle \geq 0.$$

This means $-A\omega \in B\omega$, that is, $0 \in (A+B)(\omega)$. Hence, we get $\omega \in (A+B)^{-1}(0)$. This completes the proof that $\omega \in \mathcal{F}$.

Suppose that there is another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \omega^*$. Then we can show that $\omega^* \in \mathcal{F}$ in exactly the same way. Assume that $\omega \neq \omega^*$ since $\lim_{n\to\infty} \|x_n - p\|$ exits for any $p \in F$. Put $\lim_{n\to\infty} \|x_n - \omega\| = d$. Since the space satisfies Opial's condition, we see that

$$\begin{aligned} d &= \liminf_{i\to\infty} \|x_{n_i} - \omega\| \\ &< \liminf_{i\to\infty} \|x_{n_i} - \omega^*\| \\ &= \lim_{n\to\infty} \|x_n - \omega^*\| \\ &= \liminf_{j\to\infty} \|x_{n_j} - \omega^*\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - \omega\| = d. \end{aligned}$$

This is a contradiction. This shows that $\omega = \omega^*$. This proves that the sequence $\{x_n\}$ converges weakly to $\omega \in F$. This completes the proof. □

We obtain from Theorem 3.1 the following inclusion problem.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $A : C \to H$ be an α-inverse-strongly monotone mapping, and let B be a maximal monotone operator on H such that the domain of B is included in C. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive real number sequence and let $\{e_n\}$ be a bounded error sequence in C. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n J_{\lambda_n}(x_n - \lambda_n A x_n) + \gamma_n e_n$$

for all $n \in \mathbb{N}$, where $J_{\lambda_n} = (I + \lambda_n B)^{-1}$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a \leq \beta_n \leq b < 1$,
(b) $0 < c \leq \lambda_n \leq d < 2\alpha$,
(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

where a, b, c, and d are some real numbers. Then the sequence $\{x_n\}$ converges weakly to some point in $(A+B)^{-1}(0)$.

Let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential

$$\partial f(x) = \{z \in H : f(x) + \langle y - x, z \rangle \leq f(y), \forall y \in H\}$$

for all $x \in H$. Then ∂f is a maximal monotone operator of H into itself; for more details, see [38]. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, that is,

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Furthermore, we define the normal cone $N_C(v)$ of C at v as follows:

$$N_C v = \{z \in H : \langle z, y - v\rangle \le 0, \forall y \in H\}$$

for any $v \in C$. Then $i_C : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function on H and ∂i_C is a maximal monotone operator. Let $J_\lambda x = (I + \lambda \partial i_C)^{-1} x$ for any $\lambda > 0$ and $x \in H$. From $\partial i_C x = N_C x$ and $x \in C$, we get

$$\begin{aligned} v = J_\lambda x \quad &\Leftrightarrow \quad x \in v + \lambda N_C v \\ &\Leftrightarrow \quad \langle x - v, y - v\rangle \le 0, \quad \forall y \in C, \\ &\Leftrightarrow \quad v = P_C x, \end{aligned}$$

where P_C is the metric projection from H into C. Similarly, we can get that $x \in (A + \partial i_C)^{-1}(0) \Leftrightarrow x \in \mathrm{VI}(A, C)$. Putting $B = \partial i_C$ in Theorem 3.1, we find that $J_{\lambda_n} = P_C$. The following results are not hard to derive.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Assume that $\mathcal{F} = F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive real number sequence and let $\{e_n\}$ be a bounded error sequence in C. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n S x_n + \beta_n P_C(x_n - \lambda_n A x_n) + \gamma_n e_n$$

for all $n \in \mathbb{N}$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a \le \beta_n \le b < 1$,
(b) $0 < c \le \lambda_n \le d < 2\alpha$,
(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

where a, b, c, and d are some real numbers. Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

In view of Theorem 3.3, we have the following result.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A : C \to H$ be an α-inverse-strongly monotone mapping such that $\mathrm{VI}(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a* *positive real number sequence and let* $\{e_n\}$ *be a bounded error sequence in* C. *Let* $\{x_n\}$ *be a sequence in* C *generated in the following iterative process*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n P_C(x_n - \lambda_n A x_n) + \gamma_n e_n$$

for all $n \in \mathbb{N}$. *Assume that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *satisfy the following restrictions*:

(a) $0 < a \leq \beta_n \leq b < 1$,
(b) $0 < c \leq \lambda_n \leq d < 2\alpha$,
(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$,

where a, b, c, *and* d *are some real numbers. Then the sequence* $\{x_n\}$ *converges weakly to some point in* $\mathrm{VI}(C, A)$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C.$$

In this paper, we use $EP(F)$ to denote the solution set of the equilibrium problem.

To study the equilibrium problems, we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Putting $F(x,y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem is reduced to the variational inequality (2.1).

The following lemma can be found in [39].

Lemma 3.5 *Let* C *be a nonempty closed convex subset of* H *and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). *Then, for any* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{3.13}$$

for all $r > 0$ *and* $x \in H$. *Then the following hold*:

(a) T_r *is single-valued*,
(b) T_r *is firmly nonexpansive*, *i.e.*, *for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle,$$

(c) $F(T_r) = EP(F)$,
(d) $EP(F)$ *is closed and convex*.

Lemma 3.6 [5] *Let C be a nonempty closed convex subset of a real Hilbert space H, let F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_r is defined as in (3.13).

Theorem 3.7 *Let C be a nonempty closed convex subset of a real Hilbert space H, let $S : C \to C$ be a nonexpansive mapping and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Assume that $\mathcal{F} = F(S) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive real number sequence and let $\{e_n\}$ be a bounded error sequence in C. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n S x_n + \beta_n y_n + \gamma_n e_n$$

for all $n \in \mathbb{N}$, where $y_n \in C$ such that

$$F(y_n, u) + \frac{1}{\lambda_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:
(a) $0 < a \leq \beta_n \leq b < 1$,
(b) $0 < c \leq \lambda_n \leq d < \infty$,
(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$,
where a, b, c, and d are some real numbers. Then the sequence $\{x_n\}$ converges weakly to some point in $\mathcal{F}$.

Competing interests
The author declares that he has no competing interests.

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