Cho *Fixed Point Theory and Applications* (2016) 2016:50
DOI 10.1186/s13663-016-0540-5

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized probabilistic G-contractions

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Abstract

In this paper, the notion of generalized probabilistic G-contractions in Menger probabilistic metric spaces endowed with a directed graph G is introduced and some new fixed point theorems for such mappings are established.

MSC: Primary 47H10; secondary 54H25

Keywords: fixed point; coincidence point; directed graph; Menger probabilistic metric space

1 Introduction and preliminaries

Ran and Reurings [1] gave a generalization of Banach contraction principle to partially ordered metric spaces. Since then, many authors obtained generalization and extension of the results of [2–7].

In particular, Ćirić *et al.* [3] extended the results of [1, 5, 6] to partially ordered Menger probabilistic metric spaces.

Samet *et al.* [8] introduced the notion of α-ψ-contractive type mappings and established some fixed point theorems for such mappings in complete metric spaces.

Cho [9] obtained a generalization of the results of [3] by introducing the concept of α-contractive type mappings in Menger probabilistic metric spaces.

Recently, Wu [10] obtained a generalization of the results of [3], and improved and extended the fixed point results of [4, 11, 12]. Also, Kamran *et al.* [13] introduced the notion of probabilistic G-contractions in Menger PM-spaces endowed with a graph G and obtained some fixed point results. Especially, they obtained the following result.

Theorem 1.1 *Let (X,F,Δ) be a complete Menger PM-space, where Δ is of Hadžić-type. Let $G=(V(G),E(G))$ be a directed graph such that $V(G)=X$ and $\Omega\subset E(G)$. Suppose that a map $f:X\to X$ satisfies f preserves edges and there exists $k\in(0,1)$ such that, for all $x,y\in X$ with $(x,y)\in E(G)$,*

$$F_{fx,fy}(kt)\geq F_{x,y}(t).$$

Assume that there exists $x_0\in X$ such that $(x_0,fx_0)\in E(G)$. If either f is orbitally G-continuous or G is a C-graph, then f has a fixed point in $[x_0]_{\tilde{G}}$.

Further if $(x,y)\in E(G)$ for any $x,y\in M$, where $M=\{x\in X:(x,fx)\in E(G)\}$, then f has a unique fixed point.

In this paper, we give some new fixed point theorems which are generalizations of the results of [3, 9, 10, 13], by introducing a concept of generalized probabilistic G-contractions in Menger PM-spaces with a directed graph $G = (V(G), E(G))$ such that $V(G) = X$ and $\Omega \subset E(G)$.

We recall some definitions and results which will be needed in the sequel.

A mapping $f : \mathbb{R} \to [0,\infty)$ is called a *distribution* if the following conditions hold:

(1) f is nondecreasing and left-continuous;

(2) $\sup\{f(t) : t \in \mathbb{R}\} = 1$;

(3) $\inf\{f(t) : t \in \mathbb{R}\} = 0$.

We denote by D the set of all distribution functions.

Let $\epsilon_0 : \mathbb{R} \to [0,\infty)$ be a function defined by

$$\epsilon_0(t) = \begin{cases} 0 & (t \leq 0), \\ 1 & (t > 0). \end{cases}$$

Then $\epsilon_0 \in D$.

Let $\Delta : [0,1] \times [0,1] \to [0,1]$ be a mapping such that

(1) $\Delta(a,b) = \Delta(b,a)$ for all $a, b \in [0,1]$;

(2) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for all $a,b,c \in [0,1]$;

(3) $\Delta(a,1) = a$ for all $a \in [0,1]$;

(4) $\Delta(a,b) \geq \Delta(c,d)$, whenever $a \geq c$ and $b \geq d$ for all $a,b,c,d \in [0,1]$.

Then Δ is called a *triangular norm* (for short *t-norm*).

We denote $\mathbb{N}$ by the set of all natural numbers.

For a t-norm Δ, we consider the following notation:

$$\Delta^1(t) = \Delta(t,t), \qquad \Delta^n(t) = \Delta\bigl(t, \Delta^{n-1}(t)\bigr) \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,1].$$

A t-norm Δ is said to be of *Hadžić-type* [14] whenever the family of $\{\Delta^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$.

For example, the minimum t-norm Δ_m defined by

$$\Delta_m(a,b) = \min\{a,b\}, \quad \forall a,b \in [0,1],$$

is of Hadžić-type.

It is easy to see that the following are equivalent (see [14]):

(1) for a t-norm Δ,

$$\text{it is of Hadžić-type;} \tag{1.1}$$

(2) given $\epsilon \in (0,1)$, there is a $\delta \in (0,1)$ such that $\Delta^n(x) > 1 - \epsilon$ for all $n \in \mathbb{N}$, whenever $x > 1 - \delta$.

Also, it is well known that if Δ satisfies condition $\Delta(a,a) \geq a$ for all $a \in [0,1]$, then $\Delta = \Delta_m$ (see [15]). Hence we have

$$\forall a \in [0,1], \quad \Delta(a,a) \geq a \quad \Longleftrightarrow \quad \Delta = \Delta_m.$$

Let X be a nonempty set, and let Δ be a t-norm. Suppose that a mapping $F : X \times X \to D$ (for $x, y \in X$, we denote $F(x, y)$ by $F_{x,y}$) satisfies the following conditions:

(PM1) $F_{x,y}(t) = \epsilon_0(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(PM2) $F_{x,y} = F_{y,x}$ for all $x, y \in X$;

(PM3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and all $t, s \geq 0$.

Then a 3-tuple (X, F, Δ) is called a *Menger probabilistic metric space* (briefly, *Menger PM-space*) [16, 17].

Let (X, F, Δ) be a Menger PM-space and $\in X$, and let $\epsilon > 0$ and $\lambda \in (0, 1]$.

Schweizer and Sklar [18] brought in the notion of neighborhood $U_x(\epsilon, \lambda)$ of x, where $U_x(\epsilon, \lambda)$ is defined as follows:

$$U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The family

$$\{U_x(\epsilon, \lambda) : x \in X, \epsilon > 0, \lambda \in (0, 1]\} \tag{1.2}$$

does not necessarily determine a topology on X (see [19, 20]).

It is well known that if Δ satisfies condition

$$\sup\{\Delta(t, t) : 0 < t < 1\} = 1 \tag{1.3}$$

then (1.2) determines a Hausdorff topology on X, and it is called (ϵ, λ)-topology.

So if (1.3) holds, then Menger space (X, F, Δ) is a Hausdorff topological space in the (ϵ, λ)-topology (see [18, 21]).

Remark 1.1 The following are satisfied:

(1) condition (1.3) is the weakest condition which ensure the existence of the (ϵ, λ)-topology (see [19]);

(2) condition (1.1) $\Longrightarrow$ condition (1.3) (see [22]).

Let (X, F, Δ) be a Menger PM-space, and let $\{x_n\}$ be a sequence in X and $x \in X$. Then we say that

(1) $\{x_n\}$ is *convergent* to x (we write $\lim_{n\to\infty} x_n = x$) if and only if, given $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, for all $n \geq n_0$.

(2) $\{x_n\}$ is a *Cauchy sequence* if and only if, given $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$, for all $m > n \geq n_0$.

(3) (X, F, Δ) is *complete* if and only if each Cauchy sequence in X is convergent to some point in X.

Example 1.1 Let D be a distribution function defined by

$$D(t) = \begin{cases} 0 & (t \leq 0), \\ 1 - e^{-t} & (t > 0). \end{cases}$$

Let

$$F_{x,y}(t) = \begin{cases} \epsilon_0(t) & (x = y), \\ D(\frac{t}{d(x,y)}) & (x \neq y), \end{cases}$$

for all $x, y \in X$ and $t > 0$, where d is a metric on a nonempty set X.

Then (X, F, Δ_m) is a Menger PM-space (see [18]).

Remark 1.2 If (X, d) is complete, then (X, F, Δ_m) is complete. In fact, let $\{x_n\}$ be any Cauchy sequence in (X, F, Δ_m).

Then

$$\lim_{n,m\to\infty} D\left(\frac{t}{d(x_n, x_m)}\right) = \lim_{n,m\to\infty} F_{x_n,x_m}(t) = 1$$

for all $t > 0$, which implies $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x_* \in X$ such that $\lim_{n\to\infty} d(x_n, x_*) = 0$.

Thus, we have

$$\lim_{n\to\infty} F_{x_n,x_*}(t) = \lim_{n\to\infty} D\left(\frac{t}{d(x_n, x_*)}\right) = 1$$

for all $t > 0$. Hence, (X, F, Δ_m) is complete.

From now on, let

$$\Phi = \left\{\phi : [0,\infty) \to [0,\infty) \mid \lim_{n\to\infty} \phi^n(t) = 0, \forall t > 0\right\}$$

and let

$$\Phi_w = \left\{\phi : [0,\infty) \to [0,\infty) \mid \forall t > 0, \exists r \geq t \text{ s.t. } \lim_{n\to\infty} \phi^n(r) = 0\right\}.$$

Note that $\Phi \subset \Phi_w$.

Fang [23] gave the corrected version of Theorem 12 of [11] by introducing the notion of right-locally monotone functions as follows: $\phi : [0,\infty) \to [0,\infty)$ is right-locally monotone if and only if $\forall t \geq 0$, $\exists \delta > 0$ s.t. it is monotone on $[t, t+\delta)$.

Lemma 1.1 [23] *The following are satisfied*:

(1) *If a right-locally monotone function* $\phi : [0,\infty) \to [0,\infty)$ *satisfies*

$$\phi(0) = 0, \qquad \phi(t) < t \quad \textit{and} \quad \lim_{r\to t^+} \inf \phi(r) < t \quad \textit{for all } t > 0,$$

then $\phi \in \Phi$.

(2) *If a function* $\phi : [0,\infty) \to [0,\infty)$ *satisfies*

$$\phi(t) < t \quad \textit{and} \quad \lim_{r\to t^+} \sup \phi(r) < t \quad \textit{for all } t > 0,$$

then $\phi \in \Phi_w$.

(3) *If a function* $\alpha : [0,\infty) \to [0,1)$ *is piecewise monotone and*

$$\phi(t) = \alpha(t)t \quad \textit{for all } t \geq 0,$$

then $\phi \in \Phi$.

Lemma 1.2 [23] *If* $\phi \in \Phi_w$, *then* $\forall t > 0$, $\exists r \geq t$ *s.t.* $\phi(r) < t$.

Lemma 1.3 [23] *Let* (X, F, Δ) *be a Menger PM-space, and let* $x, y \in X$. *If*

$$F_{x,y}\big(\phi(t)\big) \geq F_{x,y}(t)$$

for all $t > 0$, *where* $\phi \in \Phi_w$, *then* $x = y$.

Lemma 1.4 [18] *Let* (X, F, Δ) *be a Menger PM-space and* $x, y \in X$, *where* Δ *is continuous. Suppose that* $\{x_n\}$ *is a sequence of points in* X. *If* $\lim_{n\to\infty} x_n = x$, *then* $\lim_{n\to\infty} \inf F_{x_n,y}(t) = F_{x,y}(t)$ *for all* $t > 0$.

Lemma 1.5 *Let* (X, F, Δ) *be a Menger PM-space, where* Δ *is of Hadžić-type. Let* $\{x_n\}$ *be a sequence of points in* X *such that* $x_{n-1} \neq x_n$ *for all* $n \in \mathbb{N}$. *If there exists* $\phi \in \Phi_w$ *such that*

$$F_{x_n,x_m}\big(\phi(s)\big) \geq \min\big\{F_{x_{n-1},x_{m-1}}(s), F_{x_{n-1},x_n}(s), F_{x_{m-1},x_m}(s)\big\} \tag{1.4}$$

for all $s > 0$ *and all* $n, m \in \mathbb{N}$, *then for each* $t > 0$ *there exists* $r \geq t$ *such that*

$$F_{x_n,x_m}(t) \geq \Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big) \quad \textit{for all } m \geq n + 1. \tag{1.5}$$

Proof It is easy to see that (1.4) implies that $\phi(t) > 0$ for all $t > 0$. In fact, if there exists $t_0 > 0$ such that $\phi(t_0) = 0$, then we obtain

$$0 = F_{x_n,x_n}\big(\phi(t_0)\big) \geq F_{x_{n-1},x_n}(t_0) > 0$$

which is a contradiction.

We claim that

$$F_{x_n,x_{n+1}}(u) \geq F_{x_{n-1},x_n}(u) \quad \text{for all } u > 0 \text{ and } n \in \mathbb{N}.$$

From (1.4) we have

$$F_{x_n,x_{n+1}}\big(\phi(s)\big) \geq \min\big\{F_{x_{n-1},x_n}(s), F_{x_n,x_{n+1}}(s)\big\}$$

for all $s > 0$ and all $n \in \mathbb{N}$.

If there exists $n \in \mathbb{N}$ such that $F_{x_{n-1},x_n}(s) \geq F_{x_n,x_{n+1}}(s)$ for all $s > 0$, then $F_{x_n,x_{n+1}}(\phi(s)) \geq F_{x_n,x_{n+1}}(s)$ for all $s > 0$. Thus, $x_n = x_{n+1}$, which is a contradiction. Hence we have $F_{x_{n-1},x_n}(s) < F_{x_n,x_{n+1}}(s)$ for all $s > 0$ and $n \in \mathbb{N}$, and so

$$F_{x_n,x_{n+1}}\big(\phi(s)\big) \geq F_{x_{n-1},x_n}(s)$$

for all $s > 0$ and $n \in \mathbb{N}$.

Since $\phi \in \Phi_w$, for each $u > 0$, there exists $v \geq u$ such that

$$\phi(v) < u.$$

Hence,

$$F_{x_n,x_{n+1}}(u) \geq F_{x_n,x_{n+1}}\big(\phi(v)\big) \geq F_{x_{n-1},x_n}(v) \geq F_{x_{n-1},x_n}(u)$$

for all $u > 0$ and $n \in \mathbb{N}$. So the claim is proved.

Let $t > 0$ be given. By Lemma 1.2, there exists $r \geq t$ such that

$$\phi(r) < t. \tag{1.6}$$

By induction, we show that (1.5) holds.

Let $m = n + 1$.

Then

$$\begin{aligned} &F_{x_n,x_{n+1}}(t) \\ &\quad \geq F_{x_n,x_{n+1}}\big(t - \phi(r)\big) \\ &\quad = \Delta\big(F_{x_n,x_{n+1}}\big(t - \phi(r), 1\big)\big) \\ &\quad \geq \Delta^1\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big). \end{aligned}$$

Thus, (1.5) holds for $m = n + 1$.

Assume that (1.5) holds for some fixed $m > n + 1$. That is,

$$F_{x_n,x_m}(t) \geq \Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big) \quad \text{holds for some } m > n + 1. \tag{1.7}$$

Then

$$\begin{aligned} &F_{x_n,x_{m+1}}(t) \\ &\quad = F_{x_n,x_{m+1}}\big(t - \phi(r) + \phi(r)\big) \\ &\quad \geq \Delta\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big), F_{x_{n+1},x_{m+1}}\big(\phi(r)\big)\big). \end{aligned} \tag{1.8}$$

From (1.4) we obtain

$$\begin{aligned} &F_{x_{n+1},x_{m+1}}\big(\phi(r)\big) \\ &\quad \geq \min\big\{F_{x_n,x_m}(r), F_{x_n,x_{n+1}}(r), F_{x_m,x_{m+1}}(r)\big\}. \end{aligned}$$

By the above claim, since $F_{x_m,x_{m+1}}(t) \geq F_{x_n,x_{n+1}}(t)$, from (1.4) and (1.7) we obtain

$$\begin{aligned} &F_{x_{n+1},x_{m+1}}\big(\phi(r)\big) \\ &\quad \geq \min\big\{F_{x_n,x_m}(t), F_{x_n,x_{n+1}}(t)\big\} \\ &\quad \geq \min\big\{\Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big), F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big\} \\ &\quad = \Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t - \phi(r)\big)\big). \end{aligned} \tag{1.9}$$

Thus, from (1.8) and (1.9) we have

$$\begin{aligned} &F_{x_n,x_{m+1}}(t) \\ &\quad\geq \Delta\big(F_{x_n,x_{n+1}}\big(t-\phi(r)\big), \Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t-\phi(r)\big)\big)\big) \\ &\quad= \Delta^{m-n+1}\big(F_{x_n,x_{n+1}}\big(t-\phi(r)\big)\big). \end{aligned}$$

Hence, (1.5) holds for all $m \geq n+1$. □

Lemma 1.6 [24] *Let (X,d) be a metric space. Suppose that $F : X \times X \to D$ is a mapping defined by*

$$F(x,y)(t) = F_{x,y}(t) = \epsilon_0\big(t - d(x,y)\big)$$

for all $x, y \in X$ and all $t > 0$.

Then (X, F, Δ_m) is a Menger PM-space, which is called a Menger PM-space induced by the metric d.

Remark 1.3 Let (X,d) be a metric space. Suppose that (X, F, Δ_m) is a Menger PM-space induced by d.

Then we have the following.

(1) If $f : X \to X$ is continuous in (X,d), then it is continuous in (X, F, Δ_m).

(2) If a sequence $\{x_n\}$ is convergent to a point x in (X,d), then it is convergent to x in (X, F, Δ_m).

(3) If (X,d) is complete, then (X, F, Δ_m) is complete.

Lemma 1.7 [25] *If X is a nonempty set and $h : X \to X$ is a function, then there exists $Y \subset X$ such that $h(Y) = h(X)$ and $h : Y \to X$ is one-to-one.*

Let X be a nonempty set, and let $\Omega = \{(x,x) : x \in X\}$ the diagonal of the Cartesian product $X \times X$.

Let G be a directed graph such that the following conditions are satisfied:

(1) the set $V(G)$ of its vertices coincides with X, *i.e.* $V(G) = X$;

(2) the set $E(G)$ of its edges contains all loops, *i.e.* $\Omega \subset E(G)$.

If G has no parallel edges, then we can identify G with the pair $(V(G), E(G))$.

Let $G = (V(G), E(G))$ be a directed graph.

Then the *conversion* of the graph G (denoted by G^{-1}) is an ordered pair $(V(G^{-1}), E(G^{-1}))$ consisting of a set $V(G^{-1})$ of vertices and a set $E(G^{-1})$ of edges, where

$$V\big(G^{-1}\big) = V(G) \quad \text{and} \quad E\big(G^{-1}\big) = \big\{(x,y) \in X \times X : (y,x) \in E(G)\big\}.$$

Note that $G^{-1} = (V(G), E(G^{-1}))$.

Given a directed graph $G = (V(G), E(G))$, let $\widetilde{G} = (V(\widetilde{G}), E(\widetilde{G}))$ be a directed graph such that

$$V(\widetilde{G}) = V(G) \quad \text{and} \quad E(\widetilde{G}) = E(G) \cup E\big(G^{-1}\big).$$

For $x, y \in V(G)$, let $p = (x = x_0, x_1, x_2, \ldots, x_N = y)$ be a finite sequence such that

$$(x_{n-1}, x_n) \in E(G) \quad \text{for } n = 1, 2, \ldots, N.$$

Then p is called a path in G from x to y of length N.

Denote $\Xi(G)$ by the family of all path in G.

If, for any $x, y \in V(G)$, there is a path $p \in \Xi(G)$ from x to y, then the graph G called *connected*. A graph G is called *weakly connected*, whenever $\widetilde{G}$ is connected.

Let G be a graph such that $E(G)$ is symmetric and $x \in V(G)$.

Then the subgraph $G_x = (V(G_x), E(G_x))$ is called *component* of G containing x if and only if there is a path $p \in \Xi(G)$ beginning at x such that

$$v \in p \quad \text{for all } v \in V(G_x) \quad \text{and} \quad e \subset p \quad \text{for all } e \in E(G_x).$$

Define a relation $\Re$ on $V(G)$ as follows:

$$(y, z) \in \Re \quad \Longleftrightarrow \quad \text{there is a } p \in \Xi(G) \text{ from } y \text{ to } z.$$

Then the relation $\Re$ is an equivalence relation on $V(G)$, and $[x]_G = V(G_x)$, where $[x]_G$ is the equivalence class of $x \in V(G)$.

Note that the component G_x of G containing x is connected.

For the details of the graph theory, we refer to [26].

Let (X, F, Δ) be a Menger PM-space, and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $\Omega \subset E(G)$.

Then the graph G is said to be a *C-graph* if and only if, for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x_* \in X$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an $N \in \mathbb{N}$ such that $(x_{n_k}, x_*) \in E(G)$ (resp. $(x_*, x_{n_k}) \in E(G)$) for all $k \geq N$ whenever $(x_n, x_{n+1}) \in E(G)$ (resp. $(x_{n+1}, x_n) \in E(G)$) for all $n \in \mathbb{N}$.

The following definitions are in [13].

Let (X, F, Δ) be a Menger PM-space, and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $\Omega \subset E(G)$. Let $f : X \to X$ be a map. Then we say that:

(1) f is *continuous* if and only if, for any $x \in X$ and a sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$,

$$\lim_{n\to\infty} f x_n = f x.$$

(2) f is *G-continuous* if and only if, for any $x \in X$ and a sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} f x_n = f x.$$

(3) f is *orbitally continuous* if and only if, for all $x, y \in X$ and any sequence $\{k_n\} \subset \mathbb{N}$ with $\lim_{n\to\infty} f^{k_n} x = y$,

$$\lim_{n\to\infty} f f^{k_n} x = f y.$$

(4) f is *orbitally G-continuous* if and only if, for all $x, y \in X$ and any sequence $\{k_n\} \subset \mathbb{N}$ with $\lim_{n\to\infty} f^{k_n}x = y$ and $(f^{k_n}x, f^{k_n+1}x) \in E(G)$ for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} ff^{k_n}x = fy.$$

2 Main results

From now on, let (X, F, Δ) be a Menger PM-space, where Δ is a t-norm of Hadžić-type. Let $G = (V(G), E(G))$ be a directed graph satisfying conditions

$$V(G) = X \quad \text{and} \quad \Omega \subset E(G).$$

A map $f : X \to X$ is said to be a *generalized probabilistic G-contraction* if and only if the following conditions are satisfied:

(1) f preserves edges of G, *i.e.* $(x, y) \in E(G) \Longrightarrow (fx, fy) \in E(G)$;

(2) there exists $\phi \in \Phi_w$ such that

$$F_{fx,fy}\bigl(\phi(t)\bigr) \geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\} \tag{2.1}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and all $t > 0$.

Theorem 2.1 *Let (X, F, Δ) be complete. Suppose that a map $f : X \to X$ is a generalized probabilistic G-contraction. Assume that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. If either f is orbitally G-continuous or Δ is a continuous t-norm and G is a C-graph, then f has a fixed point in $[x_0]_{\widetilde{G}}$.*

Further if $(x, y) \in E(G)$ for any $x, y \in M$, where $M = \{x \in X : (x, fx) \in E(G)\}$, then f has a unique fixed point.

Proof Let $x_0 \in X$ be such that $(x_0, fx_0) \in E(G)$. Let $x_n = f^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = fx_{n_0}$, and so x_{n_0} is a fixed point of f.

Consider the path p in G from x_0 to x_{n_0+1}:

$$p = (x_0, x_1, x_2, \ldots, x_{n_0} = x_{n_0+1}) \in \Xi(G).$$

Then the above path is in $\widetilde{G}$. Hence, $x_{n_0} = x_{n_0+1} \in [x_0]_{\widetilde{G}}$.

Hence, the proof is finished.

Assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$.

As in the proof of Lemma 1.4, we have $\phi(t) > 0$ for all $t > 0$.

Since f is a generalized probabilistic G-contraction, $(x_n, x_{n+1}) \in E(G)$ for all $n = 0, 1, 2, \ldots,$ and from (2.1) with $x = x_{n-1}$, $y = x_n$ we have

$$\begin{aligned} F_{x_n,x_{n+1}}\bigl(\phi(t)\bigr) &= F_{fx_{n-1},fx_n}\bigl(\phi(t)\bigr) \\ &\geq \min\bigl\{F_{x_{n-1},x_n}(t), F_{x_{n-1},fx_{n-1}}(t), F_{x_n,fx_n}(t)\bigr\} \\ &= \min\bigl\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\bigr\} \end{aligned}$$

for all $t > 0$ and $n \in \mathbb{N}$.

If there exists $n \in \mathbb{N}$ such that $F_{x_{n-1},x_n}(t) \geq F_{x_n,x_{n+1}}(t)$ for all $t > 0$, then

$$F_{x_n,x_{n+1}}\bigl(\phi(t)\bigr) \geq F_{x_n,x_{n+1}}(t)$$

for all $t > 0$.

By Lemma 1.3, $x_n = x_{n+1}$, which is a contradiction. Thus, we have $F_{x_{n-1},x_n}(t) < F_{x_n,x_{n+1}}(t)$ for all $t > 0$ and $n \in \mathbb{N}$, and so

$$F_{x_n,x_{n+1}}\bigl(\phi(t)\bigr) \geq F_{x_{n-1},x_n}(t)$$

for all $t > 0$ and $n \in \mathbb{N}$. Thus, we have

$$F_{x_n,x_{n+1}}\bigl(\phi^n(t)\bigr) \geq F_{x_0,x_1}(t)$$

for all $t > 0$ and $n \in \mathbb{N}$.

We now show that

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1 \tag{2.2}$$

for all $t > 0$. Since $\lim_{t\to\infty} F_{x_0,x_1}(t) = 1$, for any $\epsilon \in (0,1)$ there exists $t_0 > 0$ such that

$$F_{x_0,x_1}(t_0) > 1 - \epsilon.$$

Because $\phi \in \Phi_w$, there exists $t_1 \geq t_0$ such that

$$\lim_{t\to\infty} \phi^n(t_1) = 0.$$

Thus, for each $t > 0$, there exists N such that $\phi^n(t_1) < t$ for all $n > N$. Hence, we have

$$F_{x_n,x_{n+1}}(t) \geq F_{x_n,x_{n+1}}\bigl(\phi^n(t_1)\bigr) \geq F_{x_0,x_1}(t_1) \geq F_{x_0,x_1}(t_0) > 1 - \epsilon$$

for all $n > N$. Thus, $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence.

Let $\epsilon \in (0,1)$ be given.

Since Δ is of Hadžić-type, there exists $\lambda \in (0,1)$ such that

$$\Delta^n(s) > 1 - \epsilon \quad \text{for all } n = 1, 2, \ldots, \text{whenever } s > 1 - \lambda. \tag{2.3}$$

Since $\phi \in \Phi_w$, for each $t > 0$, there exists $r \geq t$ such that $\phi(r) < t$. From (2.2) we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}\bigl(t - \phi(r)\bigr) = 1.$$

Thus, there exists N_1 such that

$$F_{x_n,x_{n+1}}\bigl(t - \phi(r)\bigr) > 1 - \lambda \tag{2.4}$$

for all $n > N_1$.

Since (1.4) is satisfied,

$$F_{x_n,x_m}(t) \geq \Delta^{m-n}\big(F_{x_n,x_{n+1}}\big(t-\phi(r)\big)\big) \tag{2.5}$$

holds for all $m \geq n+1$ by Lemma 1.5.

By applying (2.3) with (2.4) and (2.5),

$$F_{x_n,x_m}(t) > 1-\epsilon$$

for all $m > n > N_1$.

Thus, $\{x_n\}$ is a Cauchy sequence in X. It follows from the completeness of X that there exists $x_* \in X$ such that

$$\lim_{n\to\infty} x_n = x_*.$$

If f is orbitally G-continuous, then $\lim_{n\to\infty} x_n = fx_*$. Hence, $x_* = fx_*$.

Suppose that Δ is continuous and G is C-graph.

Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an $N \in \mathbb{N}$ such that

$$(x_{n_k}, x_*) \in E(G)$$

for all $k \geq N$. Since f is a generalized probabilistic G-contraction and $(x_{n_k}, x_*) \in E(G)$ for all $k \geq N$, from (2.1) with $x = x_{n_k}$ and $y = x_*$ we have

$$\begin{aligned}
&F_{x_{n_k+1},fx_*}\big(\phi(t)\big)\\
&\quad = F_{fx_{n_k},fx_*}\big(\phi(t)\big)\\
&\quad \geq \min\big\{F_{x_{n_k},x_*}(t), F_{x_{n_k},fx_{n_k}}(t), F_{x_*,fx_*}(t)\big\}\\
&\quad = \min\big\{F_{x_{n_k},x_*}(t), F_{x_{n_k},x_{n_k+1}}(t), F_{x_*,fx_*}(t)\big\}
\end{aligned}$$

for all $t > 0$.

By Lemma 1.4, we obtain

$$\begin{aligned}
&F_{x_*,fx_*}\big(\phi(t)\big)\\
&\quad = \liminf_{k\to\infty} F_{x_{n_k+1},fx_*}\big(\phi(t)\big)\\
&\quad \geq \liminf_{k\to\infty} \min\big\{F_{x_{n_k},x_*}(t), F_{x_{n_k},fx_{n_k}}(t), F_{x_*,fx_*}(t)\big\}\\
&\quad = \min\big\{1, 1, F_{x_*,fx_*}(t)\big\}\\
&\quad = F_{x_*,fx_*}(t)
\end{aligned}$$

for all $t > 0$. By Lemma 1.3, $x_* = fx_*$.

Consider the path q in G from x_0 to x_*:

$$q = (x_0, x_1, x_2, \ldots, x_{n_N}, x_*) \in \Xi(G).$$

Then the above path is in $\widetilde{G}$. Hence, $x_* \in [x_0]_{\widetilde{G}}$.

Suppose that $(x, y) \in E(G)$ for any $x, y \in M$.
Let x_* and y_* be two fixed point of f.
Then $x_*, y_* \in M$. By assumption, $(x_*, y_*) \in E(G)$.
From (2.1) with $x = x_*$, $y = y_*$ we have

$$\begin{aligned}
F_{x_*,y_*}\bigl(\phi(t)\bigr) &= F_{fx_*,fy_*}\bigl(\phi(t)\bigr) \\
&\geq \min\bigl\{F_{x_*,y_*}(t), F_{x_*,fx_*}(t), F_{y_*,fy_*}(t)\bigr\} \\
&= \min\bigl\{F_{x_*,y_*}(t), 1, 1\bigr\} \\
&= F_{y_*,x_*}(t)
\end{aligned}$$

for all $t > 0$. By Lemma 1.3, $x_* = y_*$. Thus, f has a unique fixed point. □

Example 2.1 Let $X = [0, \infty)$, and let $d(x, y) = |x - y|$ for all $x, y \in X$.
Let

$$F_{x,y}(t) = \begin{cases} \epsilon_0(t) & (x = y), \\ D(\frac{t}{d(x,y)}) & (x \neq y), \end{cases}$$

for all $x, y \in X$ and $t > 0$, where D is a distribution function defined by

$$D(t) = \begin{cases} 0 & (t \leq 0), \\ 1 - e^{-t} & (t > 0). \end{cases}$$

Then (X, F, Δ_m) is a complete Menger PM-space.
Let $fx = \frac{1}{2}x$ for all $x \in X$, and let

$$\phi(t) = \begin{cases} \frac{1}{2}t & (0 \leq t < 1), \\ -\frac{1}{3}t + \frac{4}{3} & (1 \leq t \leq \frac{3}{2}), \\ t - \frac{2}{3} & (\frac{3}{2} < t < \infty). \end{cases}$$

Then $\phi \in \Phi_w$ and $\phi(t) \geq \frac{1}{2}t$ for all $t \geq 0$.
Further assume that X is endowed with a graph G consisting of $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : y \preceq x\}$.
Obviously, f preserves edges, and it is orbitally G-continuous. If $x_0 = 0$, then $(x_0, fx_0) = (0, 0) \in E(G)$.
We have

$$\begin{aligned}
F_{fx,fy}\bigl(\phi(t)\bigr) &= D\biggl(\frac{\phi(t)}{|fx - fy|}\biggr) \\
&\geq D\biggl(\frac{\frac{1}{2}t}{\frac{1}{2}t|x - y|}\biggr) = D\biggl(\frac{t}{t|x - y|}\biggr) \\
&= F_{x,y}(t) \\
&\geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}
\end{aligned}$$

for all $(x, y) \in E(G)$ and $t > 0$.

Thus, (2.1) is satisfied. Hence, all the conditions of Theorem 2.1 are satisfied and f has a fixed point $x_* = 0 \in [0]_{\widetilde{G}}$. Furthermore, $M = \{0\}$ and the fixed point is unique.

Remark 2.1 Note that in Theorem 2.1 the assumption of orbitally G-continuity can be replaced by orbitally continuity, G-continuity or continuity.

Remark 2.2 Theorem 2.1 is a generalization of Theorem 3.1 in [23] to the case of a Menger PM-space endowed with a graph.

Corollary 2.2 *Let (X, F, Δ) be complete, and let $f : X \to X$ be a map. Suppose that the following are satisfied*:

(1) *f preserves edges of G*;
(2) *there exists $\phi \in \Phi$ such that*

$$F_{fx,fy}\bigl(\phi(t)\bigr) \geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and all $t > 0$.

Assume that there exists $x_0 \in X$ such that $(x_0,fx_0) \in E(G)$. If either f is orbitally G-continuous or Δ is a continuous t-norm and G is a C-graph, then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Remark 2.3

(1) Corollary 2.2, in part, is a generalization of Theorem 3.9 and Theorem 3.15 of [13].
(2) In Corollary 2.2, let $\phi(s) = ks$ for all $s \geq 0$, where $k \in (0, 1)$. If G is a graph such that $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}$, where $\alpha : X \times X \to [0,\infty)$ is a function, then Corollary 2.2 reduces to Theorem 2.1 of [9].
(3) If G is a graph such that $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$, where $\preceq$ is a partial order on X, then Corollary 2.2 become to Theorem 2.1 of [10].

Corollary 2.3 *Let (X, F, Δ) be complete. Suppose that a map $f : X \to X$ is generalized probabilistic G-contraction. Assume that either f is continuous or Δ is a continuous t-norm and G is a C-graph.*

Then f has a fixed point in $[x_0]_{\widetilde{G}}$ for some $x_0 \in Q$ if and only if $Q \neq \emptyset$, where $Q = \{x \in X : (x,fx) \in E(\widetilde{G})\}$. Further if, for any $x, y \in Q$, $(x, y) \in E(\widetilde{G})$ then f has a unique fixed point.

Proof If f has a fixed point in $[x_0]_{\widetilde{G}}$, say x_*, then $(x_*,fx_*) = (x_*, x_*) \in \Omega \subset E(\widetilde{G})$. Thus, $Q \neq \emptyset$.

Suppose that $Q \neq \emptyset$.

Then there exists $x_0 \in X$ such that $(x_0,fx_0) \in E(\widetilde{G})$.

We have two cases: $(x_0,fx_0) \in E(G)$ or $(x_0,fx_0) \in E(G^{-1})$.

If $(x_0,fx_0) \in E(G)$, then following Theorem 2.1 f has a fixed point in $[x_0]_{\widetilde{G}}$.

Assume that $(x_0,fx_0) \in E(G^{-1})$.

Then $(fx_0, x_0) \in E(G)$. Since f is preserves edges of G, $(f^{n+1}x_0,f^n x_0) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$.

In the same way as the proof of Theorem 2.1 with condition (PM2), we deduce that f has a fixed point in $[x_0]_{\widetilde{G}}$.

Suppose that, for any $x, y \in Q$, $(x, y) \in E(\widetilde{G})$.

Let x_* and y_* be two fixed points of f.

Then $x_*, y_* \in Q$. By assumption, $(x_*, y_*) \in E(\widetilde{G})$.

If $(x_*, y_*) \in E(G)$, then

$$F_{x_*,y_*}\big(\phi(t)\big) \geq \min\big\{F_{x_*,y_*}(t), F_{x_*,x_*}(t), F_{y_*,y_*}(t)\big\} = F_{x_*,y_*}(t)$$

for all $t > 0$. By Lemma 1.1, $x_* = y_*$.

Let $(x_*, y_*) \in E(G^{-1})$, then $(y_*, x_*) \in E(G)$.

Then

$$F_{y_*,x_*}\big(\phi(t)\big) \geq F_{y_*,x_*}(t)$$

for all $t > 0$. Hence, $y_* = x_*$. Thus, f has a unique fixed point. □

Remark 2.4 If $\phi \in \Phi$ and G is a graph such that $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$, where $\preceq$ is a partial order on X, then Corollary 2.3 reduces to Theorem 2.2 of [10].

In the following result, we can drop continuity of the t-norm Δ.

Corollary 2.4 *Let (X, F, Δ) be complete. Suppose that a map $f : X \to X$ satisfies*

$$F_{fx,fy}\big(\phi(t)\big) \geq F_{x,y}(t) \tag{2.6}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and all $t > 0$, where $\phi \in \Phi_w$.

Assume that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. If either f is orbitally G-continuous or G is a C-graph, then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Further if $(x, y) \in E(G)$ for any $x, y \in M$, where $M = \{x \in X : (x, fx) \in E(G)\}$, then f has a unique fixed point.

Proof Let $x_0 \in X$ be such that $(x_0, fx_0) \in E(G)$, and let $x_n = f^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$.

Note that (2.6) to be satisfied implies that (2.1) is satisfied.

As in the proof of Theorem 2.1, $x_{n-1} \neq x_n$ and $(x_{n-1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$ and there exists

$$\lim_{n\to\infty} x_n = x_* \in X.$$

If f is orbitally G-continuous, then $\lim_{n\to\infty} x_n = fx_*$, and so $x_* = fx_*$.

Assume that G is a C-graph.

Then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an $N \in \mathbb{N}$ such that

$$(x_{n_k}, x_*) \in E(G)$$

for all $k \geq N$.

Since $\phi \in \Phi_w$, for each $t > 0$, there exists $r \geq t$ such that $\phi(r) < t$.

We have

$$\begin{aligned} &F_{x_*,fx_*}(t) \\ &\quad \geq \Delta\big(F_{x_*,x_{n_k+1}}\big(t - \phi(r)\big), F_{fx_{n_k},fx_*}\big(\phi(r)\big)\big) \end{aligned}$$

$$\begin{aligned}
&\geq \Delta\big(F_{x_*,x_{n_k+1}}\big(t-\phi(r)\big), F_{x_{n_k},x_*}(r)\big) \\
&\geq \Delta\big(F_{x_*,x_{n_k+1}}\big(t-\phi(r)\big), F_{x_{n_k},x_*}(t)\big) \\
&\geq \Delta(a_n, a_n) \qquad (2.7)
\end{aligned}$$

for all $t > 0$, where $a_n = \min\{F_{x_*,x_{n_k+1}}(t-\phi(r)), F_{x_{n_k},x_*}(t)\}$.

Since $\lim_{n\to\infty} a_n = 1$ and $\Delta(t,t)$ is continuous at $t = 1$, $\lim_{n\to\infty} \Delta(a_n, a_n) = \Delta(1,1) = 1$. Hence, from (2.7) we have $F_{x_*,fx_*}(t) = 1$ for all $t > 0$, and so $x_* = fx_*$. □

Remark 2.5 Corollary 2.4 is a generalization of Theorem 3.1 in [23] to the case of a Menger PM-space endowed with a graph.

Theorem 2.5 *Let* (X, F, Δ) *be complete such that* Δ *is continuous. Let* $f, h : X \to X$ *be maps, and let* G *be a directed graph satisfying* $V(G) = h(X)$ *and* $\{(hx, hx) : x \in X\} \subset E(G)$. *Suppose that the following are satisfied*:

(1) $f(X) \subset h(X)$;
(2) $h(X)$ *is closed*;
(3) $(hx, hy) \in E(G)$ *implies* $(fx, fy) \in E(G)$;
(4) *there exists* $x_0 \in X$ *such that* $(hx_0, fx_0) \in E(G)$;
(5) *there exists* $\phi \in \Phi_w$ *such that*

$$F_{fx,fy}\big(\phi(t)\big) \geq \min\big\{F_{hx,hy}(t), F_{hx,fx}(t), F_{hy,fy}(t)\big\} \qquad (2.8)$$

for all $x, y \in X$ *with* $(hx, hy) \in E(G)$ *and all* $t > 0$;

(6) *if* $\{x_n\}$ *is a sequence in* X *such that* $(hx_n, hx_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\lim_{n\to\infty} hx_n = hu$ *for some* $u \in X$, *then* $(hx_n, hu) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *and* h *have a coincidence point in* X. *Further if* f *and* h *commute at their coincidence points and* $(hu, hhu) \in E(G)$, *then* f *and* h *have a common fixed point in* X.

Proof By Lemma 1.7, there exists $Y \subset X$ such that $h(Y) = h(X)$ and $h : Y \to X$ is one-to-one. Define a mapping $U : h(Y) \to h(Y)$ by $U(hx) = fx$. Since $h : Y \to X$ is one-to-one, U is well defined.

By (3), $(hx, hy) \in E(G)$ implies $(U(hx), U(hy)) \in E(G)$.

By (4), $(hx_0, U(hx_0)) \in E(G)$ for some $x_0 \in X$. We have

$$\begin{aligned}
&F_{U(hx),U(hy)}\big(\phi(t)\big) \\
&\quad = F_{fx,fy}\big(\phi(t)\big) \\
&\quad \geq \min\big\{F_{hx,hy}(t), F_{hx,fx}(t), F_{hy,fy}(t)\big\} \\
&\quad = \min\big\{F_{hx,hy}(t), F_{hx,U(hx)}(t), F_{hy,U(hy)}(t)\big\}
\end{aligned}$$

for all $hx, hy \in h(Y)$ with $(hx, hy) \in E(G)$. Since $h(Y) = h(X)$ is complete, by applying Theorem 2.1, there exists $u \in X$ such that $U(hu) = hu$, and so $hu = fu$. Hence, u is a coincidence point of f and h.

Suppose that f and h commute at their coincidence points and $(hu, hhu) \in E(G)$. Let $w = hu = fu$. Then $fw = fhu = hfu = hw$, and $(hu, hw) = (hu, hhu) \in E(G)$.

Applying inequality (2.8) with $x = u$, $y = w$, we have

$$
\begin{aligned}
&F_{w,fw}\bigl(\phi(t)\bigr)\\
&\quad = F_{fu,fw}\bigl(\phi(t)\bigr)\\
&\quad \geq \min\bigl\{F_{hu,hw}(t), F_{hu,fu}(t), F_{hw,fw}(t)\bigr\}\\
&\quad = \min\bigl\{F_{w,fw}(t), F_{w,w}(t), F_{fw,fw}(t)\bigr\}\\
&\quad = \min\bigl\{F_{w,fw}(t), 1, 1\bigr\}\\
&\quad = F_{fw,w}(t)
\end{aligned}
$$

for all $t > 0$.

By Lemma 1.2, $w = fw$. Hence $w = fw = hw$. Thus, w is a common fixed point of f and h. □

Remark 2.6 Theorem 2.5 is a generalization of Theorem 3.4 of [3]. If we have $\phi(s) = ks$ for all $s \geq 0$, where $k \in (0,1)$, and $V(G) = X$ and $E(G) = \{(x,y) : x \leq y\}$, where $\leq$ is a partial order on X, then Theorem 2.5 reduces to Theorem 3.4 of [3].

Theorem 2.6 *Let (X, F, Δ) be complete. Suppose that maps $f_0, f_1 : X \to X$ satisfy the following*:

$$
F_{f_0x,f_0y}\bigl(\phi(t)\bigr) \geq F_{x,y}(t), \tag{2.9}
$$

where $\phi \in \Phi_w$ and

$$
F_{f_1x,f_1y}(t) \geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\} \tag{2.10}
$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and all $t > 0$.

Suppose that f preserves edges, and assume that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$, where $f = f_0f_1$. If either f is orbitally G-continuous or Δ is a continuous t-norm and G is a C-graph, then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Further if $(x,y) \in E(G)$ for any $x, y \in M$, where $M = \{x \in X : (x, fx) \in E(G)\}$, then f_0 and f_1 have a common fixed point whenever f_0 is commutative with f_1.

Proof From (2.9) and (2.10) we have

$$
F_{fx,fy}\bigl(\phi(t)\bigr) \geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}
$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and all $t > 0$. By Theorem 2.1, f has a fixed point in $[x_0]_{\widetilde{G}}$, say x_*.

Suppose that $(x,y) \in E(G)$ for any $x, y \in M$.

Then from Theorem 2.1 f has a unique fixed point.

Since f_0 is commutative with f_1 and $fx_* = x_*$, $ff_0x_* = f_0(f_1f_0x_*) = f_0(f_0f_1x_*) = f_0fx_* = f_0x_*$. Similarly, we obtain $ff_1x_* = f_1x_*$. From the uniqueness of fixed point of f, we have $x_* = f_0x_* = f_1x_*$. □

Example 2.2 Let $X = [0,\infty)$, and let $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $x,y \in X$ and all $t > 0$, where

$$d(x,y) = \begin{cases} \max\{x,y\} & (x \neq y), \\ 0 & (\text{otherwise}). \end{cases}$$

Then (X, F, Δ_m) is a complete Menger PM-space.

Let

$$\phi(t) = \begin{cases} \frac{1}{2}t & (0 \le t < 1), \\ -\frac{1}{3}t + \frac{4}{3} & (1 \le t \le \frac{3}{2}), \\ t - \frac{2}{3} & (\frac{3}{2} < t < \infty). \end{cases}$$

Then $\phi \in \Phi_w$ and $\phi(t) \ge \frac{1}{2}t$ for all $t \ge 0$.

Further assume that X is endowed with a graph G consisting of $V(G) = X$ and $E(G) = \{(x,y) \in X \times X : y \preceq x\}$.

Obviously, G is a C-graph.

Let $f_0 : X \to X$ be a map defined by $f_0 x = \frac{1}{2}x$ for all $x \ge 0$, and define a map $f_1 : X \to X$ by

$$f_1 x = \begin{cases} \frac{x}{4(1+x)} & (0 \le x \le 2), \\ \frac{1}{12}x & (x > 2). \end{cases}$$

Then

$$fx = f_0 f_1 x = \begin{cases} \frac{x}{8(1+x)} & (0 \le x \le 2), \\ \frac{1}{24}x & (x > 2). \end{cases}$$

Obviously, f preserves edges.

Let $(x,y) \in E(G)$.

Then $y \preceq x$, and we obtain

$$\begin{aligned} F_{f_0x,f_0y}\bigl(\phi(t)\bigr) &= \frac{\phi(t)}{\phi(t) + d(\frac{1}{2}x, \frac{1}{2}y)} \\ &\ge \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{2}x} = \frac{t}{t+x} \\ &= \frac{t}{t + \max\{x,y\}} = F_{x,y}(t) \end{aligned}$$

for all $t > 0$. Hence, (2.9) is satisfied.

We consider the following three cases:

Case 1. $0 \le y < x \le 2$:

$$\begin{aligned} F_{f_1x,f_1y}(t) &= \frac{t}{t + d(\frac{x}{4(1+x)}, \frac{y}{4(1+y)})} \\ &= \frac{t}{t + \frac{x}{4(1+x)}} \ge \frac{t}{t+x} \end{aligned}$$

$$
\begin{aligned}
&= \frac{t}{t+\max\{x,y\}} = \frac{t}{t+d(x,y)} = F_{x,y}(t) \\
&\geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}
\end{aligned}
$$

for all $t > 0$.

Case 2. $2 < y < x$:

$$
\begin{aligned}
F_{f_1x,f_1y}(t) &= \frac{t}{t+d(\frac{x}{12},\frac{y}{12})} \\
&= \frac{t}{t+\frac{x}{12}} \geq \frac{t}{t+x} = \frac{t}{t+\max\{x,y\}} \\
&= \frac{t}{t+d(x,y)} = F_{x,y}(t) \\
&\geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}
\end{aligned}
$$

for all $t > 0$.

Case 3. $0 \leq y \leq 2$ and $2 < x$:

$$
\begin{aligned}
F_{f_1x,f_1y}(t) &= \frac{t}{t+d(\frac{x}{12},\frac{y}{4(1+y)})} \\
&= \frac{t}{t+\frac{x}{12}} \geq \frac{t}{t+x} = \frac{t}{t+\max\{x,y\}} \\
&= \frac{t}{t+d(x,y)} = F_{x,y}(t) \\
&\geq \min\bigl\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\bigr\}
\end{aligned}
$$

for all $t > 0$.

Thus, (2.10) is satisfied.

For $x_0 = 4$, $(x_0, fx_0) = (4, \frac{1}{6}) \in E(G)$. Hence, all the conditions of Theorem 2.6 are satisfied and f has a fixed point $x_* = 0 \in [x_0]_{\widetilde{G}}$.

Corollary 2.7 *Let (X, F, Δ) be complete. Suppose that maps $f_0, f_1 : X \to X$ satisfy the following*:

$$F_{f_0x,f_0y}\bigl(\phi(t)\bigr) \geq F_{x,y}(t), \tag{2.11}$$

where $\phi \in \Phi_w$ and

$$F_{f_1x,f_1y}(t) \geq F_{x,y}(t) \tag{2.12}$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and all $t > 0$.

Suppose that f preserves edges, and assume that there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$, where $f = f_0f_1$. If f is orbitally G-continuous or G is a C-graph, then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Further if $(x,y) \in E(G)$ for any $x, y \in M$, where $M = \{x \in X : (x, fx) \in E(G)\}$, then f_0 and f_1 have a common fixed point whenever f_0 is commutative with f_1.

Proof From (2.11) and (2.12) we have

$$F_{fx,fy}\big(\phi(t)\big) \geq F_{x,y}(t)$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and all $t > 0$. By Corollary 2.4, f has a fixed point in $[x_0]_{\widetilde{G}}$, say x_*.

Suppose that $(x, y) \in E(G)$ for any $x, y \in M$.

Then from Corollary 2.4 f has a unique fixed point.

Since f_0 is commutative with f_1, as in the proof of Theorem 2.6 we have $x_* = f_0 x_* = f_1 x_*$. □

Remark 2.7 Corollary 2.7 is a generalization of Corollary 2.1 of [23] to the case of Menger PM-space endowed with a graph.

Corollary 2.8 *Let (X, d) be a complete metric space, and let $G = (V(G), E(G))$ be a directed graph satisfying $V(G) = X$ and $\Omega \subset E(G)$. Let $f : X \to X$ be a map. Suppose that the following are satisfied*:

(1) $(x, y) \in E(G)$ *implies* $(fx, fy) \in E(G)$;

(2) *there exists* $\phi \in \Phi_w$ *such that*

$$\begin{aligned} &d(fx, fy) \\ &\quad \leq \phi\big(\max\big\{d(x, y), d(x, fx), d(y, fy)\big\}\big) \end{aligned} \tag{2.13}$$

for all $x, y \in X$ *with* $(x, y) \in E(G)$, *where* ϕ *is nondecreasing*;

(3) *there exists* $x_0 \in X$ *such that* $(x_0, fx_0) \in E(G)$;

(4a) f *is continuous*, *or*

(4b) *if* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} x_n = x_* \in X$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $(x_{n_k}, x_*) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then f *has a fixed point in* $[x_0]_{\widetilde{G}}$.

Proof Suppose that equality holds in (2.13) and $x \neq fx$ for all $x \in X$.

Let $x_0 \in X$ be fixed. Then $(x_0, x_0) \in E(G)$, and from (2.13) we have

$$\begin{aligned} 0 &= d(fx_0, fx_0) \\ &= \phi\big(\max\big\{d(x_0, x_0), d(x_0, fx_0), d(x_0, fx_0)\big\}\big) \\ &= \phi\big(d(x_0, fx_0)\big), \end{aligned}$$

which implies $d(x_0, fx_0) = 0$ and so $x_0 = fx_0$, which is a contradiction.

Thus, if equality holds in (2.13), then f has a fixed point.

Assume that equality is not satisfied in (2.13).

Let (X, F, Δ_m) be the induced Menger PM-space by (X, d).

By Lemma 1.6, (X, F, Δ_m) is complete. By Remark 1.3, (4a) implies f is continuous in (X, F, Δ_m), and (4b) implies G is C-graph.

We show that (2.1) is satisfied.

We know that the values of each distribution function $F_{u,v}(\cdot)$, $u, v \in X$, in the induced Menger PM-space only can equal 0 or 1. Hence, without loss of generality, we may assume that

$$\min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\} = 1$$

for all $x, y \in E(G)$ and $t > 0$. Then

$$t > d(x, y), \qquad t > d(x,fx) \quad \text{and} \quad t > d(y,fy).$$

Thus,

$$t > \max\{d(x, y), d(x,fx), d(y,fy)\}.$$

Since ϕ is nondecreasing,

$$\phi(\max\{d(x, y), d(x,fx), d(y,fy)\}) \le \phi(t).$$

By assumption, we have

$$d(fx,fy) < \phi(t).$$

Hence, $\phi(t) - d(fx,fy) > 0$. So $F_{fx,fy}(\phi(t)) = 1$. Thus we have

$$F_{fx,fy}(\phi(t)) \ge \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and all $t > 0$.

Hence, (2.1) is satisfied. By Theorem 2.1 and Remark 2.1, f has a fixed point in $[x_0]_{\widetilde{G}}$. □

Corollary 2.9 *Let (X, d) be a complete metric space, and let $G = (V(G), E(G))$ be a directed graph satisfying $V(G) = X$ and $\Omega \subset E(G)$. Let $f : X \to X$ be a map.*

Suppose that the following are satisfied:

(1) $(x, y) \in E(G)$ *implies* $(fx,fy) \in E(G)$;

(2) *there exists $\phi \in \Phi_w$ such that*

$$d(fx,fy) \le \phi(d(x, y))$$

for all $x, y \in X$ with $(x, y) \in E(G)$, where ϕ is nondecreasing;

(3) *there exists $x_0 \in X$ such that $(x_0,fx_0) \in E(G)$*;

(4) *either f is continuous or if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = x_* \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x_*) \in E(G)$ for all $k \in \mathbb{N}$.*

Then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Remark 2.8 Corollary 2.9 is a generalization of the results of [5]. If we have a graph G such that $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$, where $\preceq$ is a partial order on X, and $\phi(s) = ks$ for all $s \ge 0$, where $k \in [0, 1)$, then Corollary 2.9 reduces to Theorem 2.1 and Theorem 2.2 of [5].

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper himself. The author read and approved the final manuscript.

Acknowledgement
The author thanks the editor and the referees for their useful comments and suggestions.

Received: 10 August 2015 Accepted: 1 April 2016 Published online: 12 April 2016

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