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Existence of best proximity points for controlled proximal contraction

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Abstract

In this paper, we investigate the sufficient condition for the existence of best proximity points for non-self-multivalued mappings. Additionally, we discuss the stability theorem for such mappings. Our results improve and generalize some existing results on the topic in the literature, in particular, the results of Lim and of Abkar and Gabeleh.

1 Introduction and preliminaries

Let (X, d) be a metric space and A, B be subsets of X. We denote by $\mathrm{CL}(B)$, the set of all nonempty closed subsets of B. A point $x \in A$ is called a fixed point of a mapping $T : A \to \mathrm{CL}(B)$, if $x \in Tx$. The multivalued map T has no fixed point if $A \cap B = \emptyset$. In this case $d(x, Tx) > 0$ for all $x \in A$. So, one can attempt to find the necessary condition so that the minimization problem

$$\min_{x \in A} d(x, Tx)$$

has at least one solution. A point $x^* \in X$ is said to be a best proximity point of the mapping $T : A \to B$ if $d(x^*, Tx^*) = \mathrm{dist}(A, B)$. When $A = B$, the best proximity point reduces to a fixed point of the mapping T. The following well-known best approximation theorem is due to Fan.

Theorem 1.1 [1] *Let A be a nonempty compact convex subset of normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that*

$$\|x - Tx\| = \inf_{a \in A} \{ \| Tx - a \| \}.$$

In this paper, we discuss sufficient conditions which ensure the existence of best proximity points for multivalued non-self-mappings satisfying contraction condition on the closed ball of a complete metric space. Moreover, we discuss the stability of the best proximity points. Our results extend and generalize some results by Lim [2], and Abkar and Gabeleh [3]. Some important best proximity theorems can be found in [4–15].

Now we recollect some notions, definitions, and results, for easy reference. $\mathrm{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, $d(x, B) = \inf\{d(x, b) : b \in B\}$, $A_0 = \{a \in A : d(a, b) = \mathrm{dist}(A, B)$

for some $b \in B\}$, $B_0 = \{b \in B : d(a,b) = \operatorname{dist}(A,B)$ for some $a \in A\}$, $\mathrm{CB}(B)$ is the set of all nonempty closed and bounded subsets of B and $B(x_0,r) = \{x \in X : d(x_0,x) \leq r\}$.

Definition 1.2 [13] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if and only if for any $x_1,x_2 \in A$ and $y_1,y_2 \in B$,

$$\begin{cases} d(x_1,y_1) = \operatorname{dist}(A,B), \\ d(x_2,y_2) = \operatorname{dist}(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2).$$

Example 1.3 [14] Let $X = \{(0,1),(1,0),(0,3),(3,0)\}$, endowed with a metric $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,3),(3,0)\}$. Then, for

$$d\big((0,1),(0,3)\big) = \operatorname{dist}(A,B) \quad \text{and} \quad d\big((1,0),(3,0)\big) = \operatorname{dist}(A,B)$$

we have

$$d\big((0,1),(1,0)\big) < d\big((0,3),(3,0)\big).$$

Also, $A_0 \neq \emptyset$. Thus the pair (A,B) satisfies the weak P-property.

Definition 1.4 [3] An element $x^* \in A$ is said to be a best proximity point of a multivalued non-self-mapping T, if $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.

Theorem 1.5 [3] *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to \mathrm{CB}(B)$ be a mapping satisfying the following conditions*:

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$*;

(ii) *the pair (A,B) satisfies the P-property*;

(iii) *there exists $\alpha \in (0,1)$ such that, for each $x,y \in A$, we have $H(Tx,Ty) \leq \alpha d(x,y)$.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.

2 Best proximity theorems

We start this section by introducing the following definition.

Definition 2.1 Let A and B be nonempty subsets of a metric space (X,d), $x_0 \in A_0$, and $B(x_0,r)$ is a closed ball in X. A mapping $T : A \to \mathrm{CL}(B)$ is said to be a proximal contraction on $B(x_0,r)$, if there exists $\alpha \in (0,1)$ such that

$$H(Tx,Ty) \leq \alpha d(x,y) \quad \text{for each } x,y \in B(x_0,r) \cap A. \tag{2.1}$$

Lemma 2.2 [16] *Let (X,d) be a metric space, $B \in \mathrm{CL}(X)$, and $q > 1$. Then, for each $x \in X$, there exists an element $b \in B$ such that*

$$d(x,b) \leq q d(x,B). \tag{2.2}$$

Now we are in a position to state and prove our first result.

Theorem 2.3 *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T : A \to \mathrm{CL}(B)$ is a mapping satisfying the following conditions*:

(i) *for each $x \in A_0$, we have $Tx \subseteq B_0$*;
(ii) *the pair (A,B) satisfies weak P-property*;
(iii) *there exists $x_0 \in A_0$ such that T is a proximal contraction on the closed ball $B(x_0,r)$ and $d(x_0,Tx_0) + \mathrm{dist}(A,B) \leq (1-\sqrt{\alpha})r$.*

Then T has a best proximity point in $B(x_0,r) \cap A_0$.

Proof By hypothesis (iii), we have $x_0 \in A_0$ such that T is a proximal contraction on the closed ball $B(x_0,r)$ and $d(x_0,Tx_0) + \mathrm{dist}(A,B) \leq (1-\sqrt{\alpha})r$. As $x_0 \in A_0$. By (i), we have $y_0 \in Tx_0 \subseteq B_0$. Then there exists $x_1 \in A_0$ such that

$$d(x_1,y_0) = \mathrm{dist}(A,B). \tag{2.3}$$

By using the triangular inequality, hypothesis (iii) and (2.3), we have

$$d(x_0,x_1) \leq d(x_0,Tx_0) + d(Tx_0,x_1) \leq d(x_0,Tx_0) + d(y_0,x_1) \leq (1-\sqrt{\alpha})r. \tag{2.4}$$

Since $x_1 \in A_0 \subseteq A$, $x_1 \in B(x_0,r) \cap A$. From (2.1), we have

$$d(y_0,Tx_1) \leq H(Tx_0,Tx_1) \leq \alpha d(x_0,x_1). \tag{2.5}$$

As $\alpha > 0$, by Lemma 2.2, we have $y_1 \in Tx_1$ such that

$$d(y_0,y_1) \leq \frac{1}{\sqrt{\alpha}} d(y_0,Tx_1) \leq \sqrt{\alpha} d(x_0,x_1). \tag{2.6}$$

Since $Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2,y_1) = \mathrm{dist}(A,B), \tag{2.7}$$

as (A,B) satisfies the weak P-property. From (2.3) and (2.7), we have

$$d(x_1,x_2) \leq d(y_0,y_1). \tag{2.8}$$

From (2.6) and (2.8), we have

$$d(x_1,x_2) \leq \sqrt{\alpha} d(x_0,x_1). \tag{2.9}$$

Considering the triangular inequality, (2.4), and (2.9), we have

$$\begin{aligned} d(x_0,x_2) &\leq d(x_0,x_1) + d(x_1,x_2) \\ &\leq d(x_0,x_1) + \sqrt{\alpha} d(x_0,x_1) \\ &\leq (1-\alpha)r < r. \end{aligned}$$

By construction, we have $x_2 \in A_0 \subseteq A$. Thus $x_2 \in B(x_0, r) \cap A$. Again from (2.1), we have

$$d(y_1, Tx_2) \leq H(Tx_1, Tx_2) \leq \alpha d(x_1, x_2). \tag{2.10}$$

By using Lemma 2.2, we have $y_2 \in Tx_2$ such that

$$d(y_1, y_2) \leq \frac{1}{\sqrt{\alpha}} d(y_1, Tx_2) \leq \sqrt{\alpha} d(x_1, x_2). \tag{2.11}$$

Since $Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, y_2) = \operatorname{dist}(A, B), \tag{2.12}$$

as (A, B) satisfies the weak P-property. From (2.7) and (2.12), we have

$$d(x_2, x_3) \leq d(y_1, y_2). \tag{2.13}$$

From (2.11) and (2.13), we have

$$d(x_2, x_3) \leq \sqrt{\alpha} d(x_1, x_2) \leq \alpha d(x_0, x_1). \tag{2.14}$$

By considering the triangular inequality, (2.9), and (2.14), we have

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq \left[1 + \sqrt{\alpha} + (\sqrt{\alpha})^2\right] d(x_0, x_1) \\ &\leq \left[1 + \sqrt{\alpha} + (\sqrt{\alpha})^2\right](1 - \sqrt{\alpha}) r < r, \end{aligned}$$

as $x_3 \in A_0 \subseteq A$. Thus, $x_3 \in B(x_0, r) \cap A$. Continuing in the same way, we get two sequences $\{x_n\} \subseteq A_0$ with $x_n \in B(x_0, r) \cap A$ and $\{y_n\} \subseteq B_0$ with $y_n \in Tx_n$ such that

$$d(x_n, y_{n-1}) = \operatorname{dist}(A, B) \quad \text{for each } n \in \mathbb{N}. \tag{2.15}$$

Moreover,

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq (\sqrt{\alpha})^n d(x_0, x_1) \quad \text{for each } n \in \mathbb{N}. \tag{2.16}$$

For $n > m$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (\sqrt{\alpha})^i d(x_0, x_1) < \sum_{i=n}^{\infty} (\sqrt{\alpha})^i d(x_0, x_1) < \infty. \tag{2.17}$$

Hence $\{x_n\}$ is a Cauchy sequence in $B(x_0, r) \cap A \subseteq A$. A similar reasoning shows that $\{y_n\}$ is a Cauchy sequence in B. Since $B(x_0, r) \cap A$ is closed in A, and A, B are closed subsets of a complete metric space, there exist $x^* \in B(x_0, r) \cap A$ and $y^* \in B$ such that $x_n \to x^*$ and $y_n \to y^*$. By (2.15), we conclude that $d(x^*, y^*) = \operatorname{dist}(A, B)$ as $n \to \infty$. Clearly, $y^* \in Tx^*$, since $\lim_{n\to\infty} d(y_n, Tx^*) \leq \lim_{n\to\infty} H(Tx_n, Tx^*) = 0$. Hence $\operatorname{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, y^*) = \operatorname{dist}(A, B)$. Therefore, x^* is a best proximity point of the mapping T. □

Example 2.4 Let $X = \mathbb{R}^2$ be endowed with the metric $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(1,x) : x \in \mathbb{R}\}$ and $B = \{(0,x) : x \in \mathbb{R}\}$. Define $T : A \to \mathrm{CL}(B)$ by

$$T(1,x) = \begin{cases} \{(0,0)\} & \text{if } x \leq 0, \\ \{(0,0),(0,x/2)\} & \text{if } 0 \leq x \leq 10, \\ \{(0,x)\} & \text{if } x > 10. \end{cases}$$

Let us consider a ball $B(x_0,r)$ with $x_0 = (1,0.1)$ and $r = 7.5$. Then it is easy to see that T is a proximal contraction on the closed ball $B((1,0.1),7.5)$ with $\alpha = \frac{1}{2}$. Also, we have $d(x_0,Tx_0) + \operatorname{dist}(A,B) \leq (1 - \sqrt{\alpha})r$. Furthermore, $A_0 = A$, $B_0 = B$; for each $x \in A_0$ we have $Tx \subseteq B_0$ and the pair (A,B) satisfies the weak P-property. Therefore, all the conditions of Theorem 2.3 hold and T has a best proximity point.

Corollary 2.5 *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T : A \to B$ is a mapping satisfying the following conditions*:

(i) *for each $x \in A_0$, we have $Tx \in B_0$*;

(ii) *the pair (A,B) satisfies the weak P-property*;

(iii) *there exists $x_0 \in A_0$ such that T is a proximal contraction on the closed ball $B(x_0,r)$, that is,*

$$d(Tx,Ty) \leq \alpha d(x,y) \quad \text{for each } x,y \in B(x_0,r) \cap A, \tag{2.18}$$

and $d(x_0,Tx_0) + \operatorname{dist}(A,B) \leq (1 - \sqrt{\alpha})r$.

Then T has a best proximity point in $B(x_0,r) \cap A_0$.

If we assume that $X = A = B$, then Theorem 2.3 reduces to the following fixed point theorem.

Corollary 2.6 *Let (X,d) be a complete metric space and $T : X \to \mathrm{CL}(X)$ be a mapping. Assume that there exist $x_0 \in X$ and $\alpha \in (0,1)$ satisfying*

$$H(Tx,Ty) \leq \alpha d(x,y) \quad \text{for each } x,y \in B(x_0,r)$$

and $d(x_0,Tx_0) \leq (1 - \sqrt{\alpha})r$. Then T has a fixed point.

3 Stability of best proximity points

Stability of fixed point sets of multivalued mappings was initially investigated by Markin [15] and Nadler [16] with some strong conditions. Lim [2] proved the stability theorem for fixed point sets of multivalued contraction mappings by relaxing the condition assumed by Markin [15]. Abkar and Gabeleh [3] discussed the stability of best proximity point sets of non-self-multivalued mappings. In this section, we extend and generalize the stability theorems due to Abkar and Gabeleh [3], and Lim [2].

In this section, by B_{T_1} and B_{T_2} we denote the sets of best proximity points of T_1 and T_2, respectively.

Theorem 3.1 *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T_i : A \to \mathrm{CL}(B)$, $i = 1,2$ are mappings satisfying the fol-*

lowing conditions:

(i) *for each* $x \in A_0$, *we have* $T_i x \subseteq B_0$, $i = 1,2$;

(ii) *the pair* (A,B) *satisfies the weak P-property*;

(iii) *for each* $i = 1,2$, *there exists* a_i *such that* T_i *is proximal contraction on the closed ball* $B(a_i, r_i)$ *with the same* α *as a contraction constant, that is*,

$$H(T_i x, T_i y) \leq \alpha d(x,y) \quad \textit{for each } x,y \in B(a_i,r_i) \cap A, \tag{3.1}$$

and $d(a_i, T_i a_i) + \operatorname{dist}(A,B) \leq (1-\sqrt{\alpha}) r_i$.

Then

$$H(B_{T_1}, B_{T_2}) \leq \frac{1}{1-\sqrt{\alpha}} \Big[\sup_{x \in A} H(T_1 x, T_2 x) + 2 \operatorname{dist}(A,B) \Big].$$

Proof Let $x_0 \in B_{T_1}$, then we have $y_0 \in T_2 x_0$ such that

$$d(x_0, y_0) \leq H(T_1 x_0, T_2 x_0) + \operatorname{dist}(A,B).$$

Since $y_0 \in T_2 x_0 \subseteq B_0$, we have $x_1 \in A_0$ such that

$$d(x_1, y_0) = \operatorname{dist}(A,B). \tag{3.2}$$

We know that T_2 is a proximal contraction for closed ball $B(a_2, r_2)$. Without loss of generality, we take $a_2 = x_0$ and $r_2 = r$ such that $d(x_0, T_2 x_0) + \operatorname{dist}(A,B) \leq (1-\sqrt{\alpha})r$. Clearly, $x_1 \in B(x_0, r) \cap A$, since $x_1 \in A_0 \subseteq A$ and

$$d(x_0, x_1) \leq d(x_0, T_2 x_0) + d(T_2 x_0, x_1) \leq d(x_0, T_2 x_0) + d(y_0, x_1) \leq (1-\sqrt{\alpha})r. \tag{3.3}$$

By hypothesis (iii), we have

$$d(y_0, T_2 x_1) \leq H(T_2 x_0, T_2 x_1) \leq \alpha d(x_0, x_1). \tag{3.4}$$

As $\alpha > 0$, by Lemma 2.2, we have $y_1 \in T_2 x_1$ such that

$$d(y_0, y_1) \leq \frac{1}{\sqrt{\alpha}} d(y_0, T_2 x_1) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.5}$$

Since $T_2 x_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, y_1) = \operatorname{dist}(A,B), \tag{3.6}$$

as (A,B) satisfies the weak P-property. From (3.2) and (3.6), we have

$$d(x_1, x_2) \leq d(y_0, y_1). \tag{3.7}$$

From (3.5) and (3.7), we have

$$d(x_1, x_2) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.8}$$

Considering the triangular inequality, (3.3), and (3.8), we have

$$\begin{aligned} d(x_0,x_2) &\le d(x_0,x_1) + d(x_1,x_2) \\ &\le d(x_0,x_1) + \sqrt{\alpha}d(x_0,x_1) \\ &\le (1-\alpha)r < r. \end{aligned}$$

Also, $x_2 \in A_0 \subseteq A$. Thus, $x_2 \in B(x_0,r) \cap A$. Continuing in the same way, we get two sequences $\{x_n\} \subseteq A_0$ with $x_n \in B(x_0,r) \cap A$ and $\{y_n\} \subseteq B_0$ with $y_n \in T_2x_n$ such that

$$d(x_n,y_{n-1}) = \operatorname{dist}(A,B) \quad \text{for each } n \in \mathbb{N}. \tag{3.9}$$

Moreover,

$$d(x_n,x_{n+1}) \le d(y_{n-1},y_n) \le (\sqrt{\alpha})^n d(x_0,x_1) \quad \text{for each } n \in \mathbb{N}. \tag{3.10}$$

For $n > m$, we have

$$d(x_n,x_m) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{m-1} (\sqrt{\alpha})^i d(x_0,x_1) < \sum_{i=n}^{\infty} (\sqrt{\alpha})^i d(x_0,x_1) < \infty. \tag{3.11}$$

Hence $\{x_n\}$ is a Cauchy sequence in $B(x_0,r) \cap A \subseteq A$. A similar reasoning shows that $\{y_n\}$ is a Cauchy sequence in B. Since $B(x_0,r) \cap A$ is closed in A, and A, B are closed subsets of a complete metric space, there exist $u^* \in B(x_0,r) \cap A$ and $v^* \in B$ such that $x_n \to u^*$ and $y_n \to v^*$. By (3.9), we conclude that $d(u^*,v^*) = \operatorname{dist}(A,B)$ as $n \to \infty$. Clearly, $v^* \in T_2u^*$. Then we have $\operatorname{dist}(A,B) \le d(u^*, T_2u^*) \le d(u^*,v^*) = \operatorname{dist}(A,B)$. Therefore u^* is a best proximity point of T_2. Now, we have

$$\begin{aligned} d(x_0,u^*) &\le \sum_{n=0}^{\infty} d(x_n,x_{n+1}) \\ &\le \sum_{n=0}^{\infty} (\sqrt{\alpha})^n d(x_0,x_1) \\ &= \frac{1}{1-\sqrt{\alpha}} d(x_0,x_1) \\ &\le \frac{1}{1-\sqrt{\alpha}} \bigl[d(x_0,y_0) + d(y_0,x_1)\bigr] \\ &= \frac{1}{1-\sqrt{\alpha}} \bigl[d(x_0,y_0) + \operatorname{dist}(A,B)\bigr] \\ &\le \frac{1}{1-\sqrt{\alpha}} \bigl[H(T_1x_0, T_2x_0) + 2\operatorname{dist}(A,B)\bigr]. \end{aligned}$$

Similarly, if $\mathfrak{x}_o \in B_{T_2}$, then we have $\mathfrak{u}^* \in B_{T_1}$ such that

$$d(\mathfrak{x}_o,\mathfrak{u}^*) \le \frac{1}{1-\sqrt{\alpha}} \bigl[H(T_1\mathfrak{x}_o, T_2\mathfrak{x}_o) + 2\operatorname{dist}(A,B)\bigr].$$

Thus, we have

$$H(B_{T_1}, B_{T_2}) \le \frac{1}{1-\sqrt{\alpha}}\Big[\sup_{x\in A} H(T_1x, T_2x) + 2\operatorname{dist}(A,B)\Big]. \qquad \square$$

Example 3.2 Let $X = \mathbb{R}^2$ be endowed with the metric $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(1,x) : x \in \mathbb{R}\}$ and $B = \{(0,x) : x \in \mathbb{R}\}$. Define $T_1, T_2 : A \to \mathrm{CL}(B)$ by

$$T_1(1,x) = \begin{cases} \{(0,0)\} & \text{if } x \le 0, \\ \{(0,0),(0,x/2)\} & \text{if } 0 \le x \le 10, \\ \{(0,x^2)\} & \text{if } x > 10 \end{cases}$$

and

$$T_2(1,x) = \begin{cases} \{(0,1)\} & \text{if } x \le 1, \\ \{(0,1),(0,(x+1)/2)\} & \text{if } x > 1. \end{cases}$$

It is easy to see that T_1 is a proximal contraction on the closed ball $B(x_0 = (1,0.1), r = 7.5)$ with $\alpha = \frac{1}{2}$ and $d(x_0, Tx_0) + \operatorname{dist}(A,B) \le (1-\sqrt{\alpha})r$. Further, T_2 is a proximal contraction on the closed ball $B(x_1 = (1,1.25), r_1 = 8)$ with $\alpha = \frac{1}{2}$ and $d(x_1, Tx_1) + \operatorname{dist}(A,B) \le (1-\sqrt{\alpha})r_1$. Furthermore, it is easy to see that $A_0 = A$, $B_0 = B$, and for each $x \in A_0$ we have $T_ix \subseteq B_0$ for each $i = 1,2$ and the pair (A,B) satisfies the weak P-property. All the conditions of Theorem 3.1 hold. Thus the conclusion holds. That is,

$$H(B_{T_1}, B_{T_2}) \le \frac{1}{1-\sqrt{\alpha}}\Big[\sup_{x\in A} H(T_1x, T_2x) + 2\operatorname{dist}(A,B)\Big].$$

If we assume that $X = A = B$, then Theorem 3.1 reduces to the following stability result.

Corollary 3.3 *Let (X,d) be a complete metric space and $T_i : X \to \mathrm{CL}(X)$, $i = 1,2$ be mappings. Assume that there exist $\alpha \in (0,1)$ and $a_1, a_2 \in X$ such that, for each i, we have*

$$H(T_ix, T_iy) \le \alpha d(x,y) \quad \textit{for each } x,y \in B(a_i, r_i) \tag{3.12}$$

and $d(a_i, T_ia_i) \le (1-\sqrt{\alpha})r_i$. Let F_{T_1} and F_{T_2} denote the sets of fixed points of T_1 and T_2 respectively. Then

$$H(F_{T_1}, F_{T_2}) \le \frac{1}{1-\sqrt{\alpha}} \sup_{x\in A} H(T_1x, T_2x).$$

Note that in this theorem $B(a_i, r_i)$ are closed balls.

Remark 3.4 If r_1, r_2 are sufficiently large, then $B(a_1, r_1)$ and $B(a_2, r_2)$ are equal to X. In this case, from Corollary 3.3, we get the following result.

Corollary 3.5 (Lim [2], Lemma 1) *Let (X,d) be a complete metric space and $T_i : X \to \mathrm{CL}(X)$, $i = 1,2$ be α-contractions with the same α, that is,*

$$H(T_ix, T_iy) \le \alpha d(x,y) \quad \textit{for each } x,y \in X,$$

where $\alpha \in (0,1)$. *Then*

$$H(F_{T_1}, F_{T_2}) \le \frac{1}{1-\alpha} \sup_{x \in X} H(T_1 x, T_2 x).$$

Corollary 3.6 (Lim [2], Theorem 1) *Let* (X,d) *be a complete metric space and* $T_i : X \to \mathrm{CL}(X)$, $i = 1,2,\ldots$, *be* α*-contractions with the same* α. *If* $\lim_{i\to\infty} H(T_i x, T_0 x) = 0$ *uniformly for all* $x \in X$, *then* $\lim_{i\to\infty} H(F_{T_i}, F_{T_0}) = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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