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Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control

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Abstract

In this paper, we shall investigate the problem of exponential synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling and distributed time-varying delay coupling. The designed controller ensures that the synchronization of delayed complex dynamical network are proposed via either feedback control or intermittent feedback control. The constraint on the derivative of the time-varying delay is not required which allows the time-delay to be a fast time-varying function. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. This is based on the construction of an improved Lyapunov-Krasovskii functional combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Two numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme.

Keywords: exponential synchronization; complex dynamical network; mixed time-varying delays; hybrid coupling; intermittent control

1 Introduction

Complex dynamical network, as an interesting subject, has been thoroughly investigated for decades. These networks show very complicated behavior and can be used to model and explain many complex systems in nature such as computer networks [1], the world wide web [2], food webs [3], cellular and metabolic networks [4], social networks [5], electrical power grids [6] *etc.* In general, a complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. As an implicit assumption, these networks are described by the mathematical term *graph*. In such graphs, each vertex represents an individual element in the system, while edges represent the relations between them. Two nodes are joined by an edge if and only if they interact.

In the last decade, the synchronization of complex dynamic networks has attracted much attention of researchers in this field [7–18]. Because the synchronization of complex dynamical networks can well explain many natural phenomena observed and is one of the important dynamical mechanisms for creating order in complex dynamical networks, the synchronization of coupled dynamical networks has come be a focal point in the study of nonlinear science. Wang and Chen introduced a uniform dynamical network model and also investigated its synchronization [11–13]. They have shown that the synchronizability of a scale-free dynamical network is robust against random removal of nodes, and yet it is fragile to specific removal of the most highly connected nodes [12]. The authors in [14, 15] investigated synchronization of general complex dynamical network models with coupling delays. Li and Chen [8] considered the synchronization stability of complex dynamical network models with coupling delays for both continuous- and discrete-time, and they derived some synchronization conditions for both delay-independent and delay-dependent asymptotical stabilities. By utilizing Lyapunov functional method. Wang *et al.* [16] introduced several synchronization criteria for both delay-independent and delay-dependent asymptotical stability. Li and Yi [17] investigated synchronization of complex networks with time-varying couplings, the stability criteria were obtained by using Lyapunov-Krasovskii function method and subspace projection method. Yue and Li [18] studied the synchronization stability of continuous and discrete complex dynamical networks with interval time-varying delays in the dynamical nodes and the coupling term simultaneously, delay-dependent synchronization stability are derived in the form of linear matrix inequalities.

It is well known that the existence of time-delay in a system may cause instability and an example of oscillations can be found in systems such as chemical engineering systems, biological modeling, electrical networks, physical networks, and many others [19–25]. The stability criteria for a system with time-delays can be classified into two categories: delay-independent and delay-dependent. Delay-independent criteria do not employ any information on the size of the delay; while delay-dependent criteria make use of such information at different levels. Delay-dependent stability conditions are generally less conservative than delay-independent ones especially when the delay is small [25]. Recently, the delay-dependent stability for interval time-varying delay was investigated in [6, 18, 20–22]. Interval time-varying delay is a time-delay that varies in an interval in which the lower bound is not restricted to be 0. Jiang and Han [22] considered the problem of robust H_∞ control for uncertain linear systems with interval time-varying delay based on Lyapunov functional approach in which restriction on the differentiability of the interval time-varying delay was removed. Shao [24] presented a new delay-dependent stability criterion for linear systems with interval time-varying delay, and stability criteria are derived in terms of linear matrix inequalities without introducing any free-weighting matrices. In order to reduce further the conservatism introduced by the descriptor model transformation and bounding techniques, a free-weighting matrix method is proposed in [20, 26–29]. In [18], the synchronization problem has been investigated for continuous/discrete complex dynamical networks with interval time-varying delays. Based on a piecewise analysis method and the Lyapunov functional method, some new delay-dependent synchronization criteria are derived in the form of LMIs by introducing free-weighting matrices. It will be pointed out later that some existing results require more free-weighting matrix variables than our result.

Intermittent control is one of discontinuous control and has a nonzero control width. It is an engineering approach that has been widely used in engineering fields, such as manufacturing, air-quality control, transportation, and communication in practice. However, results using intermittent control to study exponential synchronization are few. In recent years, several synchronization criteria for complex dynamical networks with or without time-delays via feedback control or intermittent control have been presented; see [30–41] and the references therein. Synchronization of a complex dynamical network with delayed nodes by pinning periodically intermittent control was also reported in [31]. A periodically intermittent control was applied to the complex dynamical networks with both time-varying delays dynamical nodes and time-varying delays coupling in [32, 33]. In [34], the authors investigated exponential synchronization of a complex network with nonidentical time-delayed dynamical nodes by applying open-loop control to all nodes and adding some intermittent controllers to partial nodes. The authors in [31] investigated synchronization of a general model of complex delayed dynamical networks. The periodically intermittent control scheme is introduced to drive the network to achieve synchronization. Based on the Lyapunov stability theory and pinning control method, some novel synchronization criteria for such dynamical network are derived. To the best of the authors' knowledge, the problem of exponential synchronization for a complex dynamical network with mixed time-varying delays in the network hybrid coupling and time-varying delays in the dynamical nodes has not been fully investigated yet and remains open.

In this paper, inspired by the above discussions, we shall investigate the problem of exponential synchronization for a complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of constant coupling, interval time-varying delay coupling, and distributed time-varying delay coupling. The designed controller ensures that the synchronization of a delayed complex dynamical network is proposed via either feedback control or intermittent feedback control. The constraint on the derivative of the time-varying delay is not required, which allows the time-delay to be a fast time-varying function. We use common unitary matrices, and the problem of synchronization is transformed into the stability analysis of some linear time-varying delay systems. Based on the construction of an improved Lyapunov-Krasovskii functional is combined with the Leibniz-Newton formula and the technique of dealing with some integral terms. New synchronization criteria are derived in terms of LMIs which can be solved efficiently by standard convex optimization algorithms. Two numerical examples are included to show the effectiveness of the proposed feedback control and intermittent feedback control scheme.

The organization of the remaining part is as follows. In Section 2, a class of general complex dynamical network model with mixed time-varying and hybrid coupling delays and some useful lemmas are given. In Section 3, synchronization stability in complex dynamical network with mixed time-varying and hybrid coupling delays via feedback control and intermittent feedback control are investigated. Numerical examples illustrated the obtained results are given in Section 4. The paper ends with conclusions in Section 5.

2 Network model and mathematic preliminaries

Consider a complex dynamical network consisting of N identical coupled nodes, with each node being an n-dimensional dynamical system

$$\begin{aligned}\dot{x}_i(t) &= f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big)\\ &\quad + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k_1(t)}^{t} x_j(s)\,ds + \mathcal{U}_i(t), \quad t\geq 0, i=1,2,\ldots,N, \end{aligned} \tag{1}$$

$$x_i(t) = \phi_i(t), \quad t\in[-\tau_{\max}, 0], \tau_{\max} = \max\{h_2, d, k_1, k_2\},$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of ith node; $\mathcal{U}_i(t) \in \mathbb{R}^m$ are the control input of the node i; the constants $c_1, c_2, c_3 > 0$ are the coupling strength; $G_1 = (g_{1ij})_{n\times n}, G_2 = (g_{2ij})_{n\times n}, G_3 = (g_{3ij})_{n\times n} \in \mathbb{R}^{n\times n}$ are constant inner-coupling matrices, if some pairs (i,j), $1 \leq i,j \leq n$, with $g_{1ij} \neq 0$, $g_{2ij} \neq 0$, and $g_{3ij} \neq 0$, which means two coupled nodes are linked through their ith and jth state variables, otherwise $g_{1ij} = 0$, $g_{2ij} = 0$, $g_{3ij} = 0$; $A = (a_{ij})_{N\times N}$, $B = (b_{ij})_{N\times N}$, and $C = (c_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ are the outer-coupling matrices of the network, in which a_{ij}, b_{ij} are defined as follows: if there are a connection between node i and node j ($j \neq i$), then $a_{ij} = a_{ji} = 1$, $b_{ij} = b_{ji} = 1$, $c_{ij} = c_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$, $b_{ij} = b_{ji} = 0$, $c_{ij} = c_{ji} = 0$ ($j \neq i$), and the diagonal elements of matrices A, B, and C are defined by

$$\begin{aligned}
a_{ii} &= -\sum_{j=1,i\neq j}^{N} a_{ij} = -\sum_{j=1,i\neq j}^{N} a_{ji}, \\
b_{ii} &= -\sum_{j=1,i\neq j}^{N} b_{ij} = -\sum_{j=1,i\neq j}^{N} b_{ji}, \\
c_{ii} &= -\sum_{j=1,i\neq j}^{N} c_{ij} = -\sum_{j=1,i\neq j}^{N} c_{ji}, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{2}$$

It is assumed that network (1) is connected in the sense that there are no isolated clusters, that is, A, B, C are irreducible matrices.

Definition 2.1 [18] The delayed dynamical network (1) is said to achieve asymptotical synchronization if

$$x_1(t) = x_2(t) = \cdots = s(t) \quad \text{as } t \to \infty, \tag{3}$$

where $s(t)$ is a solution of an isolated node, satisfying

$$\dot{s}(t) = f\left(s(t), s\big(t - h(t)\big), \int_{t-k_1(t)}^{t} s(\theta)\, d\theta\right).$$

In order to stabilize the origin of dynamical network (1) by means of the state feedback controller $\mathcal{U}_i(t)$ satisfying either (H1) or (H2), for $i = 1, 2, \ldots, n$,

$$\begin{aligned}
\text{(H1):}\quad \mathcal{U}_i(t) &= D_{1i}u_i(t) + D_{2i}u_i\big(t - d(t)\big) \\
&\quad + D_{3i}\int_{t-k_2(t)}^{t} u_i(s)\, ds, \quad \forall t \geq t_0,
\end{aligned}$$

$$\text{(H2):}\quad \mathcal{U}_i(t) = \begin{cases} D_{4i}u_i(t) + D_{5i}u_i(t - d(t)) & \\ \quad + D_{6i}\int_{t-k_2(t)}^{t} u_i(s)\, ds, & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega, \end{cases}$$

where $D_{ji}, j = 1, 2, \ldots, 6$ are given matrices of appropriate dimensions, $u_i(t) = K_i(x_i(t) - s(t))$ and K_i is a constant matrix control gain, $\omega > 0$ is the control period and $\delta > 0$ is called the control width (control duration) and n is a non-negative integer. Then substituting it into dynamical network (1), it is easy to get the following:

$$\begin{aligned}\dot{x}_i(t) &= f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) \\ &\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k_1(t)}^{t} x_j(s)\,ds \\ &\quad + D_{1i}K_i\big(x_i(t)-s(t)\big) + D_{2i}u_i\big(t-d(t)\big) + D_{3i}\int_{t-k_2(t)}^{t} u_i(s)\,ds. \end{aligned} \tag{4}$$

Namely, the dynamical network (1) is governed by the following system:

$$\begin{aligned}\dot{x}_i(t) &= f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) \\ &\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k_1(t)}^{t} x_j(s)\,ds \\ &\quad + D_{4i}K_i\big(x_i(t)-s(t)\big) + D_{5i}u_i\big(t-d(t)\big) + D_{6i}\int_{t-k_2(t)}^{t} u_i(s)\,ds, \\ &\quad n\omega \le t \le n\omega + \delta, \\ \dot{x}_i(t) &= f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) \\ &\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k_1(t)}^{t} x_j(s)\,ds, \\ &\quad n\omega + \delta < t \le (n+1)\omega, i = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

It is clear that, if the zero solutions of the dynamical network (4) and (5) are globally exponentially stable, then exponential synchronization of the controlled dynamical network (1) is achieved. The time-varying delay functions $h(t)$, $d(t)$, $k_1(t)$, and $k_2(t)$ satisfy the conditions

$$0 \le h_1 \le h(t) \le h_2, \qquad 0 \le d(t) \le d, \qquad 0 \le k_1(t) \le k_1, \qquad 0 \le k_2(t) \le k_2. \tag{6}$$

The initial condition function $\phi_i(t)$ denotes a continuous vector-valued initial function of $t \in [-\tau_{\max}, 0]$.

In this paper, we assume that $s(t)$ is an orbitally stable solution of the above system. Clearly, the stability of the synchronized states (3) of network (1) is determined by the dynamics of the isolate node, the coupling strength c_1, c_2, and c_3, the inner-coupling matrices G_1, G_2, and G_3, and the outer-coupling matrices A, B, and C.

The following lemmas are used in the proof of the main result.

Lemma 2.2 [42] *Let A, B be a family of diagonalizable matrices. Then A, B is a commuting family* (*under multiplication*) *if and only if it is a simultaneously diagonalizable family.*

Lemma 2.3 [19] *For any constant symmetric matrix $M \in R^{n\times n}$, $M = M^T > 0$, $0 \le h_1 \le h(t) \le h_2$, $t \ge 0$, and any differentiable vector function $x(t) \in R^n$, we have*

$$\text{(a)} \quad \left[\int_{t-h_1}^{t} \dot{x}(s)\,ds\right]^T M \left[\int_{t-h_1}^{t} \dot{x}(s)\,ds\right] \le h_1 \int_{t-h_1}^{t} \dot{x}^T(s) M \dot{x}(s)\,ds,$$

$$\begin{aligned} \text{(b)} \quad \left[\int_{t-h(t)}^{t-h_1} \dot{x}(s)\,ds\right]^T M \left[\int_{t-h(t)}^{t-h_1} \dot{x}(s)\,ds\right] &\le \bigl(h(t) - h_1\bigr) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) M \dot{x}(s)\,ds \\ &\le (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s) M \dot{x}(s)\,ds. \end{aligned}$$

Lemma 2.4 (Cauchy inequality [19]) *For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$ we have*

$$\pm 2x^T y \le x^T N x + y^T N^{-1} y.$$

3 Synchronization of delayed complex dynamical network via delayed feedback control and intermittent control

In this section, we shall obtain some delay-dependent exponential synchronization criteria for general complex dynamical network with discrete and distributed time-varying delays and hybrid coupling delays (1) by strict LMI approaches. Let us set

$$\tilde{A}_i = J(t) + c_1\lambda_{1i}G_1, \qquad \tilde{B}_i = J_h(t) + c_2\lambda_{2i}G_2, \qquad \tilde{C}_i = J_{k_1}(t) + c_3\lambda_{3i}G_3$$

and

1. $J(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^{t} s(\xi)\,d\xi) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ at $s(t)$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ respect to $x(t)$,
2. $J_h(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^{t} s(\xi)\,d\xi) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ at $s(t-h(t))$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ respect to $x(t-h(t))$,
3. $J_{k_1}(t) = f'(s(t), s(t-h(t)), \int_{t-k_1(t)}^{t} s(\xi)\,d\xi) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ at $\int_{t-k_1(t)}^{t} s(\xi)\,d\xi$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k_1(t)}^{t} x(s)\,ds)$ respect to $\int_{t-k_1(t)}^{t} x(s)\,ds$.

Lemma 3.1 *Consider the hybrid coupling delays dynamical network in* (1). *Let $0 = \lambda_{j1} > \lambda_{j2} \ge \lambda_{j3} \ge \cdots \ge \lambda_{jN}$, $j = \{1, 2, 3\}$, be the eigenvalues of the outer-coupling matrices A, B, and C, respectively. If the $N-1$ following n-dimensional linear time-varying delays differential equations are delay-dependent exponentially stable about their zero solutions*:

$$\begin{aligned} \dot{z}_i(t) &= (\tilde{A}_i + D_{4i}K_i)z_i(t) + \tilde{B}_i z_i\bigl(t-h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds \\ &\quad + D_{5i}K_i z_i\bigl(t-d(t)\bigr) + D_{6i}K_i \int_{t-k_2(t)}^{t} z_i(s)\,ds, \quad n\omega \le t \le n\omega + \delta, i = 2, \ldots, N, \qquad (7) \\ \dot{z}_i(t) &= \tilde{A}_i z_i(t) + \tilde{B}_i z_i\bigl(t-h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds, \quad n\omega + \delta < t \le (n+1)\omega, i = 2, \ldots, N, \end{aligned}$$

then the dynamical networks (5) *is exponentially stable, and then exponential synchronization of the controlled dynamical networks* (1) *is achieved.*

Proof To investigate the stability of the synchronized states (3), set

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \ldots, N. \tag{8}$$

Substituting (8) into (5), for $1 \le i \le N$, we have

$$\begin{aligned}
\dot{e}_i(t) ={}& f\left(x_i(t), x_i\bigl(t-h(t)\bigr), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) - f\left(s(t), s\bigl(t-h(t)\bigr), \int_{t-k_1(t)}^{t} s_i(\xi)\,d\xi\right) \\
&+ c_1 \sum_{j=1}^{N} a_{ij} G_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} G_2 e_j\bigl(t-h(t)\bigr) \\
&+ c_3 \sum_{j=1}^{N} c_{ij} G_3 \int_{t-k_1(t)}^{t} e_j(s)\,ds + D_{4i} K_i\bigl(e_i(t)\bigr) + D_{5i} K_i\bigl(e_i\bigl(t-d(t)\bigr)\bigr) \\
&+ D_{6i} K_i \int_{t-k_2(t)}^{t} e_j(s)\,ds, \quad n\omega \le t \le n\omega + \delta, \\
\dot{e}_i(t) ={}& f\left(x_i(t), x_i\bigl(t-h(t)\bigr), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) - f\left(s(t), s\bigl(t-h(t)\bigr), \int_{t-k_1(t)}^{t} s_i(\xi)\,d\xi\right) \\
&+ c_1 \sum_{j=1}^{N} a_{ij} G_1 e_j(t) + c_2 \sum_{j=1}^{N} b_{ij} G_2 e_j\bigl(t-h(t)\bigr) \\
&+ c_3 \sum_{j=1}^{N} c_{ij} G_3 \int_{t-k_1(t)}^{t} e_j(s)\,ds, \quad n\omega + \delta < t \le (n+1)\omega, i = 1, 2, \ldots, N.
\end{aligned} \tag{9}$$

Since $f(\cdot)$ is continuous differentiable, it is easy to know that the origin of the nonlinear system (9) is an asymptotically stable equilibrium point if it is an asymptotically stable equilibrium point of the following linear time-varying delays systems:

$$\begin{aligned}
\dot{e}_i(t) ={}& J(t) e_i(t) + J_h(t) e_i\bigl(t-h(t)\bigr) + J_{k_1}(t) \int_{t-k_1(t)}^{t} e_i(s)\,ds \\
&+ c_1 G_1\bigl(e_1(t), e_2(t), \ldots, e_N(t)\bigr)(a_{i1}, \ldots, a_{iN})^T \\
&+ c_2 G_2\bigl(e_1\bigl(t-h(t)\bigr), \ldots, e_N\bigl(t-h(t)\bigr)\bigr)(b_{i1}, \ldots, b_{iN})^T \\
&+ c_3 G_3 \int_{t-k_1(t)}^{t} \bigl(e_1(s), e_2(s), \ldots, e_N(s)\bigr)(c_{i1}, \ldots, c_{iN})^T\,ds \\
&+ D_{4i} K_i e_i(t) + D_{5i} K_i e_i\bigl(t-d(t)\bigr) + D_{6i} K_i \int_{t-k_2(t)}^{t} e_j(s)\,ds, \\
&n\omega \le t \le n\omega + \delta, \\
\dot{e}_i(t) ={}& J(t) e_i(t) + J_h(t) e_i\bigl(t-h(t)\bigr) + J_{k_1}(t) \int_{t-k_1(t)}^{t} e_i(s)\,ds \\
&+ c_1 G_1\bigl(e_1(t), e_2(t), \ldots, e_N(t)\bigr)(a_{i1}, \ldots, a_{iN})^T \\
&+ c_2 G_2\bigl(e_1\bigl(t-h(t)\bigr), \ldots, e_N\bigl(t-h(t)\bigr)\bigr)(b_{i1}, \ldots, b_{iN})^T
\end{aligned}$$

$$+ c_3G_3\int_{t-k_1(t)}^{t}\big(e_1(s),e_2(s),\ldots,e_N(s)\big)(c_{i1},\ldots,c_{iN})^T\,ds,$$
$$n\omega+\delta<t\le(n+1)\omega.$$

Letting $e(t)=(e_1(t),\ldots,e_N(t))\in R^{n\times N}$, $e(t-h(t))=(e_1(t-h(t)),\ldots,e_N(t-h(t)))\in R^{n\times N}$, $\int_{t-k_1(t)}^{t}e(s)\,ds=\int_{t-k_1(t)}^{t}(e_1(s),e_2(s),\ldots,e_N(s))\,ds\in R^{n\times N}$, $K=\operatorname{diag}\{K_1,K_2,\ldots,K_N\}$, and $D_j=\operatorname{diag}\{D_{j1},D_{j2},\ldots,D_{jN}\}$, $j=\{4,5,6\}$, we have

$$\begin{aligned}\dot e(t)={}&\big(J(t)+DK\big)e(t)+J_h(t)e\big(t-h(t)\big)+J_{k_1}(t)\int_{t-k_1(t)}^{t}e(s)\,ds+c_1G_1e(t)A^T\\&+c_2G_2e\big(t-h(t)\big)B^T+c_3G_3\int_{t-k_1(t)}^{t}e(s)C^T\,ds+D_5Ke\big(t-d(t)\big)\\&+D_6K\int_{t-k_2(t)}^{t}e(s)\,ds,\quad n\omega\le t\le n\omega+\delta,\\\dot e(t)={}&J(t)e(t)+J_h(t)e\big(t-h(t)\big)+J_{k_1}(t)\int_{t-k_1(t)}^{t}e(s)\,ds+c_1G_1e(t)A^T\\&+c_2G_2e\big(t-h(t)\big)B^T+c_3G_3\int_{t-k_1(t)}^{t}e(s)C^T\,ds,\quad n\omega+\delta<t\le(n+1)\omega.\end{aligned}\tag{10}$$

Obviously, A, B, C are diagonalizable. If A, B, and C commute pairwise, *i.e.*, $AB=BA$, then based on Lemma 2.2, one can get a common unitary matrix $\hat U\in R^{N\times N}$ with $\hat u_i\in\mathbb{R}^n$ such that

$$\hat U^TA\hat U=\Gamma_1,\qquad \hat U^TB\hat U=\Gamma_2,\qquad \hat U^TC\hat U=\Gamma_3,$$

where $\hat U^T\hat U=I$, $\Gamma_j=\operatorname{diag}\{\lambda_{1j},\ldots,\lambda_{Nj}\}$, $j=\{1,2,3\}$. In addition, with (2) and the irreducible feature of A, B, and C we can select with $\hat u_1=\frac{1}{\sqrt N}(1,1,\ldots,1)^T$ such that $\lambda_{1j}=0$, $j=\{1,2,3\}$.

Using the nonsingular transform $e(t)\hat U=z(t)=(z_1(t),\ldots,z_N(t))\in R^{N\times N}$, from (10), we have the following matrix equation:

$$\begin{aligned}\dot z(t)={}&\big(J(t)+DK\big)z(t)+J_h(t)z\big(t-h(t)\big)+J_{k_1}(t)\int_{t-k_1(t)}^{t}z(s)\,ds+c_1G_1z(t)\Gamma_1\\&+c_2G_2z\big(t-h(t)\big)\Gamma_2+c_3G_3\int_{t-k(t)}^{t}z(s)\Gamma_3\,ds+D_5Kz\big(t-d(t)\big)\\&+D_6K\int_{t-k_2(t)}^{t}z(s)\,ds,\quad n\omega\le t\le n\omega+\delta,\\\dot z(t)={}&J(t)z(t)+J_h(t)z\big(t-h(t)\big)+J_{k_1}(t)\int_{t-k_1(t)}^{t}z(s)\,ds+c_1G_1z(t)\Gamma_1\\&+c_2G_2z\big(t-h(t)\big)\Gamma_2+c_3G_3\int_{t-k_1(t)}^{t}z(s)\Gamma_3\,ds,\quad n\omega+\delta<t\le(n+1)\omega,\end{aligned}$$

that is,

$$\begin{aligned}\dot z_i(t)={}&(\tilde A_i+D_{4i}K_i)z_i(t)+\tilde B_iz_i\big(t-h(t)\big)+\tilde C_i\int_{t-k_1(t)}^{t}z_i(s)\,ds\\&+D_{5i}K_iz_i\big(t-d(t)\big)+D_{6i}K_i\int_{t-k_2(t)}^{t}z_i(s)\,ds,\quad n\omega\le t\le n\omega+\delta,\end{aligned}$$

$$\begin{aligned}
&\dot{z}_i(t) = \tilde{A}_{1i} z_i(t) + \tilde{B}_i z_i\bigl(t - h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds, \\
&\quad n\omega + \delta < t \leq (n+1)\omega, i = 1, \ldots, N.
\end{aligned}$$

Thus, we have transformed the stability problem of the dynamical networks (5) to the stability problem of the N pieces of n-dimensional linear time-varying delays differential equations. Note that $\lambda_{1k} = 0$ corresponding to the synchronization of the dynamical networks (5), where the state $s(t)$ is an orbitally stable solution of the isolate node as assumed above in (3). If the following $N - 1$ pieces of n-dimensional linear switched time-varying delays systems:

$$\begin{aligned}
\dot{z}_i(t) &= (\tilde{A}_i + D_{4i} K_i) z_i(t) + \tilde{B}_i z_i\bigl(t - h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds \\
&\quad + D_{5i} K_i z_i\bigl(t - d(t)\bigr) + D_{6i} K_i \int_{t-k_2(t)}^{t} z_i(s)\,ds, \quad n\omega \leq t \leq n\omega + \delta, \\
\dot{z}_i(t) &= \tilde{A}_i z_i(t) + \tilde{B}_i z_i\bigl(t - h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds, \\
&\quad n\omega + \delta < t \leq (n+1)\omega, i = 2, \ldots, N,
\end{aligned}$$

are exponentially stable, then $e(t)$ will tend to the origin exponentially, which is equivalent to the synchronization of the dynamical networks (5) being exponentially stable. This completes the proof. □

Lemma 3.2 *Consider the hybrid coupling delays dynamical network in* (1). *Let* $0 = \lambda_{j1} > \lambda_{j2} \geq \lambda_{j3} \geq \cdots \geq \lambda_{jN}$, $j = \{1, 2, 3\}$, *be the eigenvalues of the outer-coupling matrices* A, B, *and* C, *respectively. If the* $N - 1$ *following* n-*dimensional linear time-varying delays differential equations are delay-dependent exponentially stable about their zero solutions*:

$$\begin{aligned}
\dot{z}_i(t) &= (\tilde{A}_i + D_{1i} K_i) z_i(t) + \tilde{B}_i z_i\bigl(t - h(t)\bigr) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds \\
&\quad + D_{2i} K_i z_i\bigl(t - d(t)\bigr) + D_{3i} K_i \int_{t-k_2(t)}^{t} z_i(s)\,ds, \quad i = 2, \ldots, N,
\end{aligned} \tag{11}$$

then the dynamical networks (4) *is exponentially stable, then exponential synchronization of the controlled dynamical networks* (1) *is achieved.*

3.1 Linear delayed feedback control

Let us denote

$$\begin{aligned}
&\|\phi_i\| = \bigl\|z_i(0)\bigr\|, \qquad \|\varphi_i\| = \sup_{-\tau_{\max} \leq s \leq 0} \bigl\|z_i(s)\bigr\|, \qquad K_i = -L_i P_i^{-1}, \\
&\gamma_i = \lambda_{\min}\bigl(P_i^{-1}\bigr), \\
&\ell_i = \lambda_{\max}\bigl(P_i^{-1}\bigr) + \bigl[2h_2 \lambda_{\max}\bigl(P_i^{-1} R_i P_i^{-1}\bigr) + h_2 \lambda_{\max}\bigl(P_i^{-1} U_i P_i^{-1}\bigr)\bigr] \frac{1 - e^{-2\alpha h_2}}{2\alpha} \\
&\quad + d \lambda_{\max}\bigl(P_i^{-1} L_i^T T_i^{-1} L_i P_i^{-1}\bigr) \frac{1 - e^{-2\alpha d}}{2\alpha},
\end{aligned}$$

$$\begin{aligned}\xi_i &= \left[2\lambda_{\max}\left(P_i^{-1}Q_iP_i^{-1}\right) + h_2\lambda_{\max}\left(P_i^{-1}R_iP_i^{-1}\right) + h_2\lambda_{\max}\left(P_i^{-1}U_iP_i^{-1}\right)\right] \\ &\quad\times \frac{1-e^{-2\alpha h_2}}{2\alpha} + k_1\lambda_{\max}\left(P_i^{-1}S_iP_i^{-1}\right)\frac{1-e^{-2\alpha k_1}}{2\alpha} \\ &\quad + d\lambda_{\max}\left(P_i^{-1}L_i^TT_i^{-1}L_iP_i^{-1}\right)\frac{1-e^{-2\alpha h_2}}{2\alpha} \\ &\quad + k_2\lambda_{\max}\left(P_i^{-1}L_i^TW_i^{-1}L_iP_i^{-1}\right)\frac{1-e^{-2\alpha d}}{2\alpha}, \\ \mathcal{N}_i &= \ell_i\|\phi_i\|^2 + \xi_i\|\varphi_i\|^2, \\ \gamma &= \min\{\gamma_i, i = 2,3,\dots,N\}, \qquad \mathcal{N} = \max\{\mathcal{N}_i, i = 2,3,\dots,N\}.\end{aligned}$$

Theorem 3.3 *For some given scalars* $0 < \alpha$, *the dynamical networks* (11) *with time-varying delay satisfying* (6) *are exponentially stable if there exist symmetric positive definite matrices* $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$, *and a matrix* L_i *appropriately dimensioned such that the following symmetric linear matrix inequality holds*:

$$\Sigma_{i1} = \Sigma_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \tag{12}$$

$$\Sigma_{i2} = \Sigma_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \tag{13}$$

$$\Sigma_{i3} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1\tilde{C}_iP_i & k_2L_i^T & 2L_i^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 \\ * & * & -k_2W_i & 0 \\ * & * & * & -2e^{-2\alpha d}T_i \end{bmatrix} < 0, \tag{14}$$

$$\Sigma_{i4} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_iP_i & d^2L_i^T & 3D_{2i}^T & 2k_2D_{3i}^T \\ * & -2k_1e^{-2\alpha k_1}S_i & 0 & 0 & 0 \\ * & * & -d^2T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d}T_i & 0 \\ * & * & * & * & -2k_2e^{-2\alpha k_2}W_i \end{bmatrix} < 0, \tag{15}$$

$i = 2,\dots,N$, *where*

$$\Sigma_i = \begin{bmatrix} \Sigma_{i11} & \Sigma_{i12} & \Sigma_{i13} & \Sigma_{i14} & \Sigma_{i15} \\ * & \Sigma_{i22} & 0 & \Sigma_{i24} & 0 \\ * & * & \Sigma_{i33} & \Sigma_{i34} & 0 \\ * & * & * & \Sigma_{i44} & \Sigma_{i45} \\ * & * & * & * & \Sigma_{i55} \end{bmatrix},$$

$$\begin{aligned}\Sigma_{i11} &= P_i^T(\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^TP_i - D_{1i}L_i - L_i^TD_{1i}^T + 3e^{2\alpha d}D_{2i}^TT_iD_{2i} \\ &\quad + 2k_2e^{2\alpha k_2}D_{3i}^TW_iD_{3i} + 2Q_i + k_1S_i - 0.5e^{-2\alpha h_1}R_i - 0.5e^{-2\alpha h_2}R_i, \\ \Sigma_{i12} &= P_i\tilde{A}_i^T, \\ \Sigma_{i13} &= e^{-2\alpha h_1}R_i, \\ \Sigma_{i14} &= \tilde{B}_iP_i, \\ \Sigma_{i15} &= e^{-2\alpha h_2}R_i, \\ \Sigma_{i22} &= h_1^2R_i + h_2^2R_i + \eta^2U_i - 1.5P_i,\end{aligned}$$

$$\begin{aligned}
\Sigma_{i24} &= \tilde{B}_i P_i, \\
\Sigma_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\
\Sigma_{i34} &= e^{-2\alpha h_2} U_i, \\
\Sigma_{i44} &= -2e^{-2\alpha h_2} U_i, \\
\Sigma_{i45} &= e^{-2\alpha h_2} U_i, \\
\Sigma_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,
\end{aligned}$$

then the dynamical networks (11) *have exponential synchronization. Moreover, the feedback control is*

$$u_i(t) = -L_i P_i^{-1} z_i(t). \tag{16}$$

Proof Let $Y_i = P_i^{-1}$, $y_i(t) = Y_i z_i(t)$. Using the feedback control (16) we consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned}
V_i\bigl(z_i(t)\bigr) = {}& V_{i1}(t) + V_{i2}(t) + V_{i3}(t) + V_{i4}(t) + V_{i5}(t) + V_{i6}(t) + V_{i7}(t) \\
& + V_{i8}(t) + V_{i9}(t),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
V_{i1}(t) &= z_i^T(t) Y_i z_i(t), \\
V_{i2}(t) &= \int_{t-h_1}^{t} e^{2\alpha(s-t)} z_i^T(s) Y_i Q_i Y_i z_i(s)\, ds, \\
V_{i3}(t) &= \int_{t-h_2}^{t} e^{2\alpha(s-t)} z_i^T(s) Y_i Q_i Y_i z_i(s)\, ds, \\
V_{i4}(t) &= h_1 \int_{-h_1}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i R_i Y_i \dot{z}_i(\tau)\, d\tau\, ds, \\
V_{i5}(t) &= h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i R_i Y_i \dot{z}_i(\tau)\, d\tau\, ds, \\
V_{i6}(t) &= (h_2 - h_1) \int_{t-h_2}^{t-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) Y_i U_i Y_i \dot{z}_i(\tau)\, d\tau\, ds, \\
V_{i7}(t) &= \int_{-k_1}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} z_i^T(\tau) Y_i S_i Y z_i(\tau)\, d\tau\, ds, \\
V_{i8}(t) &= d \int_{-d}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{z}_i^T(\tau) K_i^T T_i^{-1} K_i \dot{z}_i(\tau)\, d\tau\, ds, \\
V_{i9}(t) &= \int_{-k_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} z_i^T(\tau) K_i^T W_i^{-1} K_i z_i(\tau)\, d\tau\, ds.
\end{aligned}$$

It easy to check that

$$\gamma \bigl\| z_i(t) \bigr\|^2 \le V_i\bigl(z_i(t)\bigr), \quad \forall t \ge 0. \tag{18}$$

By taking the derivative of $V_{i1}(t)$ along the trajectories of system (11), we have the following:

$$\begin{aligned}
\dot{V}_{i1}(t) &= 2z_i^T(t)Y_i\dot{z}_i(t) \\
&= 2y_i^T(t)\biggl[(\tilde{A}_i + D_{1i}K_i)z_i(t) + \tilde{B}_i z_i\bigl(t-h(t)\bigr) + \tilde{C}_i\int_{t-k_1(t)}^t z_i(s)\,ds \\
&\quad + D_{2i}K_i z_i\bigl(t-d(t)\bigr) + D_{3i}K_i\int_{t-k_2(t)}^t z_i(s)\,ds\biggr] \\
&= y_i^T(t)\bigl[P_i\tilde{A}_i + \tilde{A}_i^T P_i\bigr]y_i(t) + 2y_i^T(t)\tilde{B}_i P_i y_i\bigl(t-h(t)\bigr) \\
&\quad + 2y_i^T(t)\tilde{C}_i P_i\int_{t-k_1(t)}^t y_i(s)\,ds - 2y_i^T(t)D_iL_i^T y_i(t) + 2y_i^T(t)D_{2i}u_i\bigl(t-d(t)\bigr) \\
&\quad + 2y_i^T(t)D_{3i}\int_{t-k_2(t)}^t u_i(s)\,ds + 2y_i^T(t)\alpha P_i y_i(t) - 2y_i^T(t)\alpha P_i y_i(t).
\end{aligned}$$

Applying Lemma 2.4 and Lemma 2.3 gives

$$\begin{aligned}
2y_i^T(t)\tilde{C}_i P_i\int_{t-k_1(t)}^t y_i(s)\,ds &\le 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_i P_i S_i^{-1}P_i\tilde{C}_i^T y_i(t) \\
&\quad + \frac{e^{-2\alpha k_1}}{2k_1}\biggl(\int_{t-k_1(t)}^t y_i(s)\,ds\biggr)^T S_i\biggl(\int_{t-k_1(t)}^t y_i(s)\,ds\biggr) \\
&\le 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_i P_i S_i^{-1}P_i\tilde{C}_i^T y_i(t) \\
&\quad + \frac{1}{2}e^{-2\alpha k_1}\int_{t-k_1(t)}^t y_i^T(s)S_i y_i(s)\,ds, \\
2y_i^T(t)D_{2i}u_i\bigl(t-d(t)\bigr) &\le 3e^{2\alpha d}y_i^T(t)D_{2i}T_iD_{2i}^T y_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}u_i\bigl(t-d(t)\bigr), \\
2y_i^T(t)D_{3i}\int_{t-k_2(t)}^t u_i(s)\,ds &\le 2k_2e^{2\alpha k_2}y_i^T(t)D_{3i}W_iD_{3i}^T y_i(t) \\
&\quad + \frac{e^{2\alpha k_2}}{2k_2}\biggl(\int_{t-k_2(t)}^t u_i(s)\,ds\biggr)^T W_i^{-1}\biggl(\int_{t-k_2(t)}^t u_i(s)\,ds\biggr) \\
&\le 2k_2e^{2\alpha k_2}y_i^T(t)D_{3i}W_iD_{3i}^T y_i(t) \\
&\quad + \frac{e^{2\alpha k_2}}{2}\int_{t-k_2(t)}^t u_i^T(s)W_i^{-1}u_i(s)\,ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{V}_{i1}(t) + 2\alpha V_{i1}(t) &\le y_i^T(t)\bigl[P_i\tilde{A}_i + \tilde{A}_i^T P_i\bigr]y_i(t) + 2y_i^T(t)\alpha P_i y_i(t) \\
&\quad + 2y_i^T(t)\tilde{B}_i P_i y_i\bigl(t-h(t)\bigr) - 2y_i^T(t)D_iL_i^T y_i(t) \\
&\quad + 2k_1e^{2\alpha k_1}y_i^T(t)\tilde{C}_i P_i S_i^{-1}P_i\tilde{C}_i^T y_i(t) \\
&\quad + \frac{1}{2}e^{-2\alpha k_1}\int_{t-k_1(t)}^t y_i^T(s)S_i y_i(s)\,ds \\
&\quad + 3e^{2\alpha d}y_i^T(t)D_{2i}T_iD_{2i}^T y_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}u_i\bigl(t-d(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ 2k_2 e^{2\alpha k_2} y_i^T(t) D_{3i} W_i D_{3i}^T y_i(t) \\
&+ \frac{e^{2\alpha k_2}}{2} \int_{t-k_2(t)}^{t} u_i^T(s) W_i^{-1} u_i(s)\, ds.
\end{aligned} \tag{19}
$$

Next, by taking the derivative of $V_{ij}(t)$, $j = 2, 3, \ldots, 9$ along the trajectories of system (11), we have the following:

$$
\begin{aligned}
\dot{V}_{i2}(t) &\le y_i^T(t) Q_i y_i(t) - e^{-2\alpha h_1} y_i^T(t-h_1) Q_i y_i(t-h_1) - 2\alpha V_{i2}(t), \\
\dot{V}_{i3}(t) &\le y_i^T(t) Q_i y_i(t) - e^{-2\alpha h_2} y_i^T(t-h_2) Q_i y_i(t-h_2) - 2\alpha V_{i3}(t), \\
\dot{V}_{i4}(t) &\le h_1^2 \dot{y}_i^T(t) R_i \dot{y}_i(t) - h_1 e^{-2\alpha h_1} \int_{t-h_1}^{t} \dot{y}_i^T(s) R_i \dot{y}_i(s)\, ds - 2\alpha V_{i4}(t), \\
\dot{V}_{i5}(t) &\le h_2^2 \dot{y}_i^T(t) R_i \dot{y}_i(t) - h_2 e^{-2\alpha h_2} \int_{t-h_2}^{t} \dot{y}_i^T(s) R_i \dot{y}_i(s)\, ds - 2\alpha V_{i5}(t), \\
\dot{V}_{i6}(t) &\le \eta^2 \dot{y}_i^T(t) U_i \dot{y}_i(t) - \eta e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{y}_i^T(s) U_i \dot{y}_i(s)\, ds - 2\alpha V_{i6}(t), \\
\dot{V}_{i7}(t) &\le k_1 y_i^T(t) S_i y_i(t) - e^{-2\alpha k_1} \int_{t-k_1(t)}^{t} y_i^T(s) S_i y_i(s)\, ds - 2\alpha V_{i7}(t), \\
\dot{V}_{i8}(t) &\le d^2 \dot{z}_i^T(t) K_i^T T_i^{-1} K_i \dot{z}_i^T(t) - d e^{-2\alpha d} \int_{t-d}^{t} \dot{z}_i^T(s) K_i^T T_i^{-1} K_i \dot{z}_i^T(s)\, ds - 2\alpha V_{i8}(t) \\
&\le d^2 \dot{y}_i^T(t) P_i K_i^T T_i^{-1} K_i P_i \dot{y}_i^T(t) - d(t) e^{-2\alpha d} \int_{t-d(t)}^{t} \dot{u}_i^T(s) T_i^{-1} \dot{u}_i(s)\, ds - 2\alpha V_{i8}(t) \\
&= d^2 \dot{y}_i^T(t) L_i^T T_i^{-1} L_i \dot{y}_i^T(t) - d(t) e^{-2\alpha d} \int_{t-d(t)}^{t} \dot{u}_i^T(s) T_i^{-1} \dot{u}_i(s)\, ds - 2\alpha V_{i8}(t), \\
\dot{V}_{i9}(t) &\le k_2 z_i^T(t) K_i^T W_i^{-1} K_i z_i^T(t) - e^{-2\alpha k_2} \int_{t-k_2}^{t} z_i^T(s) K_i^T W_i^{-1} K_i z_i^T(s)\, ds - 2\alpha V_{i9}(t) \\
&\le k_2 y_i^T(t) P_i K_i^T W_i^{-1} K_i P_i y_i^T(t) - e^{-2\alpha k_2} \int_{t-k_2(t)}^{t} u_i^T(s) W_i^{-1} u_i^T(s)\, ds - 2\alpha V_{i9}(t) \\
&\le k_2 y_i^T(t) L_i^T W_i^{-1} L_i y_i^T(t) - e^{-2\alpha k_2} \int_{t-k_2(t)}^{t} u_i^T(s) W_i^{-1} u_i^T(s)\, ds - 2\alpha V_{i9}(t).
\end{aligned} \tag{20}
$$

Applying Lemma 2.3 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_1 \int_{t-h_1}^{t} \dot{y}_i^T(s) R_i \dot{y}_i(s)\, ds &\le -\left[\int_{t-h_1}^{t} \dot{y}_i(s)\, ds\right]^T R_i \left[\int_{t-h_1}^{t} \dot{y}_i(s)\, ds\right] \\
&\le -\big[y_i(t) - y_i(t-h_1)\big]^T R_i \big[y_i(t) - y_i(t-h_1)\big] \\
&= -y_i^T(t) R_i y_i(t) + 2y_i^T(t) R_i y_i(t-h_1) \\
&\quad - y_i^T(t-h_1) R_i y_i(t-h_1)
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
-h_2 \int_{t-h_2}^{t} \dot{y}_i^T(s) R_i \dot{y}_i(s)\, ds &\le -\left[\int_{t-h_2}^{t} \dot{y}_i(s)\, ds\right]^T R_i \left[\int_{t-h_2}^{t} \dot{y}_i(s)\, ds\right] \\
&\le -\big[y_i(t) - y_i(t-h_2)\big]^T R_i \big[y_i(t) - y_i(t-h_2)\big]
\end{aligned}
$$

$$\begin{aligned} &= -y_i^T(t)R_iy_i(t) + 2y_i^T(t)R_iy_i(t-h_2) \\ &\quad - y_i^T(t-h_2)R_iy_i(t-h_2). \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} -(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &= -(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\quad - (h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &= -\big(h_2-h(t)\big)\int_{t-h_2}^{t-h(t)}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\quad - \big(h(t)-h_1\big)\int_{t-h_2}^{t-h(t)}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\quad - \big(h(t)-h_1\big)\int_{t-h(t)}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\quad - \big(h_2-h(t)\big)\int_{t-h(t)}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds. \end{aligned}$$

Using Lemma 2.3 gives

$$\begin{aligned} -\big(h_2-h(t)\big)\int_{t-h_2}^{t-h(t)}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &\le -\left[\int_{t-h_2}^{t-h(t)}\dot{y}_i(s)\,ds\right]^T U_i\left[\int_{t-h_2}^{t-h(t)}\dot{y}_i(s)\,ds\right] \\ &\le -\big[y_i\big(t-h(t)\big)-y_i(t-h_2)\big]^T U_i \\ &\quad \times \big[y_i\big(t-h(t)\big)-y_i(t-h_2)\big] \\ &= -y_i^T\big(t-h(t)\big)U_iy_i\big(t-h(t)\big) \\ &\quad + 2y_i^T\big(t-h(t)\big)U_iy_i(t-h_2) \\ &\quad - y_i^T(t-h_2)U_iy_i(t-h_2) \end{aligned} \tag{23}$$

and

$$\begin{aligned} -\big(h(t)-h_1\big)\int_{t-h(t)}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &\le -\left[\int_{t-h(t)}^{t-h_1}\dot{y}_i(s)\,ds\right]^T U_i\left[\int_{t-h(t)}^{t-h_1}\dot{y}_i(s)\,ds\right] \\ &\le -\big[y_i(t-h_1)-y_i\big(t-h(t)\big)\big]^T U_i \\ &\quad \times \big[y_i(t-h_1)-y_i\big(t-h(t)\big)\big] \\ &= -y_i^T(t-h_1)U_iy_i(t-h_1) + 2y_i^T(t-h_1)U_iy_i\big(t-h(t)\big) \\ &\quad - y_i^T\big(t-h(t)\big)U_iy_i\big(t-h(t)\big). \end{aligned} \tag{24}$$

Let $\beta = \frac{h_2-h(t)}{h_2-h_1} \le 1$. Then

$$\begin{aligned} -\big(h_2-h(t)\big)\int_{t-h(t)}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &= -\beta\int_{t-h(t)}^{t-h_1}(h_2-h_1)\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\le -\beta\int_{t-h(t)}^{t-h_1}\big(h(t)-h_1\big)\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \end{aligned}$$

$$\begin{aligned}&\le -\beta\bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr]^T U_i \\ &\quad\times \bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr]\end{aligned} \tag{25}$$

and

$$\begin{aligned}-\bigl(h(t)-h_1\bigr)\int_{t-h_2}^{t-h(t)}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &= -(1-\beta)\int_{t-h_2}^{t-h(t)}(h_2-h_1)\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\le -(1-\beta)\int_{t-h_2}^{t-h(t)}\bigl(h_2-h(t)\bigr)\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds \\ &\le -(1-\beta)\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr]^T U_i \\ &\quad\times\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr].\end{aligned} \tag{26}$$

Therefore from (23)-(26), we obtain

$$\begin{aligned}-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{y}_i^T(s)U_i\dot{y}_i(s)\,ds &\le -\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr]^T U_i \\ &\quad\times\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr] \\ &\quad-\bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr]^T U_i \\ &\quad\times\bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr] \\ &\quad-\beta\bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr]^T U_i \\ &\quad\times\bigl[y_i(t-h_1)-y_i\bigl(t-h(t)\bigr)\bigr] \\ &\quad-(1-\beta)\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr]^T U_i \\ &\quad\times\bigl[y_i\bigl(t-h(t)\bigr)-y_i(t-h_2)\bigr].\end{aligned} \tag{27}$$

From $\dot{V}_{i8}(t)$, applying Lemma 2.3 and the Leibniz-Newton formula gives

$$\begin{aligned}-d(t)e^{-2\alpha d}\int_{t-d(t)}^{t}\dot{u}_i^T(s)T_i^{-1}\dot{u}_i(s)\,ds &\le -e^{-2\alpha d}\left(\int_{t-d(t)}^{t}\dot{u}_i(s)\,ds\right)^T T_i^{-1}\left(\int_{t-d(t)}^{t}\dot{u}_i(s)\,ds\right) \\ &\le -e^{-2\alpha d}u_i^T(t)T_i^{-1}u_i(t) \\ &\quad+2e^{-2\alpha d}u_i^T(t)_iT_i^{-1}u_i\bigl(t-d(t)\bigr) \\ &\quad-e^{-2\alpha d}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}u_i\bigl(t-d(t)\bigr) \\ &\le -e^{-2\alpha d}u_i^T(t)T_i^{-1}u_i(t)+3e^{-2\alpha d}u_i^T(t)T_i^{-1}u_i(t) \\ &\quad+\frac{e^{-2\alpha d}}{3}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}T_iT_i^{-1}u_i\bigl(t-d(t)\bigr) \\ &\quad-e^{-2\alpha d}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}u_i\bigl(t-d(t)\bigr) \\ &= 2e^{-2\alpha d}z_i^T(t)K_i^TT_i^{-1}K_iz_i(t) \\ &\quad+\frac{e^{-2\alpha d}}{3}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}T_iT_i^{-1}u_i\bigl(t-d(t)\bigr) \\ &\quad-e^{-2\alpha d}u_i^T\bigl(t-d(t)\bigr)T_i^{-1}u_i\bigl(t-d(t)\bigr)\end{aligned}$$

$$\begin{aligned}
&= 2e^{-2\alpha d} y_i^T(t) L_i^T T_i^{-1} L_i y_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3} u_i^T\big(t-d(t)\big) T_i^{-1} u_i\big(t-d(t)\big) \\
&\quad - e^{-2\alpha d} u_i^T\big(t-d(t)\big) T_i^{-1} u_i\big(t-d(t)\big).
\end{aligned} \tag{28}$$

By using the following identity relation:

$$\begin{aligned}
&-\dot{z}_i(t) + (\tilde{A}_i + D_{1i}K_i) z_i(t) + \tilde{B}_i z_i\big(t-h(t)\big) + \tilde{C}_i \int_{t-k_1(t)}^{t} z_i(s)\,ds \\
&\quad + D_{2i}K_i z_i\big(t-d(t)\big) + D_{3i}K_i \int_{t-k_2(t)}^{t} z_i(s)\,ds = 0,
\end{aligned}$$

we have

$$\begin{aligned}
&-2\dot{y}_i^T(t) P_i \dot{y}_i(t) + 2\dot{y}_i^T(t) \tilde{A}_i P_i y_i(t) - 2\dot{y}_i^T(t) D_{1i} L_i y_i(t) + 2\dot{y}_i^T(t) \tilde{B}_i P_i y_i\big(t-h(t)\big) \\
&\quad + 2\dot{y}_i^T(t) \tilde{C}_i P_i \int_{t-k_1(t)}^{t} y_i(s)\,ds + 2\dot{y}_i^T(t) D_{2i} u_i\big(t-d(t)\big) \\
&\quad + 2\dot{y}_i^T(t) D_{3i} \int_{t-k_2(t)}^{t} u_i(s)\,ds = 0.
\end{aligned} \tag{29}$$

Applying Lemma 2.4 and Lemma 2.3 gives

$$\begin{aligned}
2\dot{y}_i^T(t) \tilde{C}_i P_i \int_{t-k_1(t)}^{t} y_i(s)\,ds &\le 2k_1 e^{2\alpha k_1} \dot{y}_i^T(t) \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \dot{y}_i(t) \\
&\quad + \frac{1}{2k_1} e^{-2\alpha k_1} \left(\int_{t-k_1(t)}^{t} y_i(s)\,ds \right)^T S_i \\
&\quad \times \left(\int_{t-k_1(t)}^{t} y_i(s)\,ds \right) \\
&\le 2k_1 e^{2\alpha k_1} \dot{y}_i^T(t) \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \dot{y}_i(t) \\
&\quad + \frac{1}{2} e^{-2\alpha k_1} \int_{t-k_1(t)}^{t} y_i^T(s) S_i y_i(s)\,ds,
\end{aligned} \tag{30}$$

$$\begin{aligned}
2\dot{y}_i^T(t) D_{2i} u_i\big(t-d(t)\big) &\le 3e^{2\alpha d} \dot{y}_i^T(t) D_{2i}^T T_i^{-1} D_{2i} \dot{y}_i(t) \\
&\quad + \frac{e^{-2\alpha d}}{3} u_i^T\big(t-d(t)\big) T_i^{-1} u_i\big(t-d(t)\big),
\end{aligned} \tag{31}$$

$$\begin{aligned}
3\dot{y}_i^T(t) D_{3i} \int_{t-k_2(t)}^{t} u_i(s)\,ds &\le 2k_2 e^{2\alpha k_2} y_i^T(t) D_{3i}^T W_i^{-1} D_{3i} y_i(t) \\
&\quad + \frac{1}{2k_2} e^{-2\alpha k_2} \left(\int_{t-k_2(t)}^{t} u_i(s)\,ds \right)^T W_i \\
&\quad \times \left(\int_{t-k_2(t)}^{t} u_i(s)\,ds \right) \\
&\le 2k_2 e^{2\alpha k_2} y_i^T(t) D_{3i}^T W_i^{-1} D_{3i} y_i(t) \\
&\quad + \frac{e^{2\alpha k_2}}{2} \int_{t-k_2(t)}^{t} u_i^T(s) W_i^{-1} u_i(s)\,ds.
\end{aligned} \tag{32}$$

Hence, according to (19)-(28), (30)-(32), and adding the zero items of (29) we have

$$\begin{aligned}\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) &\leq \xi_i^T(t)\big[(1-\beta)\Sigma_{1i} + \beta\Sigma_{2i}\big]\xi_i(t) + y_i^T(t)\mathcal{M}_{3i}y_i(t) \\ &\quad + \dot{y}_i^T(t)\mathcal{M}_{4i}\dot{y}_i(t),\end{aligned} \tag{33}$$

where Σ_{1i} and Σ_{2i} are defined as in (12) and (13), respectively, and

$$\begin{aligned}\xi_i^T(t) &= \begin{bmatrix} y_i^T(t) & \dot{y}_i^T(t) & y_i^T(t-h_1) & y_i^T(t-h(t)) & y_i^T(t-h_2)\end{bmatrix}, \\ \mathcal{M}_{3i} &= -0.5\big(e^{-2\alpha h_1} + e^{-2\alpha h_2}\big)R_i + 2k_1e^{2\alpha k_1}\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^T + k_2L_i^TW_i^{-1}L_i \\ &\quad + 2e^{-2\alpha d}L_i^TT_i^{-1}L_i, \\ \mathcal{M}_{4i} &= -0.5P_i + 2k_1e^{2\alpha k_1}\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^T + d^2L_i^TT_i^{-1}L_i + 3e^{2\alpha d}D_{2i}^TT_i^{-1}D_{2i} \\ &\quad + 2k_2e^{2\alpha k_2}D_{3i}^TW_i^{-1}D_{3i}.\end{aligned}$$

By $(1-\beta)\Sigma_{1i} + \beta\Sigma_{2i} < 0$ holds if and only if $\Sigma_{1i} < 0$ and $\Sigma_{2i} < 0$. Applying the Schur complement lemma, the inequalities $\mathcal{M}_{3i} < 0$ and $\mathcal{M}_{4i} < 0$ are equivalent to $\Sigma_{3i} < 0$ and $\Sigma_{4i} < 0$, respectively. Therefore, it follows from (12)-(15), and (33), we obtain

$$\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) \leq 0, \quad \forall t \geq 0. \tag{34}$$

Integrating both sides of (34) from 0 to t, we have

$$V_i(z_i(t)) \leq V_i(z_i(0))e^{-2\alpha t}, \quad \forall t \geq 0.$$

On the other hand, using the condition (18), we have

$$\|z_i(t)\| \leq \sqrt{\frac{V_i(z_i(0))}{\gamma}}e^{-\alpha t}, \quad \forall t \geq 0.$$

Estimating $V_i(z_i(0))$ gives

$$\begin{aligned}V_{i1}(z_i(0)) &= z_i^T(0)P_i^{-1}z_i(0) \leq \lambda_{\max}\big(P_i^{-1}\big)\|\phi_i\|^2, \\ V_{i2}(z_i(0)) &= \int_{-h_1}^0 e^{2\alpha s}z_i^T(s)Y_iQ_iY_iz_i(s)\,ds \leq \lambda_{\max}\big(P_i^{-1}Q_iP_i^{-1}\big)\int_{-h_1}^0 e^{2\alpha s}\,ds\|\varphi_i\|^2 \\ &= \lambda_{\max}\big(P_i^{-1}Q_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_1}}{2\alpha}\|\varphi_i\|^2 \leq \lambda_{\max}\big(P_i^{-1}Q_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\varphi_i\|^2, \\ V_{i3}(z_i(0)) &\leq \lambda_{\max}\big(P_i^{-1}Q_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\varphi_i\|^2, \\ V_{i4}(z_i(0)) &= h_1\int_{-h_1}^0\int_s^0 e^{2\alpha\tau}\dot{z}_i^T(\tau)Y_iR_iY_i\dot{z}_i(\tau)\,d\tau\,ds \\ &= h_1\int_{-h_1}^0 e^{2\alpha s}\big[z_i^T(0)Y_iR_iY_iz_i(0) - z_i^T(s)Y_iR_iY_iz_i(s)\big]\,ds \\ &\leq h_2\lambda_{\max}(Y_iR_iY_i)\int_{-h_1}^0 e^{2\alpha s}\,ds\|\phi_i\|^2 - h_2\lambda_{\max}(Y_iR_iY_i)\int_{-h_1}^0 e^{2\alpha s}\,ds\|\varphi_i\|^2\end{aligned}$$

$$\begin{aligned}
&= h_2\lambda_{\max}(Y_iR_iY_i)\frac{1-e^{-2\alpha h_1}}{2\alpha}\|\phi_i\|^2 - h_2\lambda_{\max}(Y_iR_iY_i)\frac{1-e^{-2\alpha h_1}}{2\alpha}\|\varphi_i\|^2 \\
&\le h_2\lambda_{\max}\big(P_i^{-1}R_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi_i\|^2 + h_2\lambda_{\max}\big(P_i^{-1}R_iP_i^{-1}\big) \\
&\quad\times\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\varphi_i\|^2, \\
V_{i5}\big(z_i(0)\big) &\le h_2\lambda_{\max}\big(P_i^{-1}R_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi_i\|^2 + h_2\lambda_{\max}\big(P_i^{-1}R_iP_i^{-1}\big) \\
&\quad\times\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\varphi_i\|^2, \\
V_{i6}\big(z_i(0)\big) &\le h_2\lambda_{\max}\big(P_i^{-1}U_iP_i^{-1}\big)\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\phi_i\|^2 + h_2\lambda_{\max}\big(P_i^{-1}U_iP_i^{-1}\big) \\
&\quad\times\frac{1-e^{-2\alpha h_2}}{2\alpha}\|\varphi_i\|^2, \\
V_{i7}\big(z_i(0)\big) &= \int_{-k_1}^{0}\int_{s}^{0} e^{2\alpha\tau}z_i^T(\tau)Y_iS_iY_iz_i(\tau)\,d\tau\,ds \\
&\le \int_{-k_1}^{0}\int_{-k_1}^{0} e^{2\alpha s}z_i^T(\tau)Y_iS_iY_iz_i(\tau)\,d\tau\,ds \\
&\le \lambda_{\max}(Y_iS_iY_i)\int_{-k_1}^{0}\int_{-k_1}^{0} e^{2\alpha\tau}\,d\tau\,ds\|\varphi_i\|^2 \\
&= k_1\lambda_{\max}\big(P_i^{-1}S_iP_i^{-1}\big)\frac{1-e^{-2\alpha k_1}}{2\alpha}\|\varphi_i\|^2, \\
V_{i8}\big(z_i(0)\big) &\le d\lambda_{\max}\big(P_i^{-1}L_i^TT_i^{-1}L_iP_i^{-1}\big)\frac{1-e^{-2\alpha d}}{2\alpha}\|\phi_i\|^2 \\
&\quad + d\lambda_{\max}\big(P_i^{-1}L_i^TT_i^{-1}L_iP_i^{-1}\big)\frac{1-e^{-2\alpha d}}{2\alpha}\|\varphi_i\|^2, \\
V_{i9}\big(z_i(0)\big) &\le k_2\lambda_{\max}\big(P_i^{-1}L_i^TW_i^{-1}L_iP_i^{-1}\big)\frac{1-e^{-2\alpha k_2}}{2\alpha}\|\varphi_i\|^2,
\end{aligned}$$

we have

$$\big\|z_i(t)\big\| \le \sqrt{\frac{\mathcal{N}}{\gamma}}e^{-\alpha t}, \quad \forall t \ge 0,$$

which implies the dynamical networks (11) is globally exponentially stable under the controller H1, then exponential synchronization of the controlled dynamical networks (4) is achieved. The proof is thus completed. □

3.2 Intermittent delayed feedback control

Theorem 3.4 *For some given scalars $0 < \alpha < \varepsilon$, the dynamical networks* (7) *with time-varying delay satisfying* (6) *are exponentially stable if there exist symmetric positive definite matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $S_i > 0$, $U_i > 0$, $T_i > 0$, $W_i > 0$, and a matrix L_i with appropriately dimensioned such that the following symmetric linear matrix inequality holds*:

$$\Pi_{i1} = \Pi_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2}U_i[0 \quad 0 \quad I \quad -I \quad 0] < 0, \tag{35}$$

$$\Pi_{i2} = \Pi_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2}U_i[0 \quad 0 \quad 0 \quad I \quad -I] < 0, \tag{36}$$

$$\Pi_{i3} = \tilde{\Pi}_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \tag{37}$$

$$\Pi_{i4} = \tilde{\Pi}_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \tag{38}$$

$$\Pi_{i5} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i & 2k_1\tilde{C}_i P_i & k_2 L_i^T & 2L_i^T \\ * & -2k_1 e^{-2\alpha k_1} S_i & 0 & 0 \\ * & * & -k_2 W_i & 0 \\ * & * & * & -2e^{-2\alpha d} T_i \end{bmatrix} < 0, \tag{39}$$

$$\Pi_{i6} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_i P_i & d^2 L_i^T & 3D_{5i}^T & 2k_2 D_{6i}^T \\ * & -2k_1 e^{-2\alpha k_1} S_i & 0 & 0 & 0 \\ * & * & -d^2 T_i & 0 & 0 \\ * & * & * & -3e^{-2\alpha d} T_i & 0 \\ * & * & * & * & -2k_2 e^{-2\alpha k_2} W_i \end{bmatrix} < 0, \tag{40}$$

$$\Pi_{i7} = \begin{bmatrix} -0.5(e^{-2\alpha h_1} + e^{-2\alpha h_2})R_i - 2\varepsilon P_i & 2k_1\tilde{C}_i P_i \\ * & -2k_1 e^{-2\alpha k_1} S_i \end{bmatrix} < 0, \tag{41}$$

$$\Pi_{i8} = \begin{bmatrix} -0.5P_i & 2k_1\tilde{C}_i P_i \\ * & -2k_1 e^{-2\alpha k_1} S_i \end{bmatrix} < 0 \tag{42}$$

and

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \tag{43}$$

$i = 2, \ldots, N$, *where*

$$\Pi_i = \begin{bmatrix} \Pi_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix},$$

$$\tilde{\Pi}_i = \begin{bmatrix} \tilde{\Pi}_{i11} & \Pi_{i12} & \Pi_{i13} & \Pi_{i14} & \Pi_{i15} \\ * & \Pi_{i22} & 0 & \Pi_{i24} & 0 \\ * & * & \Pi_{i33} & \Pi_{i34} & 0 \\ * & * & * & \Pi_{i44} & \Pi_{i45} \\ * & * & * & * & \Pi_{i55} \end{bmatrix},$$

$$\begin{aligned} \Pi_{i11} &= P_i^T(\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i - D_{4i}L_i - L_i^T D_{4i}^T + 3e^{2\alpha d} D_{5i}^T T_i D_{5i} \\ &\quad + 2k_2 e^{2\alpha k_2} D_{6i}^T W_i D_{6i} + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \\ \tilde{\Pi}_{i11} &= P_i^T(\tilde{A}_i + \alpha I) + (\tilde{A}_i + \alpha I)^T P_i + 2Q_i + k_1 S_i - 0.5e^{-2\alpha h_1} R_i - 0.5e^{-2\alpha h_2} R_i, \\ \Pi_{i12} &= P_i \tilde{A}_i^T, \\ \Pi_{i13} &= e^{-2\alpha h_1} R_i, \\ \Pi_{i14} &= \tilde{B}_i P_i, \\ \Pi_{i15} &= e^{-2\alpha h_2} R_i, \\ \Pi_{i22} &= h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 1.5P_i, \end{aligned}$$

$$\Pi_{i24} = \tilde{B}_i P_i,$$
$$\Pi_{i33} = -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i,$$
$$\Pi_{i34} = e^{-2\alpha h_2} U_i,$$
$$\Pi_{i44} = -2e^{-2\alpha h_2} U_i,$$
$$\Pi_{i45} = e^{-2\alpha h_2} U_i,$$
$$\Pi_{i55} = -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i,$$

then the dynamical networks (7) *have exponential synchronization. Moreover, the feedback control is*

$$u_i(t) = \begin{cases} -L_i P_i^{-1} z_i(t), & n\omega \le t \le n\omega + \delta, \\ 0, & n\omega + \delta < t \le (n+1)\omega. \end{cases} \tag{44}$$

Proof Case I: for $n\omega \le t \le n\omega + \delta$, we choose the Lyapunov-Krasovskii functional as in (17) and using the feedback control (44), we may proof this case by using a similar argument as in the proof of Theorem 3.3. By replacing D_{1i}, D_{2i} and D_{3i} in (12)-(15) with D_{4i}, D_{5i}, and D_{6i}, respectively. We have

$$\begin{aligned} \dot{V}_i\big(z_i(t)\big) + 2\alpha V_i\big(z_i(t)\big) &\le \xi_i^T(t)\big[(1-\beta)\Pi_{1i} + \beta\Pi_{2i}\big]\xi_i(t) + y_i^T(t)\mathcal{N}_{3i} y_i(t) \\ &\quad + \dot{y}_i^T(t)\mathcal{N}_{4i}\dot{y}_i(t), \end{aligned} \tag{45}$$

where Π_{1i} and Π_{2i} are defined as in (35) and (36), respectively, and

$$\xi_i^T(t) = \begin{bmatrix} y_i^T(t) & \dot{y}_i^T(t) & y_i^T(t-h_1) & y_i^T\big(t-h(t)\big) & y_i^T(t-h_2) \end{bmatrix},$$
$$\begin{aligned} \mathcal{N}_{3i} &= -0.5\big(e^{-2\alpha h_1} + e^{-2\alpha h_2}\big) R_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T \\ &\quad + k_2 L_i^T W_i^{-1} L_i + 2e^{-2\alpha d} L_i^T T_i^{-1} L_i, \\ \mathcal{N}_{4i} &= -0.5 P_i + 2k_1 e^{2\alpha k_1} \tilde{C}_i P_i S_i^{-1} P_i \tilde{C}_i^T + d^2 L_i^T T_i^{-1} L_i + 3e^{2\alpha d} D_{5i}^T T_i^{-1} D_{5i} \\ &\quad + 2k_2 e^{2\alpha k_2} D_{6i}^T W_i^{-1} D_{6i}. \end{aligned}$$

By $(1-\beta)\Pi_{1i} + \beta\Pi_{2i} < 0$ holds if and only if $\Pi_{1i} < 0$ and $\Pi_{2i} < 0$. Applying the Schur complement lemma, the inequalities $\mathcal{N}_{5i} < 0$ and $\mathcal{N}_{6i} < 0$ are equivalent to $\Pi_{5i} < 0$ and $\Pi_{6i} < 0$, respectively. Therefore, it follows from (35)-(36), (39)-(40), and (45), we obtain

$$\dot{V}_i\big(z_i(t)\big) + 2\alpha V_i\big(z_i(t)\big) \le 0 \quad \text{for } n\omega \le t \le n\omega + \delta. \tag{46}$$

Thus, by the above differential inequality (46), we have

$$V_i\big(z_i(t)\big) \le V_i\big(z_i(n\omega)\big) e^{-2\alpha(t-n\omega)} \quad \text{for } n\omega \le t \le n\omega + \delta. \tag{47}$$

Case II: for $n\omega + \delta \le t \le (n+1)\omega$, we choose the Lyapunov-Krasovskii functional having the following form:

$$V_i\big(z_i(t)\big) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t) + V_{i4}(t) + V_{i5}(t) + V_{i6}(t) + V_{i7}(t),$$

where $V_{ij}(t)$, $j = 1, 2, \ldots, 7$ are defined similar in (17). We are able to do a similar estimation as we did for Theorem 3.3, and we have the following:

$$\begin{aligned}
\dot{V}_i(z_i(t)) + 2\alpha V_i(z_i(t)) &\leq \xi_i^T(t)\big[(1-\beta)\Pi_{3i} + \beta\Pi_{4i}\big]\xi_i(t) + y_i^T(t)\mathcal{N}_{7i}y_i(t) + \dot{y}_i^T(t)\mathcal{N}_{8i}\dot{y}_i(t) \\
&\leq \xi_i^T(t)\big[(1-\beta)\Pi_{3i} + \beta\Pi_{4i}\big]\xi_i(t) + y_i^T(t)\mathcal{N}_{7i}y_i(t) \\
&\quad + \dot{y}_i^T(t)\mathcal{N}_{8i}\dot{y}_i(t) + 2\varepsilon V_i(z_i(t)) - 2\varepsilon V_{i1}(t) \\
&= \xi_i^T(t)\big[(1-\beta)\Pi_{3i} + \beta\Pi_{4i}\big]\xi_i(t) + y_i^T(t)\mathcal{N}_{7i}y_i(t) \qquad (48) \\
&\quad + \dot{y}_i^T(t)\mathcal{N}_{8i}\dot{y}_i(t) + 2\varepsilon V_i(z_i(t)) - 2\varepsilon y_i^T(t)P_iy_i(t), \\
\dot{V}_i(z_i(t)) - 2(\varepsilon-\alpha)V_i(z_i(t)) &\leq \xi_i^T(t)\big[(1-\beta)\Pi_{3i} + \beta\Pi_{4i}\big]\xi_i(t) \\
&\quad + y_i^T(t)(\mathcal{N}_{7i} - 2\varepsilon P_i)y_i(t) + \dot{y}_i^T(t)\mathcal{N}_{8i}\dot{y}_i(t),
\end{aligned}$$

where Π_{1i} and Π_{2i} are defined as in (37) and (38), respectively, and

$$\begin{aligned}
\xi_i^T(t) &= \begin{bmatrix} y_i^T(t) & \dot{y}_i^T(t) & y_i^T(t-h_1) & y_i^T(t-h(t)) & y_i^T(t-h_2) \end{bmatrix}, \\
\mathcal{N}_{7i} &= -0.1\big(e^{-2\alpha h_1} + e^{-2\alpha h_2}\big)R_i + 2k_1e^{2\alpha k_1}\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^T, \\
\mathcal{N}_{8i} &= -0.5P_i + 2k_1e^{2\alpha k_1}\tilde{C}_iP_iS_i^{-1}P_i\tilde{C}_i^T.
\end{aligned}$$

Now $(1-\beta)\Pi_{3i} + \beta\Pi_{4i} < 0$ holds if and only if $\Pi_{3i} < 0$ and $\Pi_{4i} < 0$. Applying the Schur complement lemma, the inequalities $(\mathcal{N}_{7i} - 2\varepsilon P_i) < 0$ and $\mathcal{N}_{8i} < 0$ are equivalent to $\Pi_{7i} < 0$ and $\Pi_{8i} < 0$, respectively. Therefore, it follows from (37)-(38), (41)-(42), and (48), that we obtain

$$\dot{V}_i(z_i(t)) - 2(\varepsilon-\alpha)V_i(z_i(t)) \leq 0 \quad \text{for } n\omega + \delta < t \leq (n+1)\omega. \qquad (49)$$

From the above differential inequality (49), we have

$$V_i(z_i(t)) \leq V_i(z_i(n\omega+\delta))e^{2(\varepsilon-\alpha)(t-n\omega-\delta)} \quad \text{for } n\omega + \delta < t \leq (n+1)\omega. \qquad (50)$$

By (47) and (50), we have

$$\begin{aligned}
V_i(z_i((n+1)\omega)) &\leq V_i(z_i(n\omega+\delta))e^{2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V_i(z_i(n\omega))e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(\omega-\delta)} \\
&= V_i(z_i(n\omega))e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V_i(z_i((n-1)\omega+\delta))e^{2\rho(\omega-\delta)}e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&\leq V_i(z_i((n-1)\omega))e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)}e^{-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta)} \\
&= V_i(z_i((n-1)\omega))e^{2(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))} \\
&\ \ \vdots \\
&\leq V_i(z_i(0))e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n+1)}.
\end{aligned}$$

For any $t > 0$, there is a $n_0 \geq 0$, such that $n_0\omega \leq t \leq (n_0+1)\omega$.

Case 1. For $n_0\omega + \delta \le t \le (n_0 + 1)\omega$, using condition (43), we have

$$\begin{aligned}
V_i\big(z_i(t)\big) &\le V_i\big(z_i(n_0\omega + \delta)\big)e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\le V_i\big(z_i(n_0\omega)\big)e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\le V_i\big(z_i(0)\big)e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)(t-(n_0\omega+\delta))} \\
&\le V_i\big(z_i(0)\big)e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon-\alpha)((n_0+1)\omega-(n_0\omega+\delta))} \\
&= V_i\big(z_i(0)\big)e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\
&= V_i\big(z_i(0)\big)e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\
&\le V_i\big(z_i(0)\big)e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}.
\end{aligned} \tag{51}$$

Case 2. For $n_0\omega \le t \le n_0\omega + \delta$, using condition (43), we have

$$\begin{aligned}
V_i\big(z_i(t)\big) &\le V_i\big(z_i(n_0\omega)\big)e^{-2\alpha(t-n_0\omega)} \\
&\le V_i\big(z_i(0)\big)e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0}e^{-2\alpha(t-n_0\omega)} \\
&\le V_i\big(z_i(0)\big)e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))n_0} \\
&= V_i\big(z_i(0)\big)e^{-(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)} \\
&= V_i\big(z_i(0)\big)e^{-(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\
&\le V_i\big(z_i(0)\big)e^{-(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}.
\end{aligned} \tag{52}$$

Let $\xi = e^{-(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}$. By (51) and (52), we have

$$V_i\big(z_i(t)\big) \le \xi V_i\big(z_i(0)\big)e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \ge 0.$$

On the other hand, using the condition (18), we have obtained the following:

$$\big\|z_i(t)\big\| \le \sqrt{\frac{\mathcal{N}\xi}{\gamma}}e^{\frac{(-\alpha\delta+(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \ge 0.$$

which implies the dynamical networks (7) is exponentially stable under the controller H2, then exponential synchronization of the controlled dynamical networks (5) is achieved. The proof is thus completed. □

Remark 3.5 It is clear that as $\delta \to \omega$ the intermittent feedback control will reduce to a continuous feedback. In this case, presented in Theorem 3.3.

Remark 3.6 In [14–16], the authors investigated synchronization of complex dynamical network with coupling time-delay, but the time-delay considered in these three works are assumed to be constants delay. In [9], Li *et al.* presented synchronization in complex dynamical networks with time-varying delays in the network couplings and time-varying delays in the dynamical nodes, but the time-varying delays are required to be differentiable, which is a very strict condition. Obviously, we do not need these limit condition in this paper.

Remark 3.7 If $k_1(t) = 0$, $c_1 = 0$, $c_3 = 0$, and $\mathcal{U}_i(t) = 0$, then system (1) reduces to the following system presented in [9, 18]:

$$\dot{x}_i(t) = f\big(x_i(t)\big) + c_2 \sum_{j=1}^{N} b_{ij} G_2 x_j\big(t - h(t)\big), \quad t > 0, i = 1, 2, \ldots, N. \tag{53}$$

According to Theorem 3.3, we obtain the following corollary for the synchronization of network (53).

Corollary 3.8 *For some given scalars* $0 < \alpha$, *the dynamical networks* (53) *with time-varying delay* $h(t)$ *satisfying* (6) *are exponentially synchronization if there exist symmetric positive definite matrices* $P_i > 0$, $Q_i > 0$, $R_i > 0$, $U_i > 0$, *such that the following symmetric linear matrix inequality holds*:

$$\Gamma_{i1} = \Gamma_i - [0 \quad 0 \quad I \quad -I \quad 0]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad I \quad -I \quad 0] < 0, \tag{54}$$

$$\Gamma_{i2} = \Gamma_i - [0 \quad 0 \quad 0 \quad I \quad -I]^T e^{-2\alpha h_2} U_i [0 \quad 0 \quad 0 \quad I \quad -I] < 0, \tag{55}$$

where

$$\begin{aligned}
\Gamma_{i11} &= P_i^T\big(J(t) + \alpha I\big) + \big(J(t) + \alpha I\big)^T P_i + 2Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} R_i, \\
\Gamma_{i12} &= P_i J^T(t), \\
\Gamma_{i13} &= e^{-2\alpha h_1} R_i, \\
\Gamma_{i14} &= c_2 \lambda_{2i} G_2 P_i, \\
\Gamma_{i15} &= e^{-2\alpha h_2} R_i, \\
\Gamma_{i22} &= h_1^2 R_i + h_2^2 R_i + \eta^2 U_i - 2P_i, \\
\Gamma_{i24} &= c_2 \lambda_{2i} G_2 P_i, \\
\Gamma_{i33} &= -e^{-2\alpha h_1} Q_i - e^{-2\alpha h_1} R_i - e^{-2\alpha h_2} U_i, \\
\Gamma_{i34} &= e^{-2\alpha h_2} U_i, \\
\Gamma_{i44} &= -2e^{-2\alpha h_2} U_i, \\
\Gamma_{i45} &= e^{-2\alpha h_2} U_i, \\
\Gamma_{i55} &= -2e^{-2\alpha h_2} U_i - 2e^{-2\alpha h_2} Q_i - 2e^{-2\alpha h_2} R_i.
\end{aligned}$$

Proof Similar to proof of Theorem 3.3. Indeed, by setting $S_i = 0$, $T_i = 0$, and $W_i = 0$ in (17), one may easily derive the result and hence the proof is omitted. □

Remark 3.9 In [31–34], the authors investigated synchronization of complex dynamical network with coupling time-delay based on intermittent control, but the controller is presented in terms of nominal state-delayed systems. On the other hands, we have considered more complicated problem, namely, synchronization of complex dynamical network with hybrid coupling delay and mixed time-varying delay (interval time-varying delay and distributed time-varying delay), which time-varying delay using both state-delayed feedback control as well as intermittent state-delayed feedback control. It should be pointed out that the synchronization problem for complex dynamical networks with both interval and distributed time-varying delays has not received much attention in the literature, not to mention the case when the coupling and controller are also involved.

4 Numerical examples

In this section, we now provide an example to show the effectiveness of the result in Theorem 3.3 and Theorem 3.4.

Example 4.1 We first consider the perturbed Chua circuit system with mixed time-varying delays is used as uncoupled node in the network (1) to show the effectiveness of the proposed control scheme. The perturbed Chua circuit system with mixed time-varying delays is given by [43]

$$\begin{aligned} \dot{x}_1(t) &= p\Bigl(x_2\bigl(t-h(t)\bigr) - \frac{1}{7}\bigl(2x_1^3(t) - x_1(t)\bigr)\Bigr), \\ \dot{x}_2(t) &= x_1(t) - sx_2(t) + x_3\bigl(t-h(t)\bigr), \\ \dot{x}_3(t) &= qx_2(t) + r\int_{t-k_1(t)}^{t} x_1^2(s)\,ds, \end{aligned} \tag{56}$$

where p, q, r, and s are real positive constants. It is well known that the system (56) exhibits chaotic behavior with the parameters p, q, r, and s are chosen as $p = 7$, $q = -\frac{100}{7}$, $r = 0.07$, and $s = 1.5$, the initial condition function $\phi(t) = [0.65\cos t, \sin t, \sin t]^T$, the time-varying delay functions $h(t) = 0.1 + 0.1|\sin t|$ and $k_1(t) = 0.1|\cos t|$ is shown in Figure 1. The solution of the system (56) is denoted by $s(t) = (s_1(t), s_2(t), s_3(t))^T$, which is shown in Figure 2. It is stable at the equilibrium point $s(t) = 0$, $s(t-h(t)) = 0$, $\int_{t-k_1(t)}^{t} s(\theta)\,d\theta = 0$, and the Jacobian matrices are

$$J(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.5 & 0 \\ 0 & -\frac{100}{7} & 0 \end{bmatrix}, \qquad J_h(t) = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_{k_1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

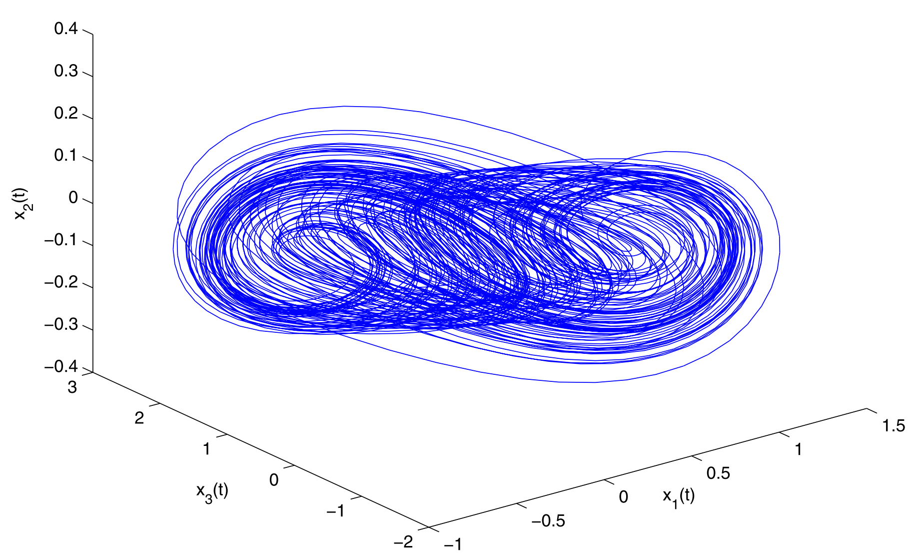

Figure 1 Chaotic behavior of the perturbed Chua circuit system with mixed time-varying delays (56).

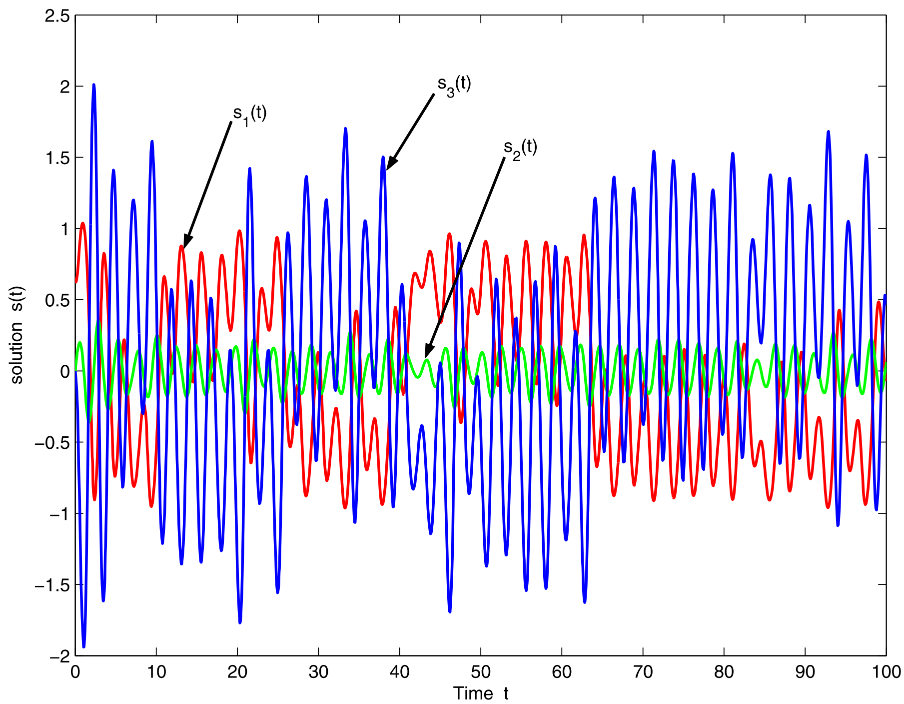

Figure 2 Solution *s*(*t*) of the perturbed Chua circuit system with mixed time-varying delays (56).

Consider a network consisting of five identical perturbed Chua circuit system with mixed time-varying and hybrid coupling delays. The corresponding controlled dynamical network (4) can be described as

$$\begin{aligned}\dot{x}_i(t) &= f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k_1(t)}^{t} x_i(s)\,ds\right) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t)\\ &\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k_1(t)}^{t} x_j(s)\,ds\\ &\quad + D_{1i}K_i\big(x_i(t)-s(t)\big) + D_{2i}K_i\big(x_i\big(t-d(t)\big)-s\big(t-d(t)\big)\big)\\ &\quad + D_{3i}K_i\left(\int_{t-k_2(t)}^{t} x_i(s)\,ds - \int_{t-k_2(t)}^{t} s(\theta)\,d\theta\right), \quad i=1,2,\ldots,5.\end{aligned}$$

Assume that $D_{1i} = \operatorname{diag}\{3,3,3\}$, $D_{2i} = \operatorname{diag}\{0.1,0.1,0.1\}$, $D_{3i} = \operatorname{diag}\{0.1,0.1,0.1\}$, $i = 1,2,\ldots,5$, the coupling strength $c_1 = 0.3$, $c_2 = 0.2$, $c_3 = 0.4$, the inner-coupling matrices are

$$G_1 = \begin{bmatrix} 2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$G_3 = \begin{bmatrix} 0.5 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 0.5\end{bmatrix},$$

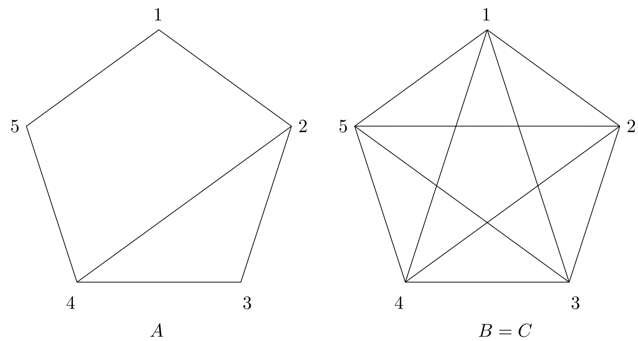

Figure 3 The topology structure of complex networks with $N = 5$.

and the outer-coupling matrices are given by the following irreducible symmetric matrices satisfying condition (2):

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \qquad B = C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix},$$

and the topology structure of complex networks is shown in Figure 3.

The eigenvalues of A, B, and C are $\lambda_1 = \{0, -1.382, -2.382, -3.618, -4.618\}$, $\lambda_2 = \{0, -5, -5, -5, -5\}$, and $\lambda_3 = \{0, -5, -5, -5, -5\}$, respectively.

Solution: From the conditions (12)-(15) of Theorem 3.3, we let $\alpha = 0.02$, $h_1 = 0.1$, $h_2 = 0.2$, $k_1 = 0.1$, $k_2 = 0.1$, $d = 0.3$, the gain matrices of the desired controllers can be obtained as follows:

$$K_1 = \begin{bmatrix} -4.6038 & -0.2266 & -0.8478 \\ -0.1812 & -15.2461 & 2.5773 \\ -1.0007 & 3.6578 & -4.2867 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -3.8398 & -0.1242 & -1.0323 \\ -0.0015 & -15.0343 & 1.9854 \\ -1.0438 & 2.7597 & -4.1461 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -3.7311 & 0.0828 & -1.2030 \\ 0.2016 & -14.8969 & 1.8453 \\ -1.1970 & 2.4629 & -4.2466 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} -3.9213 & 0.3286 & -1.4385 \\ 0.5011 & -15.4924 & 1.7291 \\ -1.4346 & 2.1419 & -4.6878 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -4.3268 & 0.5100 & -1.6409 \\ 0.7973 & -16.7178 & 1.6821 \\ -1.6546 & 1.9279 & -5.3012 \end{bmatrix}.$$

The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order. Figure 4 shows the synchronization between the states of isolate node $s(t)$ and node $x_i(t)$, $i = 1, 2, \ldots, 5$. Figure 5 shows the synchronization errors between the states of isolate node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \ldots, 5$, $j = 1, 2, 3$, without feedback control. Figure 6 shows the synchronization errors between the states of isolated node $s(t)$ and node $x_i(t)$,

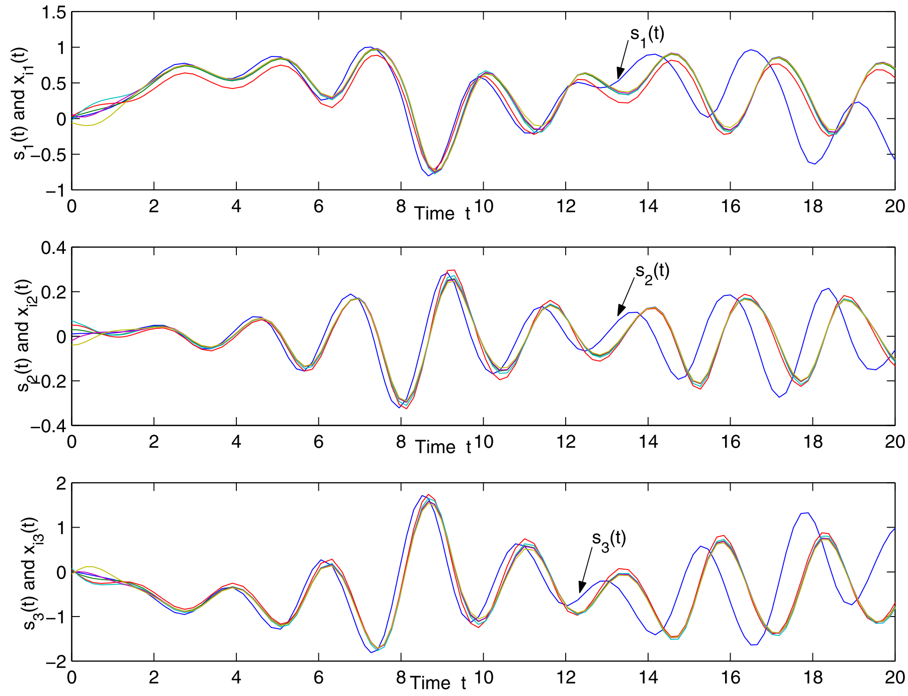

Figure 4 Synchronization curves for the states of the isolated node $s(t)$ and node $x_i(t)$, $i = 1, 2, \ldots, 5$.

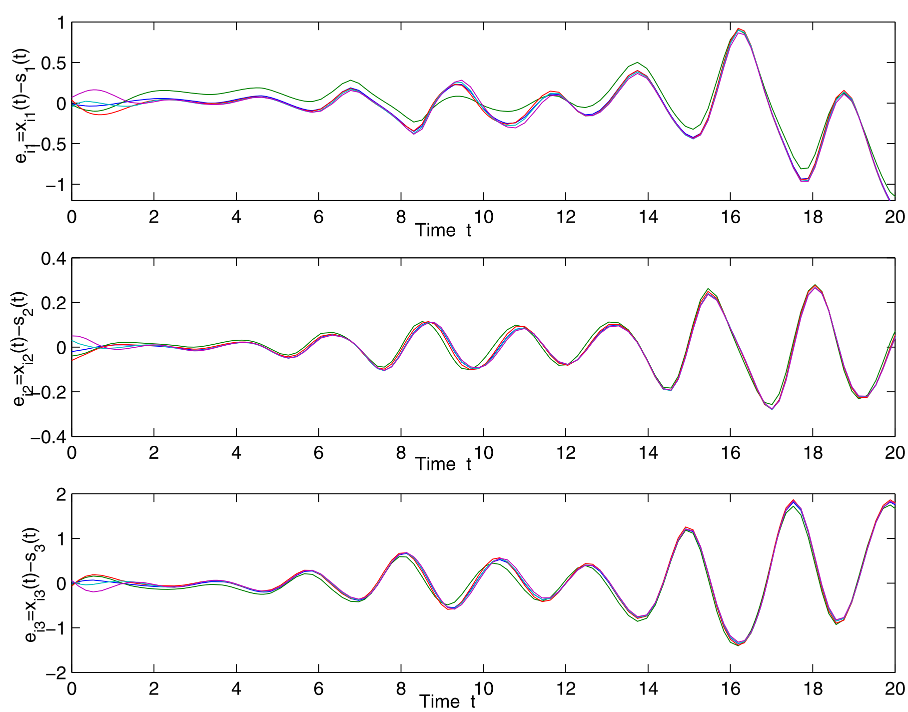

Figure 5 Synchronization error curves for the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \ldots, 5$, $j = 1, 2, 3$, without feedback control.

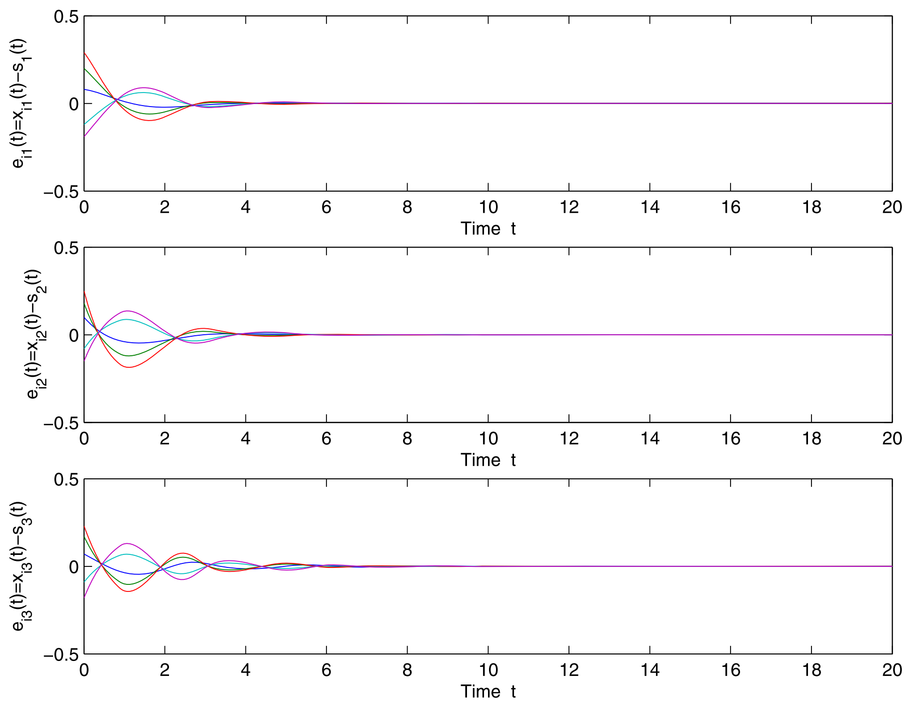

Figure 6 Synchronization error curves for the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, with feedback control.

where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, with feedback control. We see that the synchronization errors converge to zero under the above conditions.

Example 4.2 We consider the nonlinear network model with five nodes, in which each node is a Lorenz system with mixed time-varying delay described by [7]

$$\begin{aligned}
\dot{x}_{i1}(t) &= a\bigl(x_{i2}(t) - x_{i1}(t)\bigr),\\
\dot{x}_{i2}(t) &= cx_{i1}\bigl(t - h(t)\bigr) - x_{i2}(t) - x_{i1}(t)x_{i3}\bigl(t - h(t)\bigr),\\
\dot{x}_{i3}(t) &= x_{i1}(t)\int_{t-k_1(t)}^{t} x_{i2}(s)\,ds - bx_{i3}\bigl(t - h(t)\bigr),
\end{aligned} \tag{57}$$

where $a = 0.9$, $b = 1.3$, and $c = -1$. For the initial function $\phi(t) = [-3.2\cos t, 2\cos t, 5\cos t]$ the solution of system (57) is denoted by $s(t) = (s_1(t), s_2(t), s_3(t))^T$, which is shown in Figure 7. It is asymptotically stable at the equilibrium point $s(t) = 0$, $s(t - h(t)) = 0$, $\int_{t-k_1(t)}^{t} s(\theta)\,d\theta = 0$ and its Jacobian matrices are

$$J(t) = \begin{bmatrix} -0.9 & 0.9 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_{h(t)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.3 \end{bmatrix}, \qquad J_{k_1}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assume that $D_{4i} = \operatorname{diag}\{2, 2, 2\}$, $D_{5i} = \operatorname{diag}\{0.1, 0.1, 0.1\}$, $D_{6i} = \operatorname{diag}\{0.1, 0.1, 0.1\}$, $i = 1, 2, \dots, 5$, the coupling strength $c_1 = 0.1$, $c_2 = 0.2$, $c_3 = 0.3$, the inner-coupling matrices

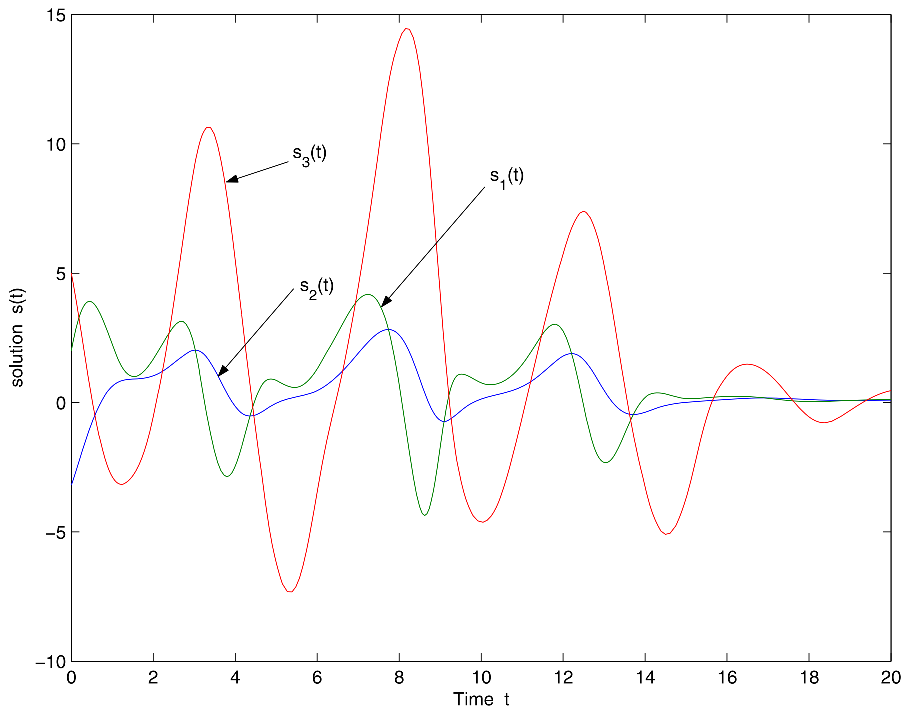

Figure 7 Solution of the Lorenz system with mixed time-varying delays (57).

are

$$G_1 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad G_3 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix},$$

and the outer-coupling matrices are given by the following irreducible symmetric matrices satisfying condition (2):

$$A = C = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix},$$

and the topology structure of complex networks is shown in Figure 8.

The eigenvalues of A, B, and C are $\lambda_1 = \{0, -1.5858, 3, -4.4142, -5\}$, $\lambda_2 = \{0, -5, -5, -5, -5\}$, and $\lambda_3 = \{0, -1.5858, 3, -4.4142, -5\}$, respectively.

Solution: From the conditions (35)-(43) of Theorem 3.4, we let $\varepsilon = 0.09$, $\alpha = 0.07$, $\omega = 4$, $\delta = 2.5$, $h_1 = 0.1$, $h_2 = 0.2$, $k_1 = 0.1$, $k_2 = 0.12$, $d = 0.3$; the gain matrices of the desired controllers can be obtained as follows:

$$K_1 = \begin{bmatrix} -0.1592 & -0.0094 & 0 \\ -0.0155 & -0.1306 & 0 \\ 0 & 0 & -0.3712 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -0.2479 & -0.0265 & 0 \\ -0.0336 & -0.2273 & 0 \\ 0 & 0 & -0.7249 \end{bmatrix},$$

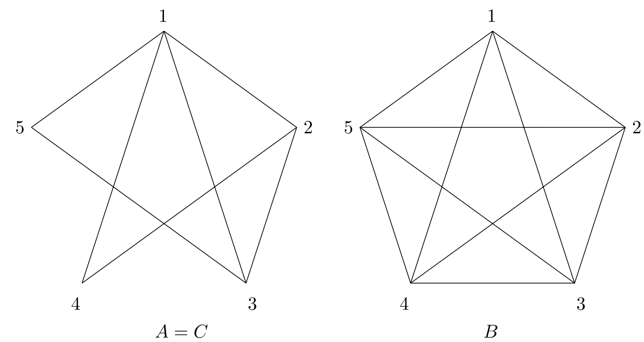

Figure 8 The topology structure of complex networks with $N = 5$.

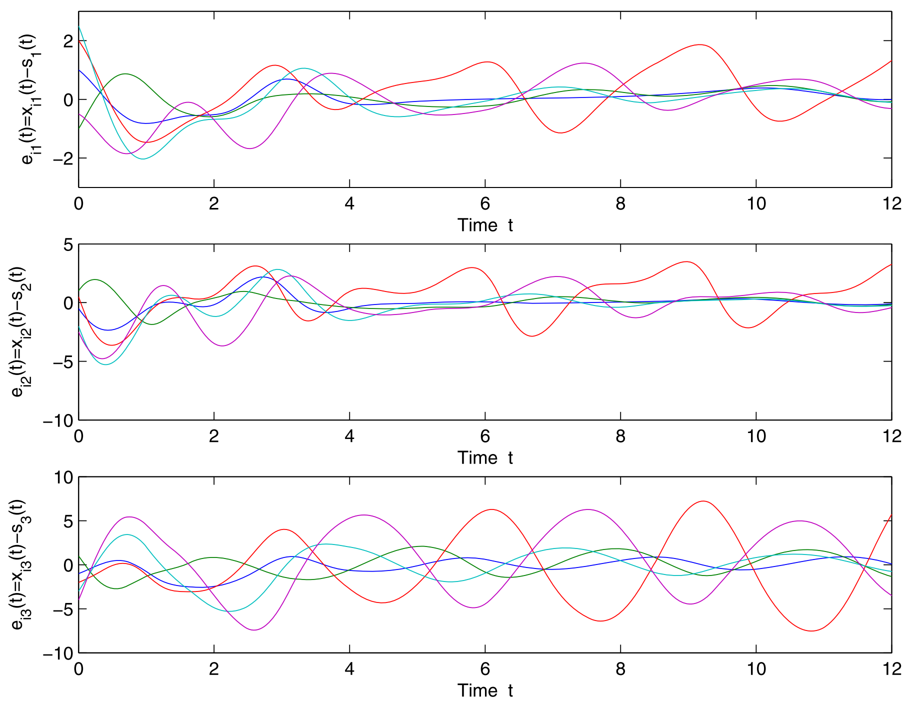

Figure 9 Synchronization error curves for the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \ldots, 5$, $j = 1, 2, 3$, without intermittent feedback control.

$$K_3 = \begin{bmatrix} -0.2325 & -0.0247 & 0 \\ -0.0306 & -0.2134 & 0 \\ 0 & 0 & -0.7079 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} -0.2188 & -0.0234 & 0 \\ -0.0286 & -0.2006 & 0 \\ 0 & 0 & -0.6986 \end{bmatrix},$$

$$K_5 = \begin{bmatrix} -0.2132 & -0.0230 & 0 \\ -0.0279 & -0.1953 & 0 \\ 0 & 0 & -0.6962 \end{bmatrix}.$$

Figure 9 shows the synchronization errors between the states of the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \ldots, 5$, $j = 1, 2, 3$, without intermittent feedback control. Figure 10 shows the synchronization errors between the states of the

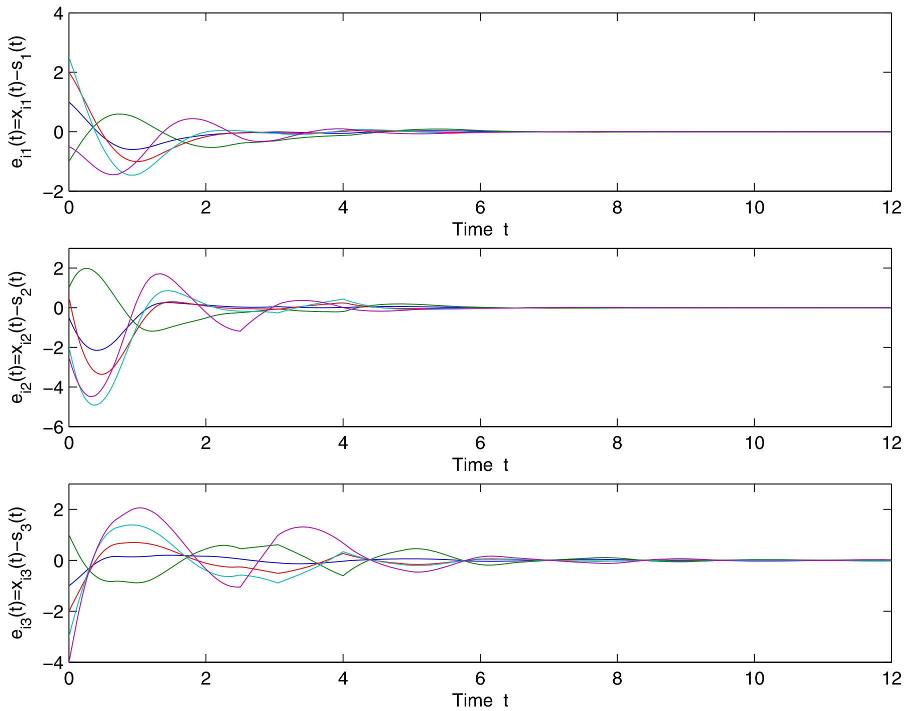

Figure 10 Synchronization error curves for the isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, with intermittent feedback control.

isolated node $s(t)$ and node $x_i(t)$, where $e_{ij}(t) = x_{ij}(t) - s_i(t)$, for $i = 1, \dots, 5$, $j = 1, 2, 3$, with intermittent feedback control. We see that the synchronization errors converge to zero under the above conditions.

Remark 4.1 In Example 4.1 and Example 4.2, each of them to consider general complex networks in which every dynamical node has mixed time-varying delays (interval time-varying delay and distributed time-varying delay), and the complex networks have state coupling, interval time-varying delay coupling and distributed time-varying delay coupling.

Example 4.3 Consider a network model with five nodes, where each node is a three-dimensional stable linear system described by [9, 18]

$$\begin{aligned} \dot{x}_{i1}(t) &= -x_{i1}(t), \\ \dot{x}_{i2}(t) &= -2x_{i2}(t), \\ \dot{x}_{i3}(t) &= -3x_{i3}(t), \end{aligned} \tag{58}$$

which is asymptotically stable at the equilibrium point $s(t) = 0$, and its Jacobian matrix is $J(t) = \text{diag}\{-1, -2, -3\}$. Assume that the network coupling is the same as that in Example 4.1. The upper bounds on the time-delay obtained from Corollary 3.8 are listed in Table 1. We see that Corollary 3.8 provides a less conservative result than those obtained via the methods of [9, 18]. When $h_m \neq 0$ especially, the result in [9] is not discussed while Corollary 3.8 in this paper also considers the case $h_m \neq 0$.

Table 1 Comparison of the maximum value h_M ($h_m = 0$) for difference c_2

c_2	0.3	0.4	0.5	0.6
Li *et al.* [9]	0.960	0.710	0.562	0.464
Yue and Li [18]	1.345	0.950	0.731	0.587
Corollary 3.8	1.9707	1.2848	0.8712	0.5941

Remark 4.2 In [9] presented the synchronization problem of general complex dynamical networks with time-varying delays in the network couplings and time-varying delays in the dynamical nodes, respectively. But the time-varying delays are required to be differentiable, however, in most cases, these conditions are difficult to satisfy. Therefore, in this paper we will employ some new techniques so that the above conditions can be removed.

5 Conclusions

This paper has investigated synchronization for complex dynamical network with mixed time-varying and hybrid coupling delays, which is composed of state coupling, interval time-varying delay coupling, and distributed time-varying delay coupling. The time-varying delay function is not necessary to be differentiable which allows the time-delay function to be a fast time-varying function. We transformed the synchronization problem of the complex network into the stability analysis of linear systems. A new class of Lyapunov-Krasovskii functionals is constructed; new delay-dependent sufficient conditions for the exponential synchronization of complex dynamical network have been derived by a set of LMIs without introducing any free-weighting matrices. The delay feedback controllers H1 and H2 designed can guarantee exponential synchronization of the complex dynamical network. Simulation results have been given to illustrate the effectiveness of the proposed method.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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