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Research article

Some Convolution Properties of Multivalent Analytic Functions

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Abstract: In this paper, we introduce a new subclass of multivalent functions associated with conic domain in an open unit disk. We study some convolution properties, sufficient condition for the functions belonging to this new class.

Keywords: Multivalent functions; Hadamard product; Conic domain; Analytic functions; Sufficient condition

1. Introduction

Let $A(p)$ denote the class of all functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \qquad (p \in N = \{1, 2, 3.....\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $E = \{z : |z| < 1\}$. For $p = 1$, $A(1) = A$. Let f, $g \in A(p)$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \qquad (z \in E).$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

Let UCV and UST denote the usual classes of uniformly convex and uniformly starlike functions and are defined by

$$UCV = \left\{ f(z) \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \right\}, \quad z \in E,$$

$$UST \;=\; \left\{ f(z) \in A : Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \right\}, \quad z \in E.$$

These classes were first introduced by Goodman [2, 3] and further investigated by [14] and [6]. Kanas and Wiśniowska [4, 5] introduced the conic domain Ω_k, $k \geq 0$ as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

For fixed k this domain represents the right half plane ($k = 0$), a parabola ($k = 1$), the right branch of hyperbola ($0 < k < 1$) and an ellipse ($k > 1$). For detail study about Ω_k and its generalizations, see [8, 9, 10]. The extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ \frac{1}{1-k^2}\cosh\left\{\left(\frac{2}{\pi}\arccos k\right)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2}, & 0 < k < 1, \\ \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{\kappa}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \frac{k^2}{k^2-1}, & k > 1, \end{cases} \tag{1.2}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa z}}, \quad z \in \mathbb{E},$$

and $\kappa \in (0, 1)$ is chosen such that $k = \cosh\left(\pi K'(\kappa)/(4K(\kappa))\right)$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$ and $K'(t)$ is the complementary integral of $K(t)$.
Now we define the following:

Definition. *Let $f \in A(p)$ given by (1.1) is said to belong to $k - UR_p$, $k \geq 0$ if it satisfies the following condition*

$$Re\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) > k\left|\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!} - 1\right|, \quad z \in E,$$

where $f^{(p)}(z)$ is the pth derivative of $f(z)$.

Special Cases:

i) For $k = 0$, we have $0 - UR_p = R_p$, introduced and studied by Noor et-al. [7].
ii) For $k = 0$, $p = 1$, we have $0 - UR_1 = R$, introduced and studied by Singh et-al. [15].

2. Preliminary Results

Lemma 2.1. [12]. For $\alpha \leq 1$ and $\beta \leq 1$

$$p(\alpha) * p(\beta) \subset p(\delta), \qquad \delta = 1 - 2(1-\alpha)(1-\beta).$$

The result is sharp.

Lemma 2.2. [1]. Let $\{d_n\}_0^\infty$ be a convex null sequence. Then the function

$$q(z) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n z^n$$

is analytic in E and $Req(z) > 0 \;\; z \in E$.

Lemma 2.3. [13]. For $0 \leq \theta \leq \pi$,

$$\frac{1}{2} + \sum_{n=1}^{m} \frac{\cos n\theta}{n+1} \geq 0.$$

Lemma 2.4. [7]. If f and g belong to the class R_p and

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z).$$

Then h also belong to the class R_p.

3. Main Result

Theorem 3.1. Let $f \in k - UR_P$ then

$$Re\left(\frac{f^{(p)}(z)}{p!}\right) > \frac{k-1+2\log 2}{k+1}.$$

Proof. Let $f \in k - UR_p$ then by definition, we have

$$Re\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) > k\left|\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!} - 1\right|.$$

After some simple computations, we have

$$Re\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) > \frac{k}{k+1}, \tag{3.1}$$

This can be written as

$$Re\left(1 + \sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{n!} a_{n+p} z^n\right) > \frac{k}{k+1}, \tag{3.2}$$

or

$$Re\left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{n!} a_{n+p} z^n\right) > \frac{2k+1}{2k+2}. \tag{3.3}$$

Consider the function

$$h(z) = 1 + 2\sum_{n=1}^{\infty} \frac{z^n}{n+1}. \tag{3.4}$$

Clearly h is analytic, $h(0) = 1$ in E and

$$Reh(z) = Re\left(1 - \frac{2}{z}[z + \log(1-z)]\right) > -1 + 2\log 2. \tag{3.5}$$

From (3.3) and (3.4), we have

$$\left(\frac{f^{(p)}(z)}{p!}\right) = \left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{n!} a_{n+p} z^n\right) * \left(1 + 2\sum_{n=1}^{\infty} \frac{z^n}{n+1}\right). \tag{3.6}$$

Now using (3.3) , (3.5) and Lemma 2.2 with $\alpha = \frac{2k+1}{2k+2}, \beta = -1 + 2\log 2$ and $\delta = \frac{k-1+2\log 2}{k+1}$, we have

$$Re\left(\frac{f^{(p)}(z)}{p!}\right) > \frac{k-1+2\log 2}{k+1}. \tag{3.7}$$

This completes the result.
For some spacial value of k and p we obtain the following known result.

Corollary 3.2. [7]. *Let $f \in R_p$ then*

$$Re\left(\frac{f^{(p)}(z)}{p!}\right) > -1 + 2\log 2.$$

Theorem 3.3. *Let $f \in k - UR_p$ then*

$$Re\left(\frac{f^{(p-1)}(z)}{z}\right) > \frac{p!(2k+1)}{2k+2}. \tag{3.8}$$

Proof. From (3.3), we have

$$Re\left(1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{n!}a_{n+p}z^n\right) > \frac{(2k+1)}{2k+2}.$$

Now consider the convex null sequence $\{d_n\}_0^\infty$ define by $d_0 = 0$, $d_n = \frac{2}{(n+1)^2}$, $n \geq 1$, using Lemma 2.2, we have

$$Re\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{(n+1)^2}z^n\right) > 0,$$

or equivalently

$$Re\left(1 + 2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right) > \frac{1}{2}. \tag{3.9}$$

From (3.3) and (3.9), we have

$$\frac{f^{(p-1)}(z)}{p!z} = \left(1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{n!}a_{n+p}z^n\right) * \left(1 + 2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right). \tag{3.10}$$

From (3.10) and Lemma (2.1) with $\alpha = \frac{2k+1}{2k+2}$ and $\beta = \frac{1}{2}$, we have

$$Re\left(\frac{f^{(p-1)}(z)}{z}\right) > \frac{p!(2k+1)}{2k+2}. \tag{3.11}$$

Which is the required result.

Corollary 3.4. [7]. *Let $f \in R_p$ then*

$$Re\left(\frac{f^{(p-1)}(z)}{z}\right) > \frac{p!}{2}, \quad z \in E.$$

Corollary 3.5. [15]. *Let $f \in R$ then*

$$Re\left(\frac{f(z)}{z}\right) > \frac{1}{2}, \quad z \in E.$$

Theorem 3.6. Let $f \in k - UR_p$ then for every $n \geq 1$, the nth partial sum of f satisfies

$$ReS_n^{(p)}(z, f) > \frac{p!k}{k+1}, \quad z \in E.$$

and hence $S_n(z, f)$ is $p-$valent in E.

Proof. From (3.2) and (3.4), we have

$$\frac{s_n^{(p)}(z, f)}{p!} = \left(1 + \sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{p!n} a_{n+p} z^n\right) * \left(1 + \sum_{n=1}^{\infty} \frac{z^n}{n+1}\right). \tag{3.12}$$

Putting $z = re^{i\theta}$, $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$ and the minimum principle for harmonic functions with Lemma 2.3, we have

$$\begin{aligned} Re\left(1 + \sum_{n=1}^{k} \frac{z^n}{n+1}\right) &= Re\left(1 + \sum_{n=1}^{k} \frac{r^n e^{in\theta}}{n+1}\right), \; 0 \leq \theta \leq \pi \\ &= Re\left(1 + \sum_{n=1}^{k} \frac{r^n}{n+1}(\cos n\theta + i \sin n\theta)\right) \\ &= \left(1 + \sum_{n=1}^{k} \frac{r^n \cos n\theta}{n+1}\right) \\ &= \left(1 + \sum_{n=1}^{k} \frac{r^n \cos n\theta}{n+1}\right) \geq \frac{1}{2}. \end{aligned} \tag{3.13}$$

Using (3.2), (3.12), (3.13) and Lemma 2.1 with $\alpha = \frac{k}{k+1}$ and $\beta = \frac{1}{2}$, we have

$$Re\left(s_n^{(p)}(z, f)\right) > \frac{p!k}{k+1}. \tag{3.14}$$

This completes the proof. From the result given by [11], we see that $s_n(z, f)$ is $p-$valent in E for every $n \geq 1$.

Corollary 3.7. [7]. *Let $f \in R_p$, then for every $n \geq 1$, the nth partial sum of f satisfies*

$$ReS_n^{(p)}(z, f) > 0, \quad z \in E$$

and hence $s_n(z, f)$ is $p-$valent in E.

For $k = 1$ we have the following corollary.

Corollary 3.8. [15]. *Let $f \in 1 - UR_p$, then for every $n \geq 1$, the nth partial sum of f satisfies*

$$ReS_n^{'}(z, f) > \frac{p!}{2}, \quad z \in E$$

and hence $s_n(z, f)$ is univalent in E.

Theorem 3.9. *Let $f \in k - UR_p$, $g \in R_p$ and*

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z).$$

Then h belong to the class $k - UR_p$.

Proof. Since

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z). \tag{3.15}$$

It follows that

$$zh^{(p)}(z) = f^{(p)}(z) * g^{(p-1)}(z). \tag{3.16}$$

After simple computations, (3.16) can be written as

$$Re\left(\frac{h^{(p)}(z) + zh^{(p+1)}(z)}{p!}\right) = Re\left(\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) * \left(\frac{g^{(p-1)}(z)}{zp!}\right)\right). \tag{3.17}$$

From (3.17), (3.1), Corollary 3.4 and Lemma 2.1 with $\alpha = \frac{k}{k+1}$ and $\beta = \frac{1}{2}$, we get the required proof.

Corollary 3.10. *[15]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to R then so does their Hadamard product*

$$h(z) = f(z) * g(z).$$

Theorem 3.11. *If $f, g \in R_p$, $h \in k - UR_p$ and*

$$\varphi^{(p-1)}(z) = h^{(p-1)}(z) * f^{(p-1)}(z) * g^{(p-1)}(z).$$

Then $\varphi \in k - UR_p$.

Proof. Suppose that

$$m^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z), \tag{3.18}$$

and it is clear from Lemma 2.4 that, $m \in R_p$. From the hypothesis and (3.18), we have

$$\varphi^{(p-1)}(z) = h^{(p-1)}(z) * m^{(p-1)}(z). \tag{3.19}$$

From (3.19) and Theorem 3.9, we get the required result.

Theorem 3.12. *If $f_1, f_2, f_3, ..., f_n$ belong to R_p, $h \in k - UR_p$ and*

$$g^{(p-1)}(z) = f_1^{(p-1)}(z) * f_2^{(p-1)}(z) * f_3^{(p-1)}(z) * ... * f_n^{(p-1)}(z) * h^{(p-1)}(z). \tag{3.20}$$

Then $g \in k - UR_p$.

Proof. For proving the above Theorem, we use the principle of mathematical induction. For $n = 2$, we have proved Theorem 3.11, thus (3.20) hold true for $n = 2$. Suppose that (3.20) hold true for $n = k$; that is,

$$g^{(p-1)}(z) = f_1^{(p-1)}(z) * f_2^{(p-1)}(z) * f_3^{(p-1)}(z) * ... * f_k^{(p-1)}(z) * h^{(p-1)}(z). \tag{3.21}$$

Then $g \in k - UR_p$.
We have to prove that (3.20) hold true for $n = k + 1$, for this, consider

$$g^{(p-1)}(z) = f_1^{(p-1)}(z) * f_2^{(p-1)}(z) * f_3^{(p-1)}(z) * ... * f_{k+1}^{(p-1)}(z) * h^{(p-1)}(z). \tag{3.22}$$

Let

$$M^{(p-1)} = f_1^{(p-1)} * f_2^{(p-1)} * f_3^{(p-1)} * \ldots\ldots\ldots * f_k^{(p-1)} * h^{(p-1)}$$

Then by hypothesis $M \in k - UR_p$. Now (3.22) becomes

$$g^{(p-1)}(z) = (M^{(p-1)} * f_{k+1}^{(p-1)})(z). \tag{3.23}$$

Using Theorem 3.9, from (3.23), we have

$$Re\left(\frac{g^{(p)}(z) + zg^{(p+1)}(z)}{p!}\right) > \frac{k}{k+1}. \tag{3.24}$$

(3.24) now implies that $g \in k - UR_p$. Therefore, the result is true for $n = k + 1$ and hence by using mathematical induction, (3.20) holds true for all $n \geq 2$. This completes the proof.

Theorem 3.13. *If f, $g \in k - UR_p$ and*

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z).$$

Then h belong to the class $k - UR_p$.

Proof. Since

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z). \tag{3.25}$$

Differentiation yields

$$zh^{(p)}(z) = f^{(p)}(z) * g^{(p-1)}(z). \tag{3.26}$$

After simplification, we have

$$Re\left(\frac{h^{(p)}(z) + zh^{(p+1)}(z)}{p!}\right) = Re\left(\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) * \left(\frac{g^{(p-1)}(z)}{zp!}\right)\right). \tag{3.27}$$

From (3.27), (3.1), (3.11) and Lemma 2.1 with $\alpha = \frac{k}{k+1}$ and $\beta = \frac{2k+1}{2k+2}$, we have

$$Re\left(\frac{h^{(p)}(z) + zh^{(p+1)}(z)}{p!}\right) > \frac{k}{k+1}. \tag{3.28}$$

(3.28) implies that h belong to $k - UR_p$.

Our next result give us a sufficient condition for the class $k - UR_p$.

Theorem 3.14. *Let $f \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty} \frac{(k-1)(n+1)(p+n)!}{p!n!} \left|a_{n+p}\right| < 1. \tag{3.29}$$

Then $f \in k - UR_p$.

Proof. To prove the required result it is sufficient to show that

$$k\left|\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right|-Re\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right)<1 \tag{3.30}$$

Now

$$\begin{aligned}
& k\left|\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right|-Re\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right)\\
\leq\ & (k-1)\left|\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right|\\
=\ & (k-1)\left|\frac{f^{(p)}(z)+zf^{(p+1)}(z)-p!}{p!}\right|\\
=\ & (k-1)\left|\sum_{n=1}^{\infty}\frac{(n+1)(p+n)!}{p!n!}a_{n+p}z^n\right|.
\end{aligned}$$

This can be written as

$$\begin{aligned}
& k\left|\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right|-Re\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}-1\right)\\
\leq\ & (k-1)\left|\sum_{n=1}^{\infty}\frac{(n+1)(p+n)!}{p!n!}a_{n+p}\right||z^n| \qquad (3.31)
\end{aligned}$$

(3.31) is bounded above by 1 if (3.29) is satisfied. This completes the proof.

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

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