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Positive periodic solution for higher-order p-Laplacian neutral singular Rayleigh equation with variable coefficient

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Abstract

In this paper, we consider the existence of a positive periodic solution for the following kind of high-order p-Laplacian neutral singular Rayleigh equation with variable coefficient:

$$(\varphi_p(x(t)-c(t)x(t-\sigma))^{(n)})^{(m)}+f(t,x'(t))+g(t,x(t))=e(t).$$

Our proof is based on Mawhin's continuation theory.

MSC: 34C25; 34K13; 34K40

Keywords: periodic solution; high-order; neutral operator; variable coefficient; singularity

1 Introduction

In this paper, we consider the following high-order p-Laplacian neutral singular Rayleigh equation with variable coefficient:

$$\left(\varphi_p\left(x(t)-c(t)x(t-\sigma)\right)^{(n)}\right)^{(m)}+f\left(t,x'(t)\right)+g\left(t,x(t)\right)=e(t), \tag{1.1}$$

where $p>1$, $\varphi_p(x)=|x|^{p-2}x$ for $x\neq 0$ and $\varphi_p(0)=0$, $c\in C^n(\mathbb{R},\mathbb{R})$ and $c(t+T)\equiv c(t)$, f is a continuous function defined in $\mathbb{R}^2$ and periodic in t with $f(t,\cdot)=f(t+T,\cdot)$ and $f(t,0)=0$, $g(t,x)=g_0(x)+g_1(t,x)$, where $g_1:\mathbb{R}\times(0,+\infty)\to\mathbb{R}$ is an L^2-Carathéodory function, $g_1(t,\cdot)=g_1(t+T,\cdot)$, $g_0\in C((0,\infty);\mathbb{R})$ has a singularity at $x=0$, $e:\mathbb{R}\to\mathbb{R}$ is a continuous periodic function with $e(t+T)\equiv e(t)$ and $\int_0^T e(t)\,dt=0$, T is a positive constant, and n and m are positive integers.

In recent years, there are many works on periodic solutions for high-order neutral differential equations (see [1–11] and the references therein). Wang and Lu [5] in 2007 investigated the existence of periodic solution for the following high-order neutral functional differential equation with distributed delay:

$$\left(x(t)-cx(t-\sigma)\right)^{(n)}+f\left(x(t)\right)x'(t)+g\left(\int_{-r}^{0}x(t+s)\,d\alpha(s)\right)=p(t). \tag{1.2}$$

Using the continuation theorem of coincidence degree theory, they obtained the existence of periodic solutions for (1.2). Afterwards, Ren et al. considered the following high-order p-Laplacian neutral differential equation

$$\left(\varphi_p\left(x(t)-cx(t-\sigma)\right)^{(l)}\right)^{(n-l)}=F\left(t,x(t),x'(t),\ldots,x^{(l-1)}(t)\right). \tag{1.3}$$

They obtained the existence of periodic solutions for (1.3) in the general case ($|c|\neq 1$) in [10] and in the critical case ($|c|=1$) in [9], respectively.

At the same time, some authors began to consider high-order neutral differential equation with singularity. Recently, applying the coincidence degree theory and some analysis skills, Xin et al. [11] discussed the existence of a positive periodic solution for the following neutral Liénard equation with singularity:

$$\left(\varphi_p\left(x(t)-cx(t-\tau)\right)^{(n)}\right)^{(m)}+f\left(x(t)\right)x'(t)+g\left(t,x(t-\sigma)\right)=e(t). \tag{1.4}$$

Inspired by these results in [5, 9–11], in this paper, we consider the existence of a positive periodic solution for (1.1) with singularity by applications of Mawhin's continuation theory. The obvious difficulty lies in the following two respects. Firstly, $(x(t)-c(t)x(t-\sigma))^{(n)}\neq x^{(n)}(t)-c(t)x^{(n)}(t-\sigma)$, and the calculation of $(x(t)-c(t)x(t-\sigma))^{(n)}$ is very complicated. Secondly, a priori bounds of periodic solutions are not easy to estimate.

2 Preparation

Firstly, we give qualitative properties of the neutral operator $(Ax)(t):=x(t)-c(t)x(t-\sigma)$.

Lemma 2.1 (see [12]) *If $|c(t)|\neq 1$, then the operator A has a continuous inverse A^{-1} on $C_T:=\{\phi\in C(\mathbb{R},\mathbb{R}):\phi(t+T)\equiv\phi(t)\}$, satisfying*

$$\left|\left(A^{-1}f\right)(t)\right|\leq\frac{|f|_\infty}{\Gamma},\quad\forall f\in C_T,$$

where $\Gamma:=\begin{cases}1-|c|_\infty & \textit{for } |c|_\infty:=\max_{t\in[0,T]}|c(t)|<1,\\ |c|_0-1 & \textit{for } |c|_0:=\min_{t\in[0,T]}|c(t)|>1.\end{cases}$

Lemma 2.2 (Gaines and Mawhin [13]) *Let X and Y be two Banach spaces, and let $L:D(L)\subset X\to Y$ be a Fredholm operator with index zero. Let $\Omega\subset X$ be an open bounded set, and let $N:\overline{\Omega}\to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx\neq\lambda Nx$, $\forall x\in\partial\Omega\cap D(L)$, $\lambda\in(0,1)$;
(2) $Nx\notin\operatorname{Im}L$, $\forall x\in\partial\Omega\cap\operatorname{Ker}L$;
(3) $\deg\{JQN,\Omega\cap\operatorname{Ker}L,0\}\neq 0$, *where $J:\operatorname{Im}Q\to\operatorname{Ker}L$ is an isomorphism.*

Then the equation $Lx=Nx$ has a solution in $\overline{\Omega}\cap D(L)$.

Lemma 2.3 (see [11]) *If $x\in C_T^1:=\{x\in C^1(\mathbb{R},\mathbb{R}):x(t+T)\equiv x(t)\}$ and there exists a point $t_0\in(0,T)$ such that $|x(t_0)|<d$, then*

$$|x|_\infty\leq d+\frac{1}{2}\int_0^T\left|x'(t)\right|dt,$$

where $|x|_\infty:=\max_{t\in\mathbb{R}}|x(t)|$.

To use the continuation degree theorem, we rewrite (1.1) in the form

$$\begin{cases} (Ax_1)^{(n)}(t) = \varphi_q(x_2(t)), \\ x_2^{(m)}(t) = -f(t, x_1'(t)) + g(t, x_1(t)) + e(t), \end{cases} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if a periodic solution of (2.1) is $x(t) := \binom{x_1}{x_2}$, then $x_1(t)$ must be a periodic solution of (1.1). Thus, the problem of finding a periodic solution for (1.1) reduces to finding a periodic solution for (2.1).

Now, set

$$X := \{x \in C(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$$

with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$ and

$$Y := \{x \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$$

with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, both X and Y are Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in C^{n+m}(\mathbb{R}, \mathbb{R}^2) : x(t + T) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)^{(n)}(t) \\ x_2^{(m)}(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f(t, x_1'(t)) - g(t, x_1(t)) + e(t) \end{pmatrix}. \tag{2.2}$$

Then (2.1) can be converted into the abstract equation $Lx = Nx$.

If $x = \binom{x_1}{x_2} \in \operatorname{Ker} L$, that is, $\begin{cases} (x_1(t) - c(t)x_1(t-\sigma))^{(n)} = 0, \\ x_2^{(m)}(t) = 0, \end{cases}$ then we have

$$\begin{cases} x_1(t) - c(t)x_1(t - \sigma) = a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0, \\ x_2(t) = b_{m-1}t^{m-1} + b_{m-2}t^{m-2} + \cdots + b_1 t + b_0, \end{cases}$$

where $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in \mathbb{R}$ are constant. From $x_1(t) - c(t)x_1(t - \sigma) \in C_T$ and $x_2(t) \in C_T$ we have $a_1 = \cdots = a_{n-1} = 0$ and $b_1 = b_2 = \cdots = b_{m-1} = 0$. Let $\phi(t) \not\equiv 0$ be a solution of $x(t) - c(t)x(t - \sigma) = 1$. Then $\operatorname{Ker} L = x = \binom{a\phi(t), a \in \mathbb{R}}{b, b \in \mathbb{R}}$. From the definition of L we can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So L is a Fredholm operator with index zero.

Next, we will consider L-compact N. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix} \quad \text{and} \quad Qy = \frac{1}{T}\int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds.$$

Then $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Denote $L_P = L|_{D(L)\cap \operatorname{Ker} P}$ and let $L_P^{-1} : \operatorname{Im} L \to D(L)$ be the inverse of L_P. Then

$$\begin{aligned} \left[L_P^{-1}y\right](t) &= \begin{pmatrix} (A^{-1}Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix}, \\ [Gy_1](t) &= \sum_{i=1}^{n-1} \frac{1}{i!} a_i t^i + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y_1(s)\, ds, \\ [Gy_2](t) &= \sum_{i=1}^{m-1} \frac{1}{i!} b_i t^i + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} y_2(s)\, ds, \end{aligned} \tag{2.3}$$

where $a_i := (Ax_1)^{(i)}(0)$ are defined as follows:

$$E_1 Z = C, \quad \text{where } E_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ e_1 & 1 & 0 & \cdots & 0 & 0 \\ e_2 & e_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ e_{n-3} & e_{n-4} & e_{n-5} & \cdots & 1 & 0 \\ e_{n-2} & e_{n-3} & e_{n-4} & \cdots & e_1 & 0 \end{pmatrix}_{(n-1)\times(n-1)},$$

$Z = (a_{n-1}, a_{n-2} \cdots, a_1)^\top$, $C = (c_1, c_2, \ldots, c_{n-1})^\top$, $c_i = -\frac{1}{i!T}\int_0^T (T-s)^i y_1(s)\, ds$, and $e_j = \frac{T^j}{(j+1)!}$, $j = 1, 2, \ldots, n-2$. Similarly, we can get $b_1 := x_2^{(i)}(0)$, $i = 1, 2, \ldots, m-1$. Therefore, from (2.2) and (2.3) we get that N is L-compact on $\bar{\Omega}$.

3 Periodic solutions for (1.1) with repulsive singularity

For convenience, we list the following assumptions, which will further used repeatedly:

(H_1) There exists a positive constant K such that $|f(t,u)| \le K$ for $(t,u) \in \mathbb{R} \times \mathbb{R}$.

(H_2) There exist positive constants α and β such that $|f(t,u)| \le \alpha|u|^{p-1} + \beta$ for $(t,u) \in \mathbb{R} \times \mathbb{R}$.

(H_3) $f(t,u) \ge 0$ for $(t,u) \in \mathbb{R} \times \mathbb{R}$;

(H_4) There exists a positive constant D such that $g(t,x) > K$ for $x > D$.

(H_5) There exists a positive constant D_1 such that $g(t,x) > |e|_\infty$ for $x > D_1$.

(H_6) There exist positive constants γ, ζ such that

$$g(t,x) \le \gamma x^{p-1} + \zeta \quad \text{for all } x > 0.$$

(H_7) (Repulsive singularity) $\int_0^1 g_0(s)\, ds = -\infty$.

Theorem 3.1 *Assume that* (H_1), (H_4), *and* (H_6)-(H_7) *hold. Then* (1.1) *has at least one* T*-periodic solution if*

$$0 < \frac{T^{2p}}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)} \frac{\gamma}{(\Gamma - \frac{T}{2}\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})^{p-1}} < 1,$$

where $c_{n-k} := \max_{t\in[0,\omega]} |c^{(n-k)}(t)|$.

Proof Consider the abstract equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)^{(n)}(t) = \lambda\varphi_q(x_2(t)), \\ x_2^{(m)}(t) = -\lambda f(t, x_1'(t)) - \lambda g(t, x_1(t)) + \lambda e(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \lambda^{1-p}\varphi_p[(Ax_1)^{(n)}(t)]$ into the second equation of (3.1), we have

$$\left(\varphi_p(Ax_1)^{(n)}(t)\right)^{(m)} + \lambda^p f\left(t, x_1'(t)\right) + \lambda^p g\left(t, x_1(t)\right) = \lambda^p e(t). \tag{3.2}$$

Integrating both sides of (3.2) from 0 to T, we have

$$\int_0^T \left(f\left(t, x_1'(t)\right) + g\left(t, x_1(t)\right)\right) dt = 0. \tag{3.3}$$

From the mean value theorem, there exists a point $\xi \in (0, T)$ such that

$$f\left(\xi, x_1'(\xi)\right) + g\left(\xi, x_1(\xi)\right) = 0.$$

Then by (H_1) we have

$$g\left(\xi, x_1(\xi)\right) = \left|-f\left(\xi, x_1'(\xi)\right)\right| \le K,$$

and in view of (H_4), we get that $x_1(\xi) \le D$. Since $x_1(t)$ is periodic with period T and $x_1(t) > 0$ for $t \in [0, T]$. Then $0 < x_1(\xi) \le D$. Therefore, from Lemma 2.3 we can get

$$|x_1|_\infty \le D + \frac{1}{2}\int_0^T \left|x_1'(s)\right| ds. \tag{3.4}$$

From (3.4) and the Wirtinger inequality (see [14], Lemma 2.4) we get

$$\begin{aligned} |x_1|_\infty &\le D + \frac{1}{2}T^{\frac{1}{2}}\left(\int_0^T \left|x_1'(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\le D + \frac{1}{2}T^{\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{n-1}\left(\int_0^T \left|x_1^{(n)}(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\le D + \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-1}\left|x_1^{(n)}\right|_\infty. \end{aligned} \tag{3.5}$$

Since $x_1^{(i-1)}(0) = x_1^{(i-1)}(T)$, $i = 1, 2 \ldots, n-1$, there exists a point $t_i^* \in [0, T]$ such that $x_1^{(i)}(t_i^*) = 0$. From the Hölder and Wirtinger inequalities, we can easily get

$$\begin{aligned}
\left|x_1^{(i)}\right|_\infty &\le \frac{1}{2}\int_0^T \left|x_1^{(i+1)}(t)\right| dt \\
&\le \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T \left|x_1^{(i+1)}(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{(n-i-1)} \left|x_1^{(n)}\right|_\infty.
\end{aligned} \tag{3.6}$$

On the other hand, since $(Ax_1)(t) = x_1(t) - c(t)x_1(t-\sigma)$, we have

$$\begin{aligned}
\left(Ax_1(t)\right)^{(n)} &= \left(x_1(t) - c(t)x_1(t-\sigma)\right)^{(n)} \\
&= x_1^{(n)}(t) - \left(c^{(n)}(t)x_1(t-\sigma) + nc^{(n-1)}(t)x_1'(t-\sigma)\right. \\
&\quad \left. + \frac{n(n-1)}{2!}c^{(n-2)}x_1''(t-\sigma) + \cdots + c(t)x_1^{(n)}(t-\sigma)\right) \\
&= x_1^{(n)}(t) - c(t)x_1^{(n)}(t-\sigma) - \left(c^{(n)}(t)x_1(t-\sigma) + nc^{(n-1)}(t)x_1'(t-\sigma)\right. \\
&\quad \left. + \frac{n(n-1)}{2!}c^{(n-2)}x_1''(t-\sigma) + \cdots + nc'(t)x_1^{(n-1)}(t-\sigma)\right).
\end{aligned}$$

So, we can get

$$\begin{aligned}
Ax_1^{(n)}(t) &= \left(Ax_1(t)\right)^{(n)} + \left(c^{(n)}(t)x_1(t-\sigma) + nc^{(n-1)}(t)x_1'(t-\sigma)\right. \\
&\quad \left. + \frac{n(n-1)}{2!}c^{(n-2)}x_1''(t-\sigma) + \cdots + nc'(t)x_1^{(n-1)}(t-\sigma)\right).
\end{aligned}$$

Applying Lemma 2.2, (3.5), and (3.6), we have

$$\begin{aligned}
\left|x_1^{(n)}\right|_\infty &= \max_{t\in[0,T]}\left|A^{-1}Ax_1^{(n)}(t)\right| \\
&\le \left(\max_{t\in[0,T]}\left|(Ax_1)^{(n)}(t) + c^{(n)}(t)x_1(t-\sigma)\right.\right. \\
&\quad \left.\left. + nc^{(n-1)}(t)x_1'(t-\sigma) + \cdots + nc'(t)x_1^{(n-1)}(t-\sigma)\right|\right)/\Gamma \\
&\le \frac{\varphi_q(|x_2|_\infty) + c_n|x_1|_\infty + nc_{n-1}|x_1'|_\infty + \cdots + nc_1|x_1^{(n-1)}|_\infty}{\Gamma} \\
&\le \left(\varphi_q\left(|x_2|_\infty\right) + c_n\left(D + \frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-1}\left|x_1^{(n)}\right|_\infty\right)\right. \\
&\quad \left. + nc_{n-1}\left(\frac{1}{2}T\left(\frac{T}{2\pi}\right)^{n-2}\left|x_1^{(n)}\right|_\infty\right) + \cdots + nc_1\frac{T}{2}\left|x_1^{(n)}\right|_\infty\right)/\Gamma
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\varphi_q\big(|x_2|_\infty\big) + \frac{T}{2}\left(\left(\frac{T}{2\pi}\right)^{n-1} c_n + nc_{n-1}\left(\frac{T}{2\pi}\right)^{n-2}\right.\right. \\
&\quad \left.\left. + \frac{n(n-1)}{2!} c_{n-2}\left(\frac{T}{2\pi}\right)^{n-3} + \cdots + nc_1\right)\big|x_1^{(n)}\big|_\infty + c_n D\right)\Big/\Gamma \\
&\le \frac{\varphi_q(|x_2|_\infty) + \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})|x_1^{(n)}|_\infty + c_n D}{\Gamma}.
\end{aligned}
$$

Since $\Gamma - \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k}) > 0$, we have

$$
\big|x_1^{(n)}\big|_\infty \le \frac{\varphi_q(|x_2|_\infty) + c_n D}{\Gamma - \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})}. \tag{3.7}
$$

In view of $\int_0^T (\varphi_q(x_2(t)))\,dt = \int_0^T (Ax_1(t))^{(n)}(t)\,dt = 0$, there exists a point $t_2 \in (0,T)$ such that $x_2(t_2) = 0$. From the Wirtinger inequality and from (3.4) we easily get

$$
\begin{aligned}
|x_2|_\infty &\le \frac{1}{2}\int_0^T \big|x_2'(t)\big|\,dt \le \frac{\sqrt{T}}{2}\left(\int_0^T \big|x_2'(t)\big|^2\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{T}}{2}\left(\frac{T}{2\pi}\right)^{m-2}\left(\int_0^T \big|x_2^{(m-1)}(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-2}\big|x_2^{(m-1)}\big|_\infty.
\end{aligned} \tag{3.8}
$$

Besides, from $x_2^{(m-2)}(0) = x_2^{(m-2)}(T)$, there exists a point $t_3 \in (0,T)$ such that $x_2^{(m-1)}(t_3) = 0$, which, together with the integration of the second equation of (3.1) on interval $[0,T]$, yield

$$
\begin{aligned}
2\big|x_2^{(m-1)}(t)\big| &\le 2\left(x_2^{(m-1)}(t_3) + \frac{1}{2}\int_0^T \big|x_2^{(m)}(t)\big|\,dt\right) \\
&\le \lambda\int_0^T \big|f\big(t,x_1'(t)\big) - g\big(t,x_1(t)\big) + e(t)\big|\,dt \\
&\le \int_0^T \big|f\big(t,x_1'(t)\big)\big|\,dt + \int_0^T \big|g\big(t,x(t)\big)\big|\,dt + \int_0^T \big|e(t)\big|\,dt \\
&\le KT + \int_0^T \big|g\big(t,x_1(t)\big)\big|\,dt + T|e|_\infty,
\end{aligned} \tag{3.9}
$$

since $|f(t,u)| \le K$ form (H_1). From (H_1) and (H_6) we have

$$
\begin{aligned}
\int_0^T \big|g\big(t,x_1(x)\big)\big|\,dt &= \int_{g(t,x_1(t))\ge 0} g\big(t,x_1(t)\big)\,dt - \int_{g(t,x_1(t))\le 0} g\big(t,x_1(t)\big)\,dt \\
&= 2\int_{g(t,x_1(t))\ge 0} g\big(t,x_1(t)\big)\,dt + \int_0^T f(t,x_1'(t)\,dt \\
&\le 2\int_0^T \big(\gamma x_1^{p-1} + \zeta\big)\,dt + \int_0^T \big|f\big(t,x_1'(t)\big)\big|\,dt \\
&\le 2\gamma|x_1|_\infty^{p-1}T + 2\zeta T + KT.
\end{aligned} \tag{3.10}
$$

Since $(1+x)^k \le 1+(1+k)x$ for $x\in[0,\delta]$, where δ is a constant, which depends only on $k>0$, substituting (3.10) into (3.9), we have

$$\begin{aligned}
2\left|x_2^{(m-1)}(t)\right| &\le 2T\gamma|x_1|_\infty^{p-1}+2\zeta T+2KT+T|e|_\infty \\
&\le 2T\gamma\left(D+\frac{1}{2}\int_0^T\left|x_1'(t)\right|dt\right)^{p-1}+N_1 \\
&= 2T\gamma\left(1+\frac{D}{\frac{1}{2}\int_0^T|x_1'(t)|\,dt}\right)^{p-1}\left(\frac{1}{2}\right)^{p-1}\left(\int_0^T\left|x_1'(t)\right|dt\right)^{p-1}+N_1 \\
&\le \frac{1}{2^{p-2}}T\gamma\left(1+\frac{2Dp}{\int_0^T|x_1'(t)|\,dt}\right)\left(\int_0^T\left|x_1'(t)\right|dt\right)^{p-1}+N_1,
\end{aligned} \tag{3.11}$$

where $N_1:=2\zeta T+2KT+T|e|_\infty$. Substituting (3.6) and (3.7) into (3.11), we have

$$\begin{aligned}
&2\left|x_2^{(m-1)}(t)\right| \\
&\quad\le \frac{T^p\gamma}{2^{p-2}}\left|x_1'\right|_\infty^{p-1}+\frac{DpT^{p-1}\gamma}{2^{p-3}}\left|x_1'\right|_\infty^{p-2}+N_1 \\
&\quad\le \frac{T^p\gamma}{2^{p-2}}\cdot\left(\frac{T}{2}\right)^{p-1}\left(\frac{T}{2\pi}\right)^{(n-2))p-1)}\left|x_1^{(n)}\right|_\infty^{p-1} \\
&\qquad+\frac{DpT^{p-1}\gamma}{2^{p-3}}\left(\frac{T}{2}\right)^{p-2}\left(\frac{T}{2\pi}\right)^{(n-2)(p-2)}\left|x_1^{(n)}\right|_\infty^{p-2}+N_1 \\
&\quad\le \frac{T^{2p-1}\gamma}{2^{2p-3}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)}\frac{(\varphi_q(|x_2|_\infty)+C_nD)^{p-1}}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}} \\
&\qquad+\frac{DpT^{2p-3}\gamma}{2^{2p-5}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-2)}\frac{(\varphi_q(|x_2|_\infty)+C_nD)^{p-2}}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-2}}+N_1.
\end{aligned} \tag{3.12}$$

Combining of (3.8) and (3.12) implies

$$\begin{aligned}
|x_2|_\infty &\le \frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-2}\left|x_2^{(m-1)}\right|_\infty \\
&\le \frac{T}{4}\left(\frac{T}{2\pi}\right)^{m-2}\left[\frac{T^{2p-1}\gamma}{2^{2p-3}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)}\frac{(\varphi_q(|x_2|_\infty)+C_nD)^{p-1}}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}}\right. \\
&\quad\left.+\frac{DpT^{2p-3}\gamma}{2^{2p-5}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-2)}\frac{(\varphi_q(|x_2|_\infty)+C_nD)^{p-2}}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-2}}+N_1\right].
\end{aligned} \tag{3.13}$$

So, we have

$$\begin{aligned}
|x_2|_\infty &\le \frac{T^{2p}\gamma}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)}\frac{|x_2|_\infty}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}} \\
&\quad+\frac{T^{2p}\gamma}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)}\frac{(\sum_{i=0}^{p-1}C_{p-1}^i(|x_2|_\infty^{q-1})^{p-1-i}(c_nD)^i)}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}}
\end{aligned}$$

$$+\frac{DpT^{2p-2}\gamma}{2^{2p-3}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-2)+(m-2)}\frac{(\sum_{i=0}^{p-1}C_{p-2}^{i}(|x_2|_\infty^{q-1})^{p-2-i}(c_nD)^i)}{(\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-2}}$$
$$+\frac{T}{4}\left(\frac{T}{2\pi}\right)^{m-2}N_1. \tag{3.14}$$

Since

$$\frac{T^{2p}}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)}\frac{\gamma}{(\Gamma-\frac{T}{2}\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k})^{p-1}}<1,$$

there exists a positive constant M_1 such that

$$|x_2|_\infty\le M_1. \tag{3.15}$$

Therefore, from (3.7) we have

$$\begin{aligned}\left|x_1^{(n)}\right|_\infty&\le\frac{\varphi_q(|x_2|_\infty)+c_nD}{\Gamma-\frac{T}{2}(\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k})}\\&\le\frac{M_1^{q-1}+c_nD}{\Gamma-\frac{T}{2}\sum_{k=0}^{n-1}C_n^kc_{n-k}(\frac{T}{2\pi})^{n-1-k}}:=M_n'.\end{aligned} \tag{3.16}$$

From (3.6) we have

$$\left|x_1'\right|_\infty\le\frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-2}\left|x_1^{(n)}\right|_\infty\le\frac{T}{2}\left(\frac{T}{2\pi}\right)^{n-2}M_n':=M_2. \tag{3.17}$$

Hence, from (3.4) we have

$$|x_1|_\infty\le D+\frac{1}{2}\int_0^T\left|x_1'(t)\right|dt\le D+\frac{TM_2}{2}:=M_3. \tag{3.18}$$

From (3.6), (3.9), and (3.10) we have

$$\begin{aligned}\left|x_2^{(m-1)}\right|_\infty&\le\frac{1}{2}\max\left|\int_0^T x_2^{(m)}(t)\,dt\right|\\&\le\frac{1}{2}\int_0^T\left|-f\big(t,x_1'(t)\big)-g\big(t,x_1(t)\big)+e(t)\right|dt\\&\le\frac{1}{2}\int_0^T\left|f(t,x_1'(t)\right|dt+\frac{1}{2}\int_0^T\left|g(t,x_1(t)\right|dt+\frac{1}{2}\int_0^T\left|e(t)\right|dt\\&\le KT+mM_3^{p-1}T+nT+\frac{1}{2}|e|_\infty T:=M_{m-1}.\end{aligned}$$

From (3.8) we get

$$\left|x_2'\right|_\infty\le\frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-3}\left|x_2^{(m-1)}\right|_\infty\le\frac{T}{2}\left(\frac{T}{2\pi}\right)^{m-3}M_{m-1}:=M_4.$$

On the other hand, since $g(t,x_1)=g_1(t,x_1(t))+g_0(x_1(t))$, (3.2) can be rewritten as

$$\big(\varphi_p(Ax_1)^{(n)}\big)^{(m)}+\lambda^pf\big(t,x_1'(t)\big)+\lambda^p\big(g_1\big(t,x_1(t)\big)+g_0\big(x(t)\big)\big)=\lambda^pe(t). \tag{3.19}$$

Let $\tau \in [0,T]$ for any $\tau \le t \le T$. Multiplying both sides of (3.19) by $x_1'(t)$ and integrating on $[\tau,t]$, we have

$$\begin{aligned}\lambda^p \int_{x_1(\tau)}^{x_1(t)} g_0(u)\,du &= \lambda^p \int_\tau^t g_0\bigl(x_1(s)\bigr)x_1'(s)\,ds \\ &= -\int_\tau^t \bigl(\varphi_p(Ax_1)^{(n)}(s)\bigr)^{(m)} x_1'(s)\,ds - \lambda^p \int_\tau^t f\bigl(s,x_1'(s)\bigr)x_1'(s)\,ds \\ &\quad - \lambda^p \int_\tau^t g_1\bigl(s,x_1(s)\bigr)x_1'(s)\,ds + \lambda^p \int_\tau^t e(s)x_1'(s)\,ds. \end{aligned} \tag{3.20}$$

By (3.2), (3.12), (3.17), and (3.18) we have

$$\begin{aligned}&\left|\int_\tau^t \bigl(\varphi_p(Ax_1)^{(n)}(s)\bigr)^{(m)} x_1(s)\,ds\right| \\ &\quad\le \int_\tau^t \bigl|\bigl(\varphi_p(Ax_1)^{(n)}(s)\bigr)^{(m)}\bigr|\bigl|x_1'(s)\bigr|\,ds \\ &\quad\le \bigl|x_1'\bigr|_\infty \int_\tau^t \bigl|\bigl(\varphi_p(Ax_1)^{(n)}(s)\bigr)^{(m)}\bigr|\,ds \\ &\quad\le \lambda^p M_2\left(\int_0^T \bigl|f\bigl(t,x_1'(t)\bigr)\bigr|\,dt + \int_0^T \bigl|g\bigl(t,x_1(t)\bigr)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt\right) \\ &\quad\le \lambda^p M_2\bigl(2KT + 2mM_3^{p-1}T + 2nT + |e|_\infty T\bigr). \end{aligned} \tag{3.21}$$

Moreover, we have

$$\begin{aligned}&\left|\int_\tau^t f\bigl(s,x_1'(s)\bigr)x_1'(s)\,ds\right| \le \bigl|x_1'\bigr|_\infty \int_0^T \bigl|f\bigl(t,x_1'(t)\bigr)\bigr|\,dt \le M_2KT, \\ &\left|\int_\tau^t g_1\bigl(s,x_1(s)\bigr)x_1'(s)\,ds\right| \le \bigl|x_1'\bigr|_\infty \int_0^T \bigl|g_1\bigl(t,x_1(t)\bigr)\bigr|\,dt \le M_2\sqrt{T}\|g_{M_3}\|_2, \\ &\left|\int_\tau^t e(s)x_1'(s)\,ds\right| \le \bigl|x_1'\bigr|_\infty \int_0^T \bigl|e(t)\bigr|\,dt \le M_2|e|_\infty T, \end{aligned} \tag{3.22}$$

where $g_{M_3} := \max_{0<x\le M_3} |g_1(t,u)| \in L^2(0,T)$ and $\|g_{M_3}\|_2 := (\int_0^T |g_1(t,x_1'(t))|^2\,dt)^{\frac{1}{2}}$. Substituting (3.21) and (3.22) into (3.20), we have

$$\left|\int_{x_1(\tau)}^{x_1(t)} g_0(x)\,dx\right| \le M_2\bigl(3KT + 2mM_3^{p-1}T + 2nT + \sqrt{T}\|g_{M_3}\|_2 + 2|e|_\infty T\bigr) := M_5^*.$$

From repulsive singular condition (H_7) we know that there exists a constant $M_5 > 0$ such that

$$x_1(t) \ge M_5, \quad \forall t \in [\tau,T]. \tag{3.23}$$

The case $t \in [0,\tau]$ can be treated similarly.

Let

$$\Omega_2 = \bigl\{x = (x_1,x_2)^\top : E_5 < x_1(t) < E_1, \bigl|x_1'\bigr|_\infty < E_2, |x_2|_\infty < E_3, \bigl|x_2'\bigr|_\infty < E_4\bigr\},$$

where $0 < E_5 < M_5$, $E_1 > \max\{D, M_3\}$, $E_2 > M_2$, $E_3 > M_1$, and $E_4 > M_4$. Next, we shall prove that Ω_2 is a bounded set. In fact, for all $x \in \Omega_2$, $x_2 = 0$, $x_1 = a_0\phi(t)$, and $a_0 \in \mathbb{R}^+$, we have

$$0 = \int_0^T g\bigl(t, a_0\phi(t)\bigr)\,dt.$$

From assumption (H_1) we have $0 < a_0\phi(t) \le D$. So Ω_2 is a bounded set.

Let $\Omega = \{x \in (x_1, x_2)^\top : \|x\| \le M\}$, where $M = \max\{E_1, E_2, E_3, E_4\}$. Then $\Omega_1 \cup \Omega_2 \subset \Omega$, and, as it follows from the above proof, $Lx \ne \lambda Nx$ for all $(x, \lambda) \in \partial\Omega \times (0, 1)$, so that conditions (1) and (2) of Lemma 2.2 are both satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x_1, x_2)^\top = (x_2, -x_1)^\top.$$

Let $H(\mu, x) = -\mu x + (1 - \mu)JQNx$, $(\mu, x) \in [0, 1] \times \Omega$. Then, for all $(\mu, x) \in (0, 1) \times (\partial\Omega \cap \operatorname{Ker} L)$,

$$H(\mu, x) = \begin{pmatrix} -\mu x_1(t) - \frac{1-\mu}{T}\int_0^T g(t, x_1(t))\,dt \\ -\mu x_2(t) - (1-\mu)\varphi_q(x_2(t)) \end{pmatrix},$$

since $\int_0^T e(t)\,dt = 0$ and $f(t, 0) = 0$. From (H_4) it is obvious that $x^\top H(\mu, x) < 0$ for all $(\mu, x) \in (0, 1) \times (\partial\Omega \cap \operatorname{Ker} L)$. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(0, x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(1, x), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \ne 0. \end{aligned}$$

So condition (3) of Lemma 2.2 is satisfied. Applying Lemma 2.2, we conclude that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ on $\bar{\Omega} \cap D(L)$, that is, (1.1) has a T-periodic solution $x_1(t)$. □

Theorem 3.2 *Assume that* (H_2)-(H_3) *and* (H_5)-(H_7) *hold. Then* (1.1) *has at least a non-constant T-periodic solution if*

$$0 < \frac{(\frac{T^{2p}\gamma}{2^{2p-1}} + \frac{T^{p+1}\alpha}{2^p})(\frac{T}{2\pi})^{m+n-4}}{(\Gamma - \frac{T}{2}\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})^{p-1}} < 1.$$

Proof We follow the same strategy and notation as the proof of Theorem 3.1. Now, we consider $\|x'\| \le M_2$.

We first claim that there is a constant $\xi^* \in [0, T]$ such that

$$0 < x_1\bigl(\xi^*\bigr) \le D_1. \tag{3.24}$$

Since $\int_0^T (\varphi_p(Ax_1)'(t))'dt = 0$, there exist two points $\xi^*, \xi_* \in [0, T]$ such that

$$\bigl(\varphi_p(Ax_1)'\bigl(\xi^*\bigr)\bigr)' \ge 0 \quad \text{and} \quad \bigl(\varphi_p(Ax_1)'(\xi_*)\bigr)' \le 0.$$

From (H_3) and (3.2) we have

$$g(\xi^*, x_1(\xi^*)) - e(\xi^*) \le -f(\xi^*, x_1'(\xi^*)) \le 0,$$

since $f(\xi^*, x_1'(\xi^*)) > 0$. Therefore, we get

$$g(\xi^*, x_1'(\xi^*)) \le e(\xi^*) \le |e|_\infty.$$

From (H_5) we have

$$x_1(\xi) \le D_1.$$

Since $x(t) > 0$, we get $0 < x_1(\xi^*) \le D_1$. This proves (3.24).

Similarly, from (3.4) we have

$$\left|x_1(t)\right| \le D_1 + \frac{1}{2}\int_0^T \left|x_1'(t)\right| dt. \tag{3.25}$$

From (3.9) and (H_2) we get

$$\begin{aligned} 2\left|x_2^{(m-1)}(t)\right| &\le 2\left(x_2^{(m-1)}(t_3) + \frac{1}{2}\int_0^T \left|x_2^{(m)}(t)\right| dt\right) \\ &\le \lambda \int_0^T \left|f(t, x_1'(t)) - g(t, x_1(t)) + e(t)\right| dt \\ &\le \int_0^T \left|f(t, x_1'(t))\right| dt + \int_0^T \left|g(t, x(t))\right| dt + \int_0^T \left|e(t)\right| dt \\ &\le \alpha \int_0^T \left|x_1'(t)\right|^{p-1} dt + \beta T + \int_0^T \left|g(t, x_1(t))\right| dt + T|e|_\infty. \end{aligned} \tag{3.26}$$

From (3.10), (H_2), and (H_7) we have

$$\begin{aligned} \int_0^T \left|g(t, x_1(t))\right| dt &= \int_{g(t,x_1(t))\ge 0} g(t, x_1(t))\, dt - \int_{g(t,x_1(t))<0} g(t, x_1(t))\, dt \\ &= 2\int_{g(t,x_1(t))\ge 0} g(t, x_1(t))\, dt + \int_0^T f(t, x_1'(t))\, dt \\ &\le 2\int_{g(t,x_1(t))\ge 0} \left(\gamma x_1^{p-1}(t) + \zeta\right) dt + \int_0^T \left|f(t, x_1'(t))\right| dt \\ &\le 2\gamma |x_1|^{p-1} T + 2\zeta T + \alpha \int_0^T \left|x_1'(t)\right|^{p-1} dt + \beta T. \end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26), from (3.11) we have

$$\begin{aligned} 2\left|x_2^{(m-1)}(t)\right| &\le 2\gamma \int_0^T \left|x(t)\right|^{p-1} dt + 2\zeta T + 2\alpha \int_0^T \left|x'(t)\right|^{p-1} dt + 2\beta T + |e|_\infty T \\ &\le \left(\frac{T^p \gamma}{2^{p-2}} + 2\alpha T\right) \left|x_1'\right|_\infty^{p-1} + \frac{DpT^{p-1}\gamma}{2^{p-3}} \left|x_1'\right|_\infty^{p-2} + N_2, \end{aligned} \tag{3.28}$$

where $N_2 = 2T(\zeta + \beta) + \|e\|T$. From (3.12), (3.13), and (3.14) we get

$$\begin{aligned}|x_2|_\infty \le{}& \left(\frac{T^{2p}\gamma}{2^{2p-1}} + \frac{T^{p+1}\alpha}{2^p}\right)\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)} \frac{|x_2|_\infty}{(\Gamma - \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}}\\ &+ \frac{T^{2p}\gamma}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)} \frac{(\sum_{i=0}^{p-1} C_{p-1}^i (|x_2|_\infty^{q-1})^{p-1-i}(c_nD)^i)}{(\Gamma - \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-1}}\\ &+ \frac{DpT^{2p-2}\gamma}{2^{2p-3}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-2)+(m-2)} \frac{(\sum_{i=0}^{p-1} C_{p-2}^i (|x_2|_\infty^{q-1})^{p-2-i}(c_nD)^i)}{(\Gamma - \frac{T}{2}(\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k}))^{p-2}}\\ &+ \frac{T}{4}\left(\frac{T}{2\pi}\right)^{m-2} N_2.\end{aligned}$$

Since $\frac{(\frac{T^{2p}\gamma}{2^{2p-1}} + \frac{T^{p+1}\alpha}{2^p})(\frac{T}{2\pi})^{(n-2)(p-1)+(m-2)}}{(\Gamma - \frac{T}{2}\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})^{p-1}} < 1$, it is easy to see that there exists a positive constant M_2 such that

$$\|x'\| \le M_2.$$

The rest of the proof is the same as in Theorem 3.1. □

We illustrate our results with an example.

Example 3.1 Consider the neutral functional differential

$$\begin{aligned}&\left(\varphi_p\left(x(t) - \frac{1}{64}\sin(4t)x(t-\sigma)\right)'''\right)''' + \cos^2(2t)\sin x'(t) + \frac{1}{4\pi}\big(\sin(4t)+3\big)x^3(t) - \frac{1}{x^\mu}\\ &\quad = 5\cos(4t), \end{aligned} \tag{3.29}$$

where $p = 4$, σ and μ are constants, and $0 < \sigma < T$. It is clear that $T = \frac{\pi}{2}$, $n = 3$, $m = 3$, $c(t) = \frac{1}{64}\sin 4t$, $e(t) = 5\cos 4t$, $c_1 = \max_{t\in[0,T]} |\frac{1}{16}\cos 4t| = \frac{1}{16}$, $c_2 = \max_{t\in[0,T]} |-\frac{1}{4}\sin 4t| = \frac{1}{4}$, and $c_3 = \max_{t\in[0,T]} |-\cos 4t| = 1$. In this case, $f(t,u) = \cos^2(2t)\sin u$, $f(t,0) = 0$, $|f(t,u)| = |\cos^2(2t)\sin u| \le 1$, $K = 1$; and $g(t,x) = \frac{1}{4\pi}(\sin 4t + 3)x^3(t) - \frac{1}{x^\mu} \le \frac{1}{\pi}x^3 + 1$, $\gamma = \frac{1}{\pi}$, $\zeta = 1$; Obviously, conditions (H_1) and (H_6)-(H_7) hold. Choose $D = 4\pi$ such that (H_4) holds. Now we consider the following condition:

$$\begin{aligned}&\frac{T^{2p}}{2^{2p-1}}\left(\frac{T}{2\pi}\right)^{(n-2)(p-1)+(m-2)} \frac{\gamma}{(\Gamma - \frac{T}{2}\sum_{k=0}^{n-1} C_n^k c_{n-k}(\frac{T}{2\pi})^{n-1-k})^{p-1}}\\ &\quad = \frac{(\frac{\pi}{2})^8}{2^7}\left(\frac{\frac{\pi}{2}}{2\pi}\right)^4 \frac{\frac{1}{\pi}}{(\frac{63}{64} - \frac{\pi}{4}\times(1\times\frac{1}{16} + 3\times\frac{1}{4}\times\frac{1}{4} + 3\times\frac{1}{16}))^3}\\ &\quad \approx \frac{\pi^8}{2^{26}} < 1.\end{aligned}$$

So, by Theorem 3.1, (3.29) has at least one nonconstant $\frac{\pi}{2}$-periodic solution.

4 Conclusions

In summary, a periodic solution of (1.1) with singularity is illustrated by Theorems 3.1 and 3.2. In Theorem 3.1, we consider the existence of a periodic solution for (1.1) in the case

$|f(t,u)| \leq K$. Furthermore, in Theorem 3.2, we give a condition on $f(t,u)$ that is weaker than $|f(,u)| \leq K$ in Theorem 3.1, that is, we obtain the existence of periodic solution for (1.1) in the case where $|f(t,u)| \leq \alpha|u|^{p-1} + \beta$. From the mathematical point of view, the results are valuable to understand the periodic solutions for high-order neutral differential equations.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YX, SWY, and ZBC worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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