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Numerical solution of Volterra partial integro-differential equations based on sinc-collocation method

Atefeh Fahim[1], Mohammad Ali Fariborzi Araghi[1*], Jalil Rashidinia[1,2] and Mehdi Jalalvand[3]

*Correspondence:
fariborzi.araghi@gmail.com
[1]Department of Mathematics, Central Tehran Branch, Islamic Azad University, Simaye Iran Street, Tehran, 14676-86831, Iran
Full list of author information is available at the end of the article

Abstract
We provide the numerical solution of a Volterra integro-differential equation of parabolic type with memory term subject to initial boundary value conditions. Finite difference method in combination with product trapezoidal integration rule is used to discretize the equation in time and sinc-collocation method is employed in space. A weakly singular kernel has been viewed as an important case in this study. The convergence analysis has been discussed in detail, which shows that the approach exponentially converges to the solution. Furthermore, numerical examples and illustrations are presented to prove the validity of the suggested method.

MSC: 65R99; 45A99; 45K05

Keywords: partial integro-differential equation; sinc-collocation method; finite difference method; product trapezoidal integration rule

1 Introduction

We consider a Volterra integro-differential equation with memory term of the form

$$u_t(x,t) = \int_0^t k_0(t-s)u_{xx}(x,s)\,ds + f(x,t), \quad x \in \Omega, t \in J, \tag{1}$$

subjected to initial and boundary conditions

$$\begin{aligned} &u(a,t) = u(b,t) = 0, \quad t \in J, \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{2}$$

where $\Omega = [a,b] \subseteq \mathbb{R}$ and $J = [0,T]$. Here $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and k_0 is a real-valued and positive definite kernel, that is,

$$\int_0^T \varphi(t) \int_0^t k_0(t-s)\varphi(s)\,ds\,dt \geq 0 \tag{3}$$

for all $T > 0$ and any continuous $\varphi : [0,T] \longrightarrow \mathbb{R}$, and f is a real-valued function. If k_0 is a smooth function on $\mathbb{R}^+$, equation (1) is hyperbolic, whereas if k_0 has a weak singularity at

0, such as $k_0(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $0 < \beta < 1$, then it adopts a parabolic behavior [1–3]. With $u_{xx} = \nabla^2 u$, the evolution equation (1) is sometimes called a fractional wave equation [4], because in the limiting case where $\beta = 1$, after differentiation with respect to t, we obtain

$$u_{tt}(x,t) = \nabla^2 u(x,t) + f'(x,t),$$

and as $\beta \to 0$, we get the heat equation

$$u_t(x,t) = \nabla^2 u(x,t) + f(x,t).$$

Modeling phenomena in viscoelasticity, biological models, chemical kinetics, heat conduction in materials with memory, population dynamics, fluid dynamics and nuclear reactor dynamics, mathematical biology, financial mathematics, compression of viscoelastic media, and other similar areas are all done by partial integro-differential equations of type (1). See, for example, [5] and the references therein. This problem governs many physical systems occurring in diffusion problems as a particular case [6].

To treat the partial integro-differential equations (PIDEs), a substantial number of methods have been applied. For example, the pseudo-spectral Legendre-Galerkin method for solving a parabolic PIDE with convolution-type kernel was presented in [7]. Combination of radial basis functions and finite difference for solving nonlinear-type PIDEs with smooth kernel containing an unknown function was considered in [8]. Also, a spectral method was proposed in [9] for the PIDEs with a weakly singular kernel.

The numerical solution of equation (1) with a weakly singular kernel was considered by many authors, such as finite-element methods [3, 10], finite-difference methods [11, 12], compact difference schemes [13], spectral collocation methods [14], orthogonal spline collocation methods [15], variational iteration and Adomian decomposition methods [16], radial basis functions methods [17], and quasi-wavelet methods [18]. However, construction of precise numerical methods for integro-differential equations is still a challenge owing to the weak singularity of the kernel k_0 that contains sharp states of transitions in the solution. This lack of smoothness of the solution near $t = 0$ results in a decay in the order of the practical performance of familiar timestepping methods for equation (1). For instance, the trapezoidal rule with product integration of the quadrature term does not produce expected $\mathcal{O}(\Delta t^2)$ errors [19].

The sinc approximation has been studied by many authors to solve various equations such as integral equations [20], ordinary differential equations [21], partial differential equations [22–24], integro-differential equations [25], and so on, due to high accuracy, exponential rate of convergence, and near optimality of this method [26]. With these backgrounds, we extend the sinc-collocation method for solving partial integro-differential equations of type (1).

In this paper, the time discretization method to solve equation (1) is effected by a combination of finite difference and quadrature. For this purpose, we apply the backward Euler method in addition to the product trapezoidal integration rule [19] for the integral term. Consequently, equation (1) is reduced to a system of ordinary differential equations (ODEs), which is discretized with the sinc-collocation method. In addition, the accuracy and efficiency of the suggested method is tested with some examples and illustrations.

This paper is organized as follows. Section 2 provides some basic definitions, assumptions, and preliminaries of sinc approximation. In Section 3, we develop the sinc collocation method to solve Volterra partial integro-differential equations. In Section 4, we discuss the convergence analysis of the proposed method. Finally, in Section 5, numerical examples are solved to verify the accuracy and efficiency of the proposed approach.

2 Preliminaries

The goal of this section is to recall notation and definitions of the sinc function and state some known theorems important for the rest of this paper, which were discussed thoroughly in [27, 28].

The sinc method is basically defined on the real line. So, the sinc function is defined on the whole real line by

$$\operatorname{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

and the translated sinc functions with evenly spaced nodes are given as

$$S(j,h)(z) = \operatorname{sinc}\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots. \tag{4}$$

The sinc function at the interpolating points $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

They are based on the infinite strip D_d in the complex plane

$$D_d = \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

Let f be a function defined on $\mathbb{R}$, and let $h > 0$ be the mesh size. Then the Whittaker cardinal function is defined by the infinite series as follows:

$$C(f,h,x) = \sum_{j=-\infty}^{\infty} f(jh) S(j,h)(x).$$

However, in practice, the finite number of terms are used in this series such as $j = -N, \ldots, N$, where $2N + 1$ is the number of sinc grid points. So,

$$C(f,h,x) \approx \sum_{j=-N}^{N} f(jh) S(j,h)(x),$$

where h is suitably selected depending on the properties of the function f and given positive integer N.

To construct an approximation on the interval $\Gamma = [a,b]$, we consider the conformal map

$$\phi(z) = \log\left(\frac{z-a}{b-z}\right). \tag{5}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \leq \frac{\pi}{2}\right\}$$

onto D_d such that $\phi(a) = -\infty$, $\phi(b) = \infty$, where a, b are the boundary points of D_E with $a, b \in \partial D_E$. For the sinc method on the interval $\Gamma = [a,b]$, basis functions are derived from the composite translated sinc functions

$$S_j(z) = S(j,h) \circ \big(\phi(z)\big) = \operatorname{sinc}\left(\frac{\phi(z) - jh}{h}\right), \quad j = 0, \pm1, \pm2, \ldots.$$

The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w}.$$

Let ψ denote the inverse map of ϕ, so we define the range of ϕ^{-1} on the real line as

$$\Gamma = \big\{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\big\} = [a,b].$$

For $h > 0$, let the points x_k on Γ be given by

$$x_k = \psi(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k \in \mathbb{Z}. \tag{6}$$

Definition 1 ([27], p. 59) Let $B(D_E)$ denote the class of functions f analytic in D_E such that, for some constant γ with $0 \leq \gamma < 1$,

$$\int_{\psi(u+\sum)} \big|f(z)\,dz\big| = \mathcal{O}\big(|x|^\gamma\big), \quad u \to \pm\infty,$$

where $\sum = \{i\eta : |\eta| < d \leq \frac{\pi}{2}\}$, and, for a simple closed contour δ in D_E,

$$N(f, D_E) \equiv \lim_{\delta \to \partial D_E} \int_\delta \big|f(z)\,dz\big| < \infty,$$

where ∂D_E represents the boundary of D_E.

Definition 2 ([28], p. 180) By $L_\alpha(D_E)$ we denote the set of all analytic functions f for which there exists a constant, C such that

$$\big|f(z)\big| \leq C\frac{|\rho(z)|^\alpha}{(1 + |\rho(z)|)^{2\alpha}}, \quad z \in D_E, 0 < \alpha \leq 1, \tag{7}$$

where $\rho(z) = e^{\phi(z)}$.

The following theorem presents the convergence result on the approximation of derivatives particularly useful for approximate solving some differential equations.

Theorem 1 ([28], p. 208) *If $\phi' u \in B(D_E)$ and*

$$\sup_{\frac{-\pi}{h} \le t \le \frac{\pi}{h}} \left| \left(\frac{d}{dx}\right)^l e^{it\phi(x)} \right| \le C_1 h^{-l}, \quad x \in \Gamma,$$

for $l = 0, 1, \ldots, m$ with a constant C_1 depending only on m and ϕ. If $u \in L_\alpha(D_E)$, then taking $h = \sqrt{\pi d/\alpha N}$ it follows that

$$\sup_{x\in\Gamma} \left| u^{(l)}(x) - \left(\frac{d}{dx}\right)^l \sum_{j=-N}^{N} u(x_j) S_j(x) \right| \le C N^{(l+1)/2} exp\bigl(-(\pi d\alpha N)^{1/2}\bigr),$$

where C is a constant depending only on u, d, m, ϕ, and α.

The sinc-collocation method requires the derivatives of the composite sinc function to be evaluated at the nodes. So, we need to recall the following lemma.

Lemma 1 ([27], p. 106) *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d given by* (5). *Then*

$$\delta_{jk}^{(0)} = \bigl[S(j,h) \circ \phi(x)\bigr]\Big|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \ne k, \end{cases} \tag{8}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi}\bigl[S(j,h) \circ \phi(x)\bigr]\Big|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \ne k, \end{cases} \tag{9}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2}\bigl[S(j,h) \circ \phi(x)\bigr]\Big|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \ne k. \end{cases} \tag{10}$$

In equations (8)-(10), *h is step size, and x_k is the sinc grid given by* (6).

3 Description of the method

In this section, we give the sinc-collocation method for solving the partial integro-differential equation with kernel $k_0(t-s) = (t-s)^{-\beta}$:

$$u_t(x,t) = \int_0^t (t-s)^{-\beta} u_{xx}(x,s)\, ds + f(x,t), \quad 0 < x < 1, t \in J, \tag{11}$$

with boundary and initial conditions

$$\begin{aligned} &u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \\ &u(x,0) = u_0(x), \quad 0 \le x \le 1. \end{aligned} \tag{12}$$

The parameter β shows the order of singularity at the point $s = x$, and we assume that $0 < \beta < 1$. Since the integral kernel has this kind of singularity, equation (11) is said to have a weakly singular kernel. First of all, a description of the spatial-temporal discretization for this type of equations is provided in detail. The sinc-collocation algorithm is then described for solving equation (11).

3.1 Discretization in time

Now, the backward Euler method is applied for time derivatives in equation (11). Let $t_n = n\Delta t$ with time step Δt, $u^n = u(x, t_n)$, and $f^n = f(x, t_n)$ for $n = 0, 1, \dots, M$, $M = [\frac{T}{k}]$, $k \in \mathbb{N}$. By substituting $t = t_{n+1}$ into the left-hand side of (11) for the first term, we have

$$u_t(x, t_{n+1}) \approx \frac{u^{n+1}(x) - u^n(x)}{\Delta t} + R_{n+1,1}, \quad 0 < x < 1, n \geq 0, \tag{13}$$

where $R_{n+1,1} = \mathcal{O}(\Delta t)$ is the order of the backward Euler method. The integral term of (11) can be approximated by unusual quadrature approximation, that is, a kind of the product trapezoidal integration rule [19] as follows:

$$\begin{aligned} \int_0^{t_{n+1}} (t_{n+1} - s)^{-\beta} u_{xx}(x, s)\, ds &= \sum_{l=0}^{n} \int_{t_l}^{t_{l+1}} (t_{n+1} - s)^{-\beta} u_{xx}(x, s)\, ds \\ &\approx \sum_{l=0}^{n} \int_{t_l}^{t_{l+1}} (t_{n+1} - s)^{-\beta} \left\{ \frac{t_{l+1} - s}{\Delta t} u_{xx}^{l}(x) + \frac{s - t_l}{\Delta t} u_{xx}^{l+1}(x) \right\} ds \\ &\approx \frac{1}{\Delta t} \sum_{l=0}^{n} \bigl(A_{n,l} u_{xx}^{l}(x) + B_{n,l} u_{xx}^{l+1}(x) \bigr) + R_{n+1,2}, \end{aligned} \tag{14}$$

where $R_{n+1,2} = \mathcal{O}(\Delta t^{2-\beta})$ is the order of the product trapezoidal integration rule, proved by Dixon [29], and

$$\begin{aligned} A_{n,l} &= \int_{t_l}^{t_{l+1}} (t_{n+1} - s)^{-\beta} (t_{l+1} - s)\, ds, \\ B_{n,l} &= \int_{t_l}^{t_{l+1}} (t_{n+1} - s)^{-\beta} (s - t_l)\, ds. \end{aligned} \tag{15}$$

Substituting equations (13) and (14) into equation (11), we get the temporal semi-discrete form of (11) as follows:

$$\begin{aligned} &u^{n+1}(x) - B_{n,n} u_{xx}^{n+1}(x) = \Delta t f^{n+1}(x) + u^n(x) + \sum_{l=0}^{n} \rho_{n,l} u_{xx}^{l}(x) + R_{n+1}, \\ &u^{n+1}(0) = 0, \qquad u^{n+1}(1) = 0, \end{aligned} \tag{16}$$

where

$$|R_{n+1}| \leq \min\bigl\{ |R_{n+1,1}|, |R_{n+1,2}| \bigr\},$$

and

$$\begin{aligned} \rho_{n,0} &= A_{n,0}, \\ \rho_{n,l} &= A_{n,l} + B_{n,l-1}, \quad l = 1, 2, \dots, n, \end{aligned} \tag{17}$$

and with additional initial condition

$$u^0(x) = u_0(x). \tag{18}$$

Ignoring the small error term R_{n+1}, we arrive at the semidiscrete scheme

$$u^{n+1}(x) - B_{n,n}u_{xx}^{n+1}(x) = \Delta t f^{n+1}(x) + u^n(x) + \sum_{l=0}^{n} \rho_{n,l} u_{xx}^l(x), \quad 0 < x < 1, n \geq 0. \tag{19}$$

The scheme (19) is implicit because the integral term depends on u^{n+1} and is accurate of order $R_{n+1} = \mathcal{O}(\Delta t)$. In fact, we find that

$$u^1(x) - B_{0,0}u_{xx}^1(x) = \Delta t f^1(x) + u^0(x),$$

and, for $n \geq 1$, by applying (19) at each step the right-hand side involves the solution at all previous time levels. As a consequence, we have a linear ordinary differential equation in the form (19) with boundary conditions (16) in each time level. Now, in each time level, we can use the sinc-collocation method to estimate the solution of the linear boundary value problem (19)-(16).

3.2 Discretization in space: sinc-collocation method

We discretize the spatial direction by the described sinc-collocation method. Assume that the approximate solution of (19) defined by

$$u_m^n(x) = \sum_{j=-N}^{N} c_j^n S(j,h) \circ \phi(x), \quad m = 2N + 1, \tag{20}$$

and

$$\phi(x) = \log\left(\frac{x}{1-x}\right) \tag{21}$$

and that the unknown coefficients c_j^n in (20) are determined by the sinc-collocation method. The points in the sinc-collocation method are

$$x_k = \frac{e^{kh}}{1 + e^{kh}}, \quad k = -N, \dots, N, h = \sqrt{\frac{\pi d}{\alpha N}}, \tag{22}$$

so

$$\begin{aligned} \frac{d^2}{dx^2} u_m^n(x) &= \sum_{j=-N}^{N} c_j^n \frac{d^2}{dx^2} \bigl[S(j,h) \circ \phi(x)\bigr] \\ &= \sum_{j=-N}^{N} c_j^n \bigl[\phi''(x) S_j^{(1)}(x) + \bigl(\phi'(x)\bigr)^2 S_j^{(2)}(x)\bigr], \end{aligned} \tag{23}$$

where

$$S_j^{(l)}(x) = \frac{d^{(l)}}{d\phi^{(l)}} \bigl[S(j,h) \circ \phi(x)\bigr], \quad l = 1, 2. \tag{24}$$

Thus, by Theorem 2,

$$\frac{d^2}{dx^2}u_m^n(x_i) = \sum_{j=-N}^{N} c_j^n \left[\phi''(x_i)\frac{\delta_{ji}^{(1)}}{h} + \left(\phi'(x_i)\right)^2 \frac{\delta_{ji}^{(2)}}{h^2} \right]. \tag{25}$$

By substituting (20) and (25) into (19) we have

$$\begin{aligned} &\sum_{j=-N}^{N} c_j^{n+1}\delta_{ji}^{(0)} - B_{n,n}\sum_{j=-N}^{N} c_j^{n+1}\left[\phi''(x_i)\frac{\delta_{ji}^{(1)}}{h} + \left(\phi'(x_i)\right)^2 \frac{\delta_{ji}^{(2)}}{h^2} \right] \\ &\quad = \Delta t f_i^{n+1} + \sum_{j=-N}^{N} c_j^n \delta_{ji}^{(0)} + \sum_{l=0}^{n}\sum_{j=-N}^{N} \rho_{n,l} c_j^l \left[\phi''(x_i)\frac{\delta_{ji}^{(1)}}{h} + \left(\phi'(x_i)\right)^2 \frac{\delta_{ji}^{(2)}}{h^2} \right] \end{aligned} \tag{26}$$

with additional initial condition

$$c_i^0 = u_0(x_i), \quad i = -N, \ldots, N. \tag{27}$$

Note that $\delta_{ji}^{(0)} = \delta_{ij}^{(0)}$, $\delta_{ji}^{(1)} = -\delta_{ij}^{(1)}$, and $\delta_{ji}^{(2)} = \delta_{ij}^{(2)}$. We denote $I^{(r)} = [\delta_{ij}^{(r)}]$, $r = 0, 1, 2$, where $I^{(0)}$ is the identity matrix, and $I^{(1)}$ and $I^{(2)}$ are symmetric and skew-symmetric Toplitz matrices of order $2N+1$, respectively. We define the $(2N+1)\times(2N+1)$ diagonal matrix as follows:

$$D\bigl(g(x)\bigr)_{ij} = \begin{cases} g(x_i), & i = j, \\ 0, & i \neq j. \end{cases} \tag{28}$$

By multiplying both sides of (26) by $\frac{1}{(\phi'(x_i))^2}$ we have

$$\begin{aligned} &\frac{1}{(\phi'(x_i))^2} c_i^{n+1} - B_{n,n}\sum_{j=-N}^{N}\left[\left(\frac{-\phi''(x_i)}{(\phi'(x_i))^2} \right)\frac{\delta_{ij}^{(1)}}{h} + \frac{1}{h^2}\delta_{ij}^{(2)} \right] c_j^{n+1} \\ &\quad = \frac{\Delta t}{(\phi'(x_i))^2} f_i^{n+1} + \frac{1}{(\phi'(x_i))^2} c_i^n + \sum_{l=0}^{n}\sum_{j=-N}^{N} \rho_{n,l}\left[\left(\frac{-\phi''(x_i)}{(\phi'(x_i))^2} \right)\frac{\delta_{ij}^{(1)}}{h} + \frac{1}{h^2}\delta_{ij}^{(2)} \right] c_j^l. \end{aligned} \tag{29}$$

Therefore, system (29) can be written in a matrix form as

$$\begin{aligned} &\left(D\left(\left(\frac{1}{\phi'}\right)^2 \right) C^{n+1} - B_{n,n}\left[\frac{1}{h} D\left(\left(\frac{1}{\phi'}\right)' \right) I^{(1)} + \frac{1}{h^2} I^{(2)} \right] \right) C^{n+1} \\ &\quad = \Delta t D\left(\left(\frac{1}{\phi'}\right)^2 \right) F^{n+1} + D\left(\left(\frac{1}{\phi'}\right)^2 \right) C^n \\ &\qquad + \sum_{l=0}^{n} \rho_{n,l}\left[\frac{1}{h} D\left(\left(\frac{1}{\phi'}\right)' \right) I^{(1)} + \frac{1}{h^2} I^{(2)} \right] C^l \end{aligned} \tag{30}$$

or in a compact form as

$$PC^{n+1} = R\bigl(\Delta t F^{n+1} + C^n\bigr) + \sum_{l=0}^{n} \rho_{n,l} Q C^l, \tag{31}$$

where

$$\begin{aligned} Q &= \frac{1}{h} D\left(\left(\frac{1}{\phi'}\right)'\right) I^{(1)} + \frac{1}{h^2} I^{(2)}, \\ R &= D\left(\left(\frac{1}{\phi'}\right)^2\right), \\ P &= R - B_{n,n} Q, \end{aligned} \tag{32}$$

and

$$C^{n+1} = \left(c_{-N}^{n+1}, c_{-N+1}^{n+1}, \dots, c_{N}^{n+1}\right)^t, \qquad F^{n+1} = \left(f_{-N}^{n+1}, f_{-N+1}^{n+1}, \dots, f_{N}^{n+1}\right)^t. \tag{33}$$

If we set

$$G^{n+1} = R\left(\Delta t F^{n+1} + C^n\right) + \sum_{l=0}^{n} \rho_{n,l} Q C^l, \tag{34}$$

then the system of equations can be written as follows:

$$PC^{n+1} = G^{n+1} \tag{35}$$

with additional initial condition

$$C^0 = \left(u_0(x_{-N}), u_0(x_{-N+1}), \dots, u_0(x_N)\right)^t. \tag{36}$$

For each n, system (35) is a linear system of equations consisting of $2N + 1$ equations and $2N + 1$ unknowns. The coefficients c_j^n in the approximate solution (20) can be determined by solving this linear system.

4 Convergence analysis

In this section, we consider the ODE (19), and for simplicity, we can rewrite it as

$$u^{n+1}(x) - B_{n,n} \frac{d^2}{dx^2}\left(u^{n+1}(x)\right) = g(x), \tag{37}$$

where

$$g(x) = \Delta t f^{n+1}(x) + u^n(x) + \sum_{l=0}^{n} \rho_{n,l} \frac{d^2}{dx^2}\left(u^l(x)\right),$$

associated with boundary conditions

$$u^{n+1}(0) = u^{n+1}(1) = 0.$$

Let $u^{n+1}(x)$ be the exact solution of ODE (37), that is, the solution of given equations (11)-(12) at time level $(n + 1)$th. Also, we assume that $u_m^{n+1}(x)$ is the approximate solution of equation (37) by using the sinc-collocation (20). The computed solution of equations (11)-(12) at point x_j can be obtained by

$$w_m^{n+1}(x) = \sum_{j=-N}^{N} u^{n+1}(x_j) S_j(x). \tag{38}$$

We need to derive an upper bound for $\|P^{-1}\|_2$, which is given in the following lemma.

Lemma 2 *Let the matrix P be defined by equation* (32). *For $x \in \phi^{-1}((-\infty,\infty))$, we can obtain*

$$\frac{P+P^*}{2} = H - \frac{B_{n,n}}{h^2} I^{(2)},$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix, and

$$H = D\left(\mathrm{Re}\left(\left(\frac{1}{\phi'}\right)^2\right)\right) - \frac{B_{n,n}}{2h}\left\{D\left(\left(\frac{1}{\phi'}\right)'\right) I^{(1)} - I^{(1)} D\left(\left(\overline{\frac{1}{\phi'}}\right)'\right)\right\}.$$

If the eigenvalues of matrix H are nonnegative, then there exists a constant c_0, independent of N, such that

$$\left\|P^{-1}\right\|_2 \leq \frac{4dN}{\alpha\pi B_{n,n}}\left(1 + \frac{c_0}{N}\right) \tag{39}$$

for a sufficiently large N.

Proof Let $\lambda_i(\cdot)$, $i = 1, 2, \ldots, 2N+1$, be the eigenvalues of a matrix ordered as $\lambda_i(\cdot) \leq \lambda_{i+1}(\cdot)$, and let σ_i be the singular values of the matrix P satisfying $\sigma_i \leq \sigma_{i+1}$. Note that the matrix $I^{(2)}$ is a symmetric, negative definite Toeplitz matrix with bounded eigenvalues and matrix $I^{(1)}$ is a skew-symmetric Toeplitz matrix with complex eigenvalues ([27], p. 151-152). From ([30], p. 327, [23]) we have

$$\begin{aligned}
\sigma_1(P) &= \min_{1\leq i\leq 2N+1} \sigma_i(P) \geq \min_{1\leq i\leq 2N+1}\left|\lambda_i\left(\frac{P+P^*}{2}\right)\right| = \min_{1\leq i\leq 2N+1}\left|\lambda_i\left(H - \frac{B_{n,n}}{h^2} I^{(2)}\right)\right| \\
&\geq \frac{B_{n,n}}{h^2} \min_{1\leq i\leq 2N+1}\left|\lambda_i\left(I^{(2)}\right)\right| \geq \frac{4B_{n,n}}{h^2}\sin^2\left(\frac{\pi}{4(N+1)}\right),
\end{aligned}$$

and setting $h = \sqrt{\pi d/\alpha N}$ leads to

$$\left\|P^{-1}\right\|_2 = \frac{1}{\sigma_1(P)} \leq \frac{h^2}{4B_{n,n}\sin^2(\frac{\pi}{4(N+1)})} \leq \frac{4h^2N^2}{\pi^2 B_{n,n}}\left(1 + \frac{c_0}{N}\right) = \frac{4dN}{\alpha\pi B_{n,n}}\left(1 + \frac{c_0}{N}\right),$$

where $B_{n,n}$ is given by equation (15). □

The following theorem gives a bound for $|u_m^{n+1}(x) - w_m^{n+1}(x)|$.

Theorem 2 *Let $u_m^{n+1}(x)$ be an approximate solution of equation* (37), *and let $w_m^{n+1}(x)$ be an approximate solution of equations* (11)-(12). *Then, there exists a constant c_4, independent of N, such that*

$$\sup_{x\in\Gamma}\left|u_m^{n+1}(x)-w_m^{n+1}(x)\right|\leq c_4N^3\exp\bigl(-(\pi d\alpha N)^{1/2}\bigr). \tag{40}$$

Proof By equations (20) and (38) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|u_m^{n+1}(x)-w_m^{n+1}(x)\right| &= \left|\sum_{j=-N}^{N}c_j^{n+1}S_j(x)-\sum_{j=-N}^{N}u^{n+1}(x_j)S_j(x)\right|\\ &\leq\left(\sum_{j=-N}^{N}\left|c_j^{n+1}-u^{n+1}(x_j)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j=-N}^{N}\left|S_j(x)\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $(\sum_{j=-N}^{N}|S_j(x)|^2)^{\frac{1}{2}}\leq c_1$, where c_1 is a constant independent of N, we get

$$\left|u_m^{n+1}(x)-w_m^{n+1}(x)\right|\leq c_1\left\|C^{n+1}-V^{n+1}\right\|_2, \tag{41}$$

where C^{n+1} is given by (35) and denoting the vector V^{n+1} by

$$V^{n+1}=\bigl(u^{n+1}(x_{-N}),u^{n+1}(x_{-N+1}),\ldots,u^{n+1}(x_N)\bigr)^t. \tag{42}$$

Using equation (35) in (41), we have

$$\left\|C^{n+1}-V^{n+1}\right\|_2=\left\|P^{-1}\bigl(PC^{n+1}-PV^{n+1}\bigr)\right\|_2\leq\left\|P^{-1}\right\|_2\left\|PV^{n+1}-G^{n+1}\right\|_2. \tag{43}$$

Now, we must get a bound for $\|PV^{n+1}-G^{n+1}\|_2$. For simplicity, we denote

$$r_k=\bigl(PV^{n+1}-G^{n+1}\bigr)_k,\quad k=-N,\ldots,N,$$

and using equation (37), we obtain

$$\begin{aligned}|r_k| &= \left|g(x_k)-g_m(x_k)\right|\\ &= \left|u^{n+1}(x_k)-B_{n,n}\frac{d^2}{dx^2}\bigl(u^{n+1}(x_k)\bigr)-u_m^{n+1}(x_k)+B_{n,n}\frac{d^2}{dx^2}\bigl(u_m^{n+1}(x_k)\bigr)\right|\\ &\leq\left|u^{n+1}(x_k)-u_m^{n+1}(x_k)\right|+B_{n,n}\left|\frac{d^2}{dx^2}\bigl(u^{n+1}(x_k)\bigr)-\frac{d^2}{dx^2}\bigl(u_m^{n+1}(x_k)\bigr)\right|.\end{aligned} \tag{44}$$

Now, using Theorem 1, we obtain

$$\begin{aligned}\|r_k\| &\leq c_2N^{1/2}\exp\bigl(-(\pi d\alpha N)^{1/2}\bigr)+B_{n,n}c_3N^{3/2}\exp\bigl(-(\pi d\alpha N)^{1/2}\bigr)\\ &\leq\exp\bigl(-(\pi d\alpha N)^{1/2}\bigr)\bigl(c_2N^{3/2}+B_{n,n}c_3N^{3/2}\bigr)\\ &= KN^{3/2}\exp\bigl(-(\pi d\alpha N)^{1/2}\bigr),\end{aligned} \tag{45}$$

where c_2 and c_3 are constants independent of N, and $K = c_2 + B_{n,n}c_3$. We know that

$$\|PV^{n+1} - G^{n+1}\|_2 \le \sqrt{2N+1}\|PV^{n+1} - G^{n+1}\|_\infty,$$

and using inequality (45), we obtain

$$\|PV^{n+1} - G^{n+1}\|_2 \le \sqrt{2}KN^2 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr). \tag{46}$$

Now, using Lemma 2 and inequality (46) in (43), we have

$$\|C^{n+1} - V^{n+1}\|_2 \le \frac{4\sqrt{2}dK(1+c_0)}{\alpha\pi B_{n,n}} N^3 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr). \tag{47}$$

So, from (41) and (47) we get

$$\sup_{x\in\Gamma}\bigl|u_m^{n+1}(x) - w_m^{n+1}(x)\bigr| \le c_4 N^3 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr),$$

where $c_4 = \frac{4\sqrt{2}dK(1+c_0)c_1}{\alpha\pi B_{n,n}}$. □

Theorem 3 *Let $u^{n+1}(x)$ be the exact solution of ODE* (37), *and let $u_m^{n+1}(x)$ be its sinc approximation defined by Eq.* (20). *Then, under the assumptions of Theorems* 1 *and* 2, *there exists a constant c_7, independent of N, such that*

$$\sup_{x\in\Gamma}\bigl|u^{n+1}(x) - u_m^{n+1}(x)\bigr| \le c_7 N^3 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr). \tag{48}$$

Proof Applying the triangular inequality,

$$\bigl|u^{n+1}(x) - u_m^{n+1}(x)\bigr| \le \bigl|u^{n+1}(x) - w_m^{n+1}(x)\bigr| + \bigl|w_m^{n+1}(x) - u_m^{n+1}(x)\bigr|. \tag{49}$$

After Applying Theorem 1, there exists a constant c_5 independent of N such that

$$\bigl|u^{n+1}(x) - w_m^{n+1}(x)\bigr| \le c_5 N^{1/2} \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr). \tag{50}$$

Also, using Theorem 2, we obtain

$$\bigl|w_m^{n+1}(x) - u_m^{n+1}(x)\bigr| \le c_6 N^3 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr), \tag{51}$$

where c_6 is a constant independent of N. Finally, applying solutions to (50) and (51), we conclude

$$\sup_{x\in\Gamma}\bigl|u^{n+1}(x) - u_m^{n+1}(x)\bigr| \le c_7 N^3 \exp\bigl(-(\pi d\alpha N)^{1/2}\bigr),$$

where $c_7 = \max\{c_5, c_6\}$. □

Remark We know that the time discretization is affected by a combination of the backward Euler method and product trapezoidal integration rule with orders of accuracy

$\mathcal{O}(\Delta t)$ and $\mathcal{O}(\Delta t^{2-\beta})$, respectively [1–3, 10, 29]. Then, by applying (40) the truncation error of the proposed approach for solution of equations (11)-(12) can be written as follows:

$$\left\| u(x,t) - u_m(x,t) \right\|_\infty \le \gamma \left(N^3 \exp\left(-(\pi d \alpha N)^{1/2} \right) + \Delta t \right),$$

where γ is a constant independent of N.

5 Numerical results

In this section, we provide numerical experiments of the suggested method. In all examples, we set the parameters $d = \frac{\pi}{2}$ and $\alpha = 1$ and denote the computational solution and exact analytical solution by u_{app} and u_{ex}, respectively. The error estimation is given to show the accuracy of approximation, and the following maximum pointwise error between the exact and approximate solution is given:

$$\| \cdot \|_\infty = \operatorname*{Max}_{i,n} \left| u_{\text{app}}(x_i, t_n) - u_{\text{ex}}(x_i, t_n) \right|, \quad i = -N, \ldots, N, n = 0, 1, \ldots, M.$$

To implement the method, the following algorithm is given.

The linear algebraic system in step 6 of Algorithm 1 is solved directly by using 'linsolve' command from 'LinearAlgebra' package in Matlab R2014a software, and to overcome the ill-conditioning faced in this problem, we used the following Tikhonov regularization [31], which states that 'solve the system $Ax = b$ by replacing $\min_{x \in \mathbb{R}^n} \|AX - b\|_2$ by the least square problem $\min_{x \in \mathbb{R}^n} \{ \|AX - b\|_2^2 + \mu^2 \|X\|_2^2 \}$'. All the calculations were supported by Intel CORE Dual-Core at 2.20 GHz CPU with 4 GB RAM.

Algorithm 1 Implementation of the proposed approach

1: Input M, N, n, $u_0(x)$, $f(x,t)$, $u_{\text{ex}}(x,t)$,
2: Set $x_i := \frac{e^{ih}}{1+e^{ih}}$, $i = -N, \ldots, N$,
3: Set $t_j := j\Delta t$, $j = 0, 1, \ldots, n$,
4: Compute $u_{\text{ex}}(x_i, t_j)$,
5: Compute $u_{\text{app}}(x_i, t_j)$ as follows:
6: Set $u_{\text{app}}(x_i, t_0) := C^0$, $i = -N, \ldots, N$, based on Eq. (36)
  for $j = 0 : n - 1$ do
    for $i = -N : N$ do
      $u_{\text{app}}(x_i, t_{j+1}) := C^{j+1}$ by applying Eqs. (32), (34), and (35)
      error$(x_i, t_j) := |u_{\text{ex}}(x_i, t_j) - u_{\text{app}}(x_i, t_j)|$,
    end do
  end do
7: Print MaxError := max(error(x_i, t_j)), $i = -N, \ldots, N$, $j = 0, 1, \ldots, n$.

Example 1 Consider the following homogenous Volterra partial integro-differential equation [11]:

$$
\begin{aligned}
&u_t(x,t) = \int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds, \quad 0 < x < 1, 0 < t < 1,\\
&u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1,\\
&u(x,0) = \sin(\pi x), \quad 0 \le x \le 1,
\end{aligned}
$$

with analytic solution [32, 33]

$$
u(x,t) = \sum_{k=0}^{\infty} (-1)^k \Gamma\left(\frac{3}{2}k+1\right)^{-1} \left(\pi^{5/2} t^{3/2}\right)^k \sin(\pi x).
$$

To evaluate the analytic solution practically at a specific point, we truncate this infinite series by the term $k = 21$. In Table 1, the outcomes of the three-point explicit method (TPEM), three-point implicit method (TPIM), Crank-Nicolson method (CNM), Crandall method (CM) (see [11]) with $\Delta t = 10^{-4}$ are presented in order to compare with the sinc-collocation method solving the arising system solved by the Linsolve package (SMLP) and the sinc-collocation method solving the arising system by the Tikhonov regularization (SMTR) with $\Delta t = 10^{-2}$, $\Delta t = 10^{-3}$, and $\Delta t = 10^{-4}$. In Figure 1 and Figure 2, we can also observe that the computational solution is highly consistent with the truncated analytical solution when Δt is selected small enough. Furthermore, in Table 2, the maximum pointwise errors and condition numbers for various values of N at $t = 0.01$, $\Delta t = 10^{-4}$, and $T = 1$ for SMLP and SMTR are reported, which shows the improved rate of convergence when the number of sinc points increases. Also, the global maximum pointwise errors at $N = 4$ and $\Delta t = 10^{-3}$ are plotted in Figure 3(a) for SMLP and in Figure 3(b) for SMTR in order to compare with the thin plate spline-radial basis function method (TPS-RBF), inverse multiquadric-radial basis function method (IMQ-RBF), and hyperbolic secant-radial basis function method (Sech-RBF) (see [17]) with $\Delta t = 10^{-3}$ at $N = 25$. These figures show that our method achieved more accurate results with less data grid points. Convergence curves of Table 2 are plotted in Figure 4. This figure indicates that the maximum errors decline at an exponential rate with respect to N for both SMLP and SMTR, and these graphs confirm the theoretical results.

Example 2 Consider the following nonhomogenous Volterra partial integro-differential equation [1, 18]:

$$
\begin{aligned}
&u_t(x,t) = \int_0^t (t-s)^{-1/2} u_{xx}(x,s)\,ds + f(x,t), \quad 0 < x < 1, 0 < t < 1,\\
&u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1,\\
&u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.
\end{aligned}
$$

In the case of

$$
f(x,t) = \frac{2t^{1/2}}{\sqrt{\pi}} \left(\pi^2 \sin \pi x - \sin 2\pi x\right) - 2\pi^2 t^2 \sin 2\pi x,
$$

Table 1 Comparison of estimated maximum pointwise errors of Example 1 for $N = 4$, $T = 1$ at $t = 1$ and different values of x

		x								
		0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
$\Delta t = 0.01$	SMLP	1.9×10^{-3}	3.4×10^{-3}	4.4×10^{-3}	5.1×10^{-3}	5.3×10^{-3}	5.1×10^{-3}	4.4×10^{-3}	3.4×10^{-3}	1.9×10^{-3}
	SMTR	7.6×10^{-4}	8.7×10^{-4}	8.9×10^{-4}	9.1×10^{-4}	9.3×10^{-4}	9.2×10^{-4}	8.6×10^{-4}	7.5×10^{-4}	6.7×10^{-4}
$\Delta t = 0.001$	SMLP	4.1×10^{-4}	6.1×10^{-4}	5.3×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	5.3×10^{-4}	6.1×10^{-4}	4.1×10^{-4}
	SMTR	1.2×10^{-4}	1.8×10^{-4}	1.4×10^{-4}	2.1×10^{-4}	9.2×10^{-5}	9.6×10^{-5}	8.9×10^{-5}	1.6×10^{-4}	2.3×10^{-4}
$\Delta t = 0.0001$	TPEM	7.5×10^{-3}	7.5×10^{-3}	7.6×10^{-3}	7.4×10^{-3}	7.5×10^{-3}	7.4×10^{-3}	7.3×10^{-3}	7.7×10^{-3}	7.8×10^{-3}
	TPIM	7.1×10^{-3}	7.2×10^{-3}	7.4×10^{-3}	7.5×10^{-3}	7.5×10^{-3}	7.3×10^{-3}	7.2×10^{-3}	7.4×10^{-3}	7.6×10^{-3}
	CNM	5.1×10^{-3}	5.2×10^{-3}	5.3×10^{-3}	5.2×10^{-3}	5.4×10^{-3}	5.3×10^{-3}	5.5×10^{-3}	5.3×10^{-3}	5.2×10^{-3}
	CM	6.2×10^{-4}	6.1×10^{-4}	6.5×10^{-4}	6.6×10^{-5}	6.6×10^{-5}	6.6×10^{-5}	6.4×10^{-4}	6.5×10^{-4}	6.4×10^{-4}
	SMLP	2.6×10^{-4}	3.1×10^{-4}	1.3×10^{-4}	1.1×10^{-5}	1.7×10^{-5}	1.1×10^{-5}	1.3×10^{-4}	3.1×10^{-4}	2.6×10^{-4}
	SMTR	7.5×10^{-5}	8.1×10^{-5}	5.7×10^{-5}	9.2×10^{-6}	9.5×10^{-6}	9.1×10^{-6}	7.8×10^{-5}	8.6×10^{-5}	8.2×10^{-5}

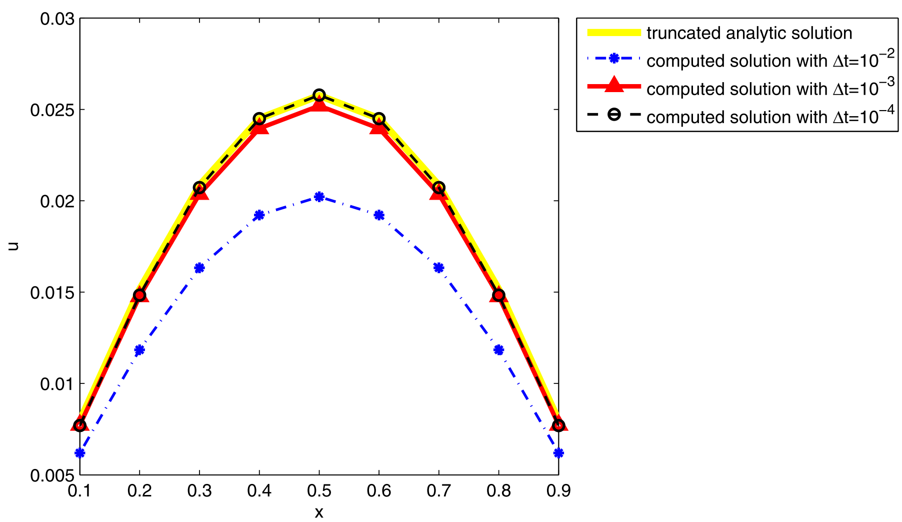


Figure 1 Truncated analytic and computed solutions of Example 1 with $N = 4$ at $t = 1$ by using the Linsolve package.

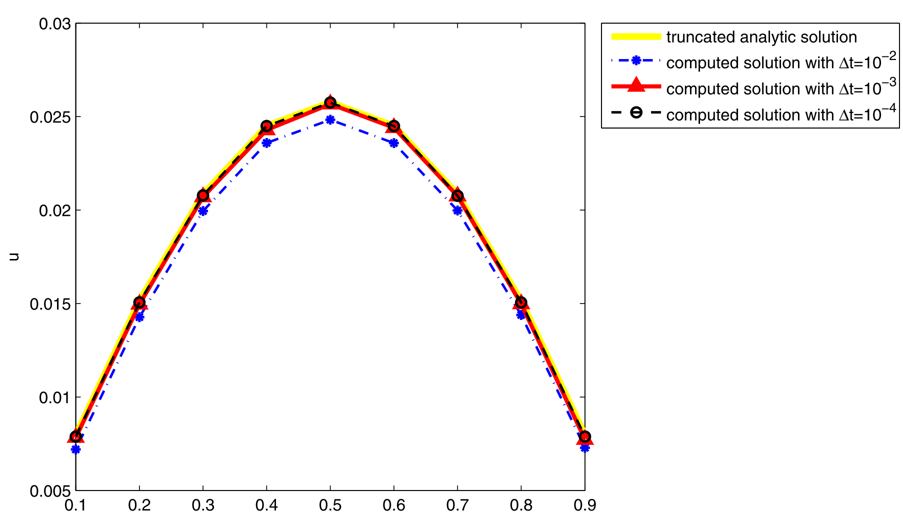


Figure 2 Truncated analytic and computed solutions of Example 1 with $N = 4$ at $t = 1$ by using Tikhonov regularization.

Table 2 Results for Example 1 at $t = 0.01$

N	SMLP	SMTR	Cond$(P) = \|\|P\|\| \|\|P^{-1}\|\|$
4	1.10×10^{-2}	8.27×10^{-3}	4.61×10^{2}
8	2.85×10^{-3}	7.94×10^{-4}	6.71×10^{3}
16	4.68×10^{-4}	9.32×10^{-5}	3.88×10^{4}
32	9.75×10^{-5}	4.71×10^{-5}	1.30×10^{5}

the analytic solution is given by $u(x,t) = \sin \pi x - \frac{4t^{3/2}}{3\sqrt{\pi}} \sin 2\pi x$. We apply our presented methods SMLP and SMTR to this example for comparison with the quasi-wavelet method (QWM) [18]. We used $N = 32$ and $\Delta t = 10^{-5}, 10^{-6}$. The global maximum pointwise errors

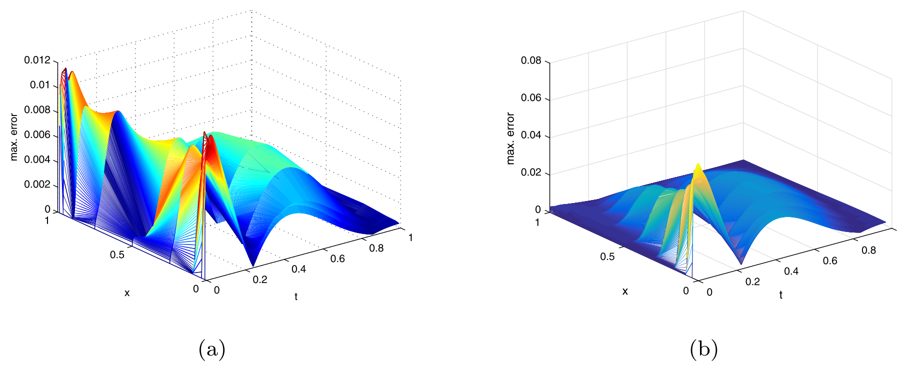


Figure 3 The global maximum pointwise errors at $N = 4$ and $\Delta t = 10^{-3}$ (a) by using the Linsolve package, (b) by using of Tikhonov regularization.

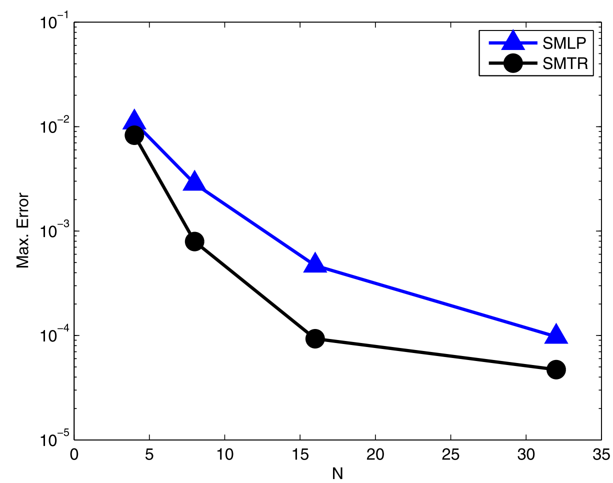
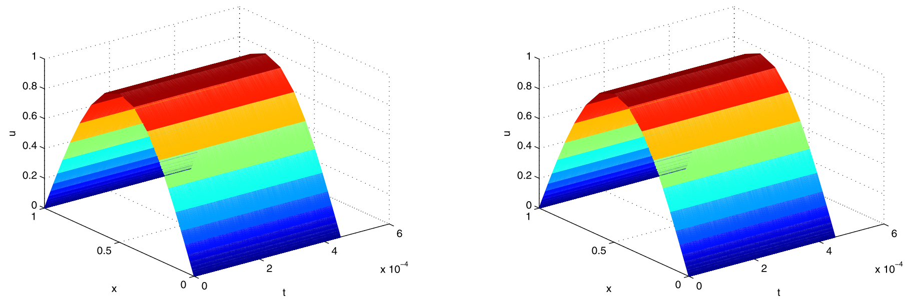


Figure 4 Convergence of the SMLP and SMTR methods for various values of N at $t = 0.01$, $\Delta t = 10^{-4}$, and $T = 1$.

Table 3 Results for Example 2

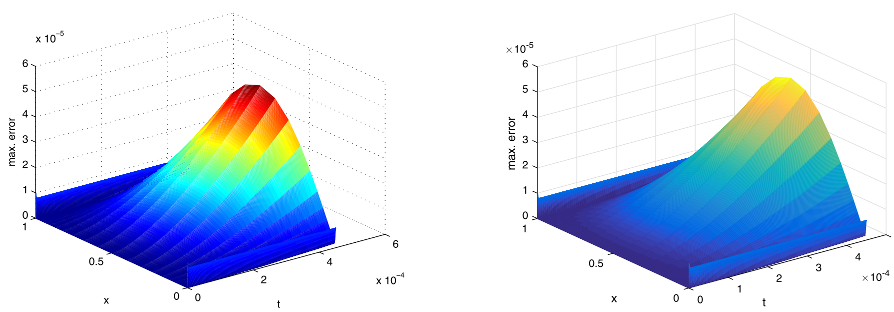
n	$\Delta t = 10^{-5}$			$\Delta t = 10^{-6}$		
	QWM	SMLP	SMTR	QWM	SMLP	SMTR
50	4.9343e–004	5.1621e–005	7.5561e–006	1.5630e–005	9.8142e–006	2.7356e–006
150	2.5228e–003	2.9006e–004	8.6902e–005	8.0470e–005	9.8142e–006	2.9441e–006
250	5.3616e–003	6.4177e–004	1.1924e–004	1.7272e–004	2.0303e–005	9.1242e–006
350	8.7631e–003	1.0796e–003	6.6327e–004	2.8572e–004	3.4165e–005	1.5836e–005
450	1.2588e–002	1.5898e–003	1.1745e–004	4.1611e–004	5.0328e–005	2.3042e–005

in the solutions have been computed for 50th, 150th, 250th, 350th, and 450th time levels and tabulated in Table 3, which shows that the sinc method in comparison with QWM is considerably accurate. The analytic and exact solutions are compared in Figure 5 for $N = 32$ and $\Delta t = 10^{-6}$ by using the SMLP. In addition, the maximum pointwise errors in the solution by SMLP and SMTR in Table 3 are plotted in Figure 6 and Figure 7.



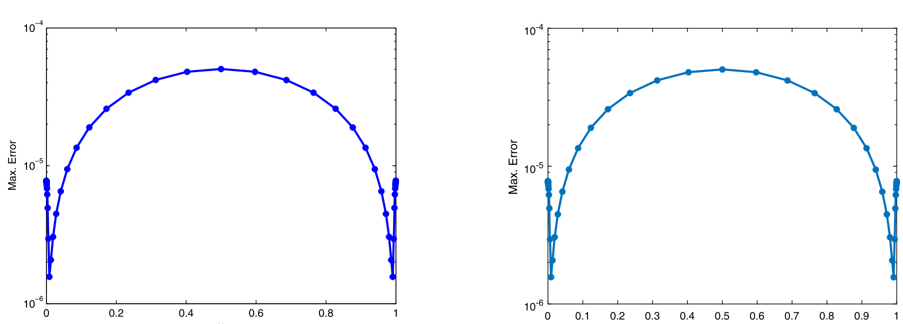
(a) The analytic solution (b) The computed solution

Figure 5 **Analytic and computed solutions of Example 2 with $\Delta t = 10^{-6}$ and $N = 32$ by using the Linsolve package.**



(a) (b)

Figure 6 **The global maximum pointwise errors at $N = 32$ and $\Delta t = 10^{-6}$ (a) by using the Linsolve package, (b) by using Tikhonov regularization.**



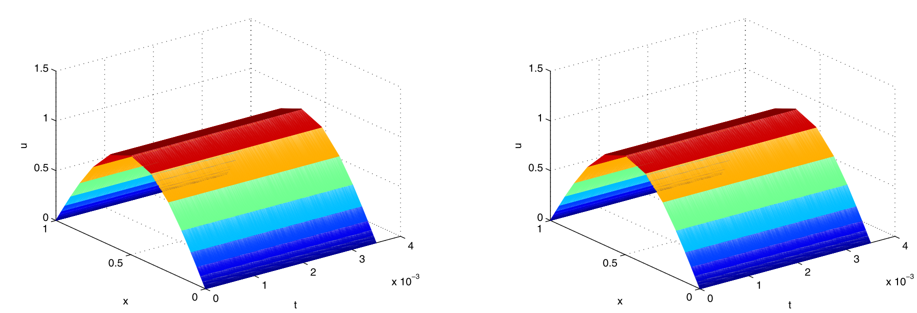
(a) (b)

Figure 7 **The global maximum pointwise errors for $N = 32$ at $t = 0.00045$, $\Delta t = 10^{-6}$, and $T = 1$ (a) by using the SMLP method, (b) by using the SMTR method.**

Example 3 Consider equation (1) in the nonhomogenous form when $k_0(t-s) = (\pi(t-s))^{-\frac{1}{2}}$, $0 \le x \le 1$, $0 \le t \le 1$, $u_0(x) = \sin(\pi x)$, and $f(x,t) = \sin(\pi x)$. Thus, the analytical solu-

Table 4 Results for Example 3

N	$\Delta t = 10^{-5}$		$\Delta t = 10^{-6}$		$\Delta t = 10^{-7}$	
	SMLP	SMTR	SMLP	SMTR	SMLP	SMTR
8	2.75×10^{-3}	8.52×10^{-4}	1.05×10^{-3}	8.28×10^{-4}	3.57×10^{-6}	9.81×10^{-7}
	2.83×10^{-3}	8.94×10^{-4}	2.58×10^{-3}	8.41×10^{-4}	1.65×10^{-4}	7.34×10^{-6}
	2.87×10^{-3}	9.86×10^{-4}	2.61×10^{-3}	9.73×10^{-4}	4.35×10^{-4}	6.52×10^{-5}
	2.88×10^{-3}	9.71×10^{-3}	2.61×10^{-3}	8.84×10^{-3}	6.13×10^{-4}	1.41×10^{-4}
16	2.75×10^{-4}	7.32×10^{-5}	2.63×10^{-4}	7.28×10^{-5}	9.00×10^{-5}	6.87×10^{-7}
	2.92×10^{-4}	6.54×10^{-5}	2.66×10^{-4}	1.79×10^{-5}	2.56×10^{-4}	5.62×10^{-6}
	6.45×10^{-4}	8.30×10^{-5}	2.67×10^{-4}	5.48×10^{-5}	2.61×10^{-4}	6.42×10^{-6}
	1.08×10^{-3}	9.24×10^{-5}	2.67×10^{-4}	6.74×10^{-5}	2.61×10^{-4}	7.83×10^{-6}
32	5.28×10^{-5}	9.12×10^{-6}	9.81×10^{-6}	5.41×10^{-6}	8.66×10^{-6}	1.19×10^{-7}
	2.92×10^{-4}	4.37×10^{-5}	9.81×10^{-6}	4.09×10^{-6}	8.66×10^{-6}	4.72×10^{-7}
	6.45×10^{-4}	5.83×10^{-5}	2.04×10^{-5}	6.49×10^{-6}	8.66×10^{-6}	7.28×10^{-7}
	1.08×10^{-3}	7.34×10^{-5}	3.43×10^{-5}	8.51×10^{-6}	8.66×10^{-6}	8.63×10^{-7}
64	5.28×10^{-5}	3.41×10^{-6}	1.67×10^{-6}	9.86×10^{-7}	6.11×10^{-8}	7.53×10^{-8}
	2.92×10^{-4}	1.92×10^{-5}	9.24×10^{-6}	5.17×10^{-6}	2.28×10^{-7}	8.91×10^{-8}
	6.45×10^{-4}	4.56×10^{-5}	2.04×10^{-5}	6.84×10^{-6}	6.45×10^{-7}	9.52×10^{-8}
	1.08×10^{-3}	5.37×10^{-5}	3.43×10^{-5}	7.13×10^{-6}	9.44×10^{-7}	1.54×10^{-7}



(a) The truncated analytic solution (b) The computed solution

Figure 8 Truncated analytic and computed solutions of Example 3 with $\Delta t = 0.00001$ and $N = 16$ by using the Linsolve package.

tion is given by [3]

$$u(x,t) = \left\{ \sum_{k=0}^{\infty} (-1)^k \frac{(\pi^2 t^{3/2})^k}{\Gamma(1+\frac{3}{2}k)} + t \sum_{k=0}^{\infty} (-1)^k \frac{(\pi^2 t^{3/2})^k}{\Gamma(2+\frac{3}{2}k)} \right\} \sin(\pi x).$$

To evaluate the analytic solution practically at a specific point, the infinite series given above is truncated by the term $k = 21$. In Table 4, we show the results of the 50th, 150th, 250th, and 350th time levels of the three different grid sizes $\Delta t = 10^{-5}$, $\Delta t = 10^{-6}$, and $\Delta t = 10^{-7}$ for SMLP and SMTR methods when $N = 8, 16, 32, 64$, which verify that the sinc method is accurate enough. Besides, we can also see in Figure 8 that the computational solution is consistent with the truncated analytical solution. In addition, the maximum pointwise errors in the solution by SMLP and SMTR in Table 4 are plotted in Figure 9 and Figure 10.

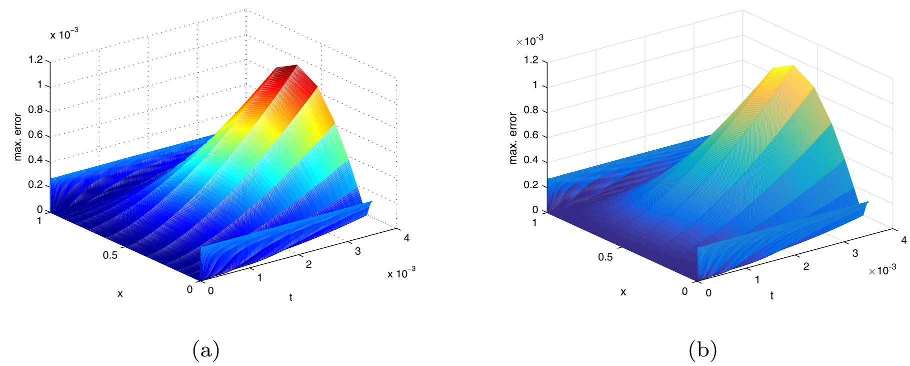


Figure 9 The global maximum pointwise errors at $N = 16$ and $\Delta t = 10^{-5}$ (a) by using the Linsolve package, (b) by using Tikhonov regularization.

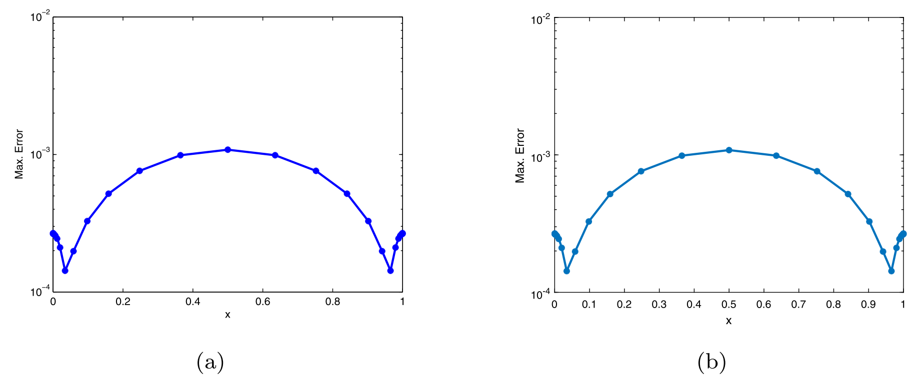


Figure 10 The global maximum pointwise errors for $N = 16$ at $t = 0.0035$, $\Delta t = 10^{-5}$, and $T = 1$ (a) by using the Linsolve package, (b) by using Tikhonov regularization.

6 Conclusions

In this paper, the sinc-collocation method was applied to solve linear Volterra partial integro-differential equations by using the Linsolve package and Tikhonov regularization methods for a final ill-conditioned system. To illustrate the effectiveness of the method, some examples were solved based on the proposed algorithm. Also, the convergence of the method was given. The results show that the proposed method is practically reliable and consistent in comparison with other mentioned methods, and using the Tikhonov regularization method for solving the final ill-conditioned algebraic system, the rate of convergence improved.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Central Tehran Branch, Islamic Azad University, Simaye Iran Street, Tehran, 14676-86831, Iran. [2]School of Mathematics, Iran University of Science and Technology, Daneshgah Street, Tehran, 16846-13114, Iran. [3]Department of Mathematics, Shahid Chamran University of Ahvaz, Golestan Blvd, Ahvaz, 61357-83151, Iran.

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