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Second-order differential operators with interior singularity

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Abstract

The purpose of this study is to investigate a new class of boundary value transmission problems (BVTPs) for a Sturm-Liouville equation on two separate intervals. We introduce a modified inner product in the direct sum space $L_2[a,c) \oplus L_2(c,b] \oplus C^2$ and define a symmetric linear operator in it in such a way that the considered problem can be interpreted as an eigenvalue problem of this operator. Then, by suggesting own approaches, we construct the Green's function for the BVTP under consideration and find the resolvent function for the corresponding inhomogeneous problem.

Keywords: Sturm-Liouville problems; Green's function; transmission conditions; resolvent operator

1 Introduction

Many interesting applications of Sturm-Liouville theory arise in quantum mechanics. For instance, for a single quantum-mechanical particle of mass m moving in one space dimension in a potential $V(x)$, the time-dependent Schrödinger equation is

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi.$$

Looking for separable solutions $\psi(x,t) = \varphi(x)e^{-iEt/\hbar}$, we find that $\varphi(x)$ satisfies the differential equation

$$-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi.$$

That is a Sturm-Liouville equation of the form

$$-y'' + qy = \lambda y.$$

The coefficient q is proportional to the potential V, and the eigenvalue parameter λ is proportional to the energy E. Physical problems such as this and those involving sound, surface waves, heat conduction, electromagnetic waves, and gravitational waves, for example, can be solved using the mathematical theory of boundary value problems. Boundary value problems can be investigated also through the methods of Green's function and eigenfunction expansion. The main tool for solvability analysis of such problems is the concept of

Green's function. The concept of Green's function is very close to physical intuition (see [1]). If one knows the Green's function of a problem, one can write down its solution in a closed form as linear combinations of integrals involving the Green's function and the functions appearing in the inhomogeneities. Green's functions can often be found in an explicit way, and in these cases it is very efficient to solve the problem in this way. Determination of Green's functions is also possible using Sturm-Liouville theory. This leads to a series representation of Green's functions (see [2]).

2 Statement of the problem

In this study we shall investigate a new class of BVPs which consist of the Sturm-Liouville equation

$$\ell(y) := -p(x)y''(x) + q(x)y(x) = \lambda y(x) \tag{1}$$

to hold in a finite interval $[a, b]$ except at one inner point $c \in (a, b)$, where discontinuities in y and y' are prescribed by the transmission conditions at the interior point $x = c$,

$$V_j(y) := \beta_{j1}^- y'(c-) + \beta_{j0}^- y(c-) + \beta_{j1}^+ y'(c+) + \beta_{j0}^+ y(c+) = 0, \quad j = 1, 2, \tag{2}$$

together with eigenparameter-dependent boundary conditions at end points $x = a, b$,

$$U_1(y) := \alpha_{10} y(a) - \alpha_{11} y'(a) - \lambda\big(\alpha'_{10} y(a) - \alpha'_{11} y'(a)\big) = 0, \tag{3}$$

$$U_2(y) := \alpha_{20} y(b) - \alpha_{21} y'(b) + \lambda\big(\alpha'_{20} y(b) - \alpha'_{21} y'(b)\big) = 0, \tag{4}$$

where $p(x) = p^- > 0$ for $x \in [a, c)$, $p(x) = p^+ > 0$ for $x \in (c, b]$, the potential $q(x)$ is a real-valued function continuous in each of the intervals $[a, c)$ and $(c, b]$, and it has a finite limit $q(c \mp 0)$; λ is a complex spectral parameter, α_{ij}, $\beta_{ij}^{\pm}$, α'_{ij} ($i = 1, 2$ and $j = 0, 1$) are real numbers. We want to emphasize that the boundary value problem studied here differs from standard boundary value problems in that it contains transmission conditions and the eigenvalue parameter appears not only in the differential equation, but also in the boundary conditions. Moreover, the coefficient functions may have discontinuity at one interior point. Naturally, eigenfunctions of this problem may have discontinuity at the one inner point of the considered interval. The problems with transmission conditions have become an important area of research in recent years because of the needs of modern technology, engineering and physics. Many of the mathematical problems encountered in the study of boundary-value-transmission problem cannot be treated with the usual techniques within the standard framework of a boundary value problem (see [3]). Note that some special cases of this problem arise after an application of the method of separation of variables to a varied assortment of physical problems. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems [4], in vibrating string problems when the string is loaded additionally with point masses [5], in diffraction problems [6]. Such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel bases of a system of root functions of similar boundary value problems with transmission conditions and its applications to the corresponding initial boundary value problems for parabolic equations were investigated in [7–10]. Also some problems with transmission conditions which arise in mechanics (thermal conduction problems for a thin laminated plate) were studied in [11]. Boundary value problems with transmission conditions were investigated extensively in the recent years (see, for example, [3, 7–9, 12–21]).

3 The 'basic' solutions and a characteristic function

With a view to constructing the characteristic function $\omega(\lambda)$, we shall define two basic solutions $\varphi^-(x,\lambda)$ and $\psi^-(x,\lambda)$ on the left interval $[a,c)$ and two basic solutions $\varphi^+(x,\lambda)$ and $\psi^+(x,\lambda)$ on the right interval $(c,b]$ by the following procedure. Let $\varphi^-(x,\lambda)$ and $\psi^+(x,\lambda)$ be the solutions of equation (1) on $[a,c)$ and $(c,b]$ satisfying the initial conditions

$$\varphi^-(a,\lambda) = \alpha_{11} - \lambda\alpha'_{11}, \qquad \frac{\partial\varphi^-(a,\lambda)}{\partial x} = \alpha_{10} - \lambda\alpha'_{10}, \tag{5}$$

$$\psi^+(b,\lambda) = \alpha_{21} + \lambda\alpha'_{21}, \qquad \frac{\partial\psi^+(b,\lambda)}{\partial x} = \alpha_{20} + \lambda\alpha'_{20}, \tag{6}$$

respectively. In terms of these solutions, we shall define the other solutions $\varphi^+(x,\lambda)$ and $\psi^-(x,\lambda)$ by the initial conditions

$$\varphi^+(c+,\lambda) = \frac{1}{\Delta_{12}}\left(\Delta_{23}\varphi^-(c-,\lambda) + \Delta_{24}\frac{\partial\varphi^-(c-,\lambda)}{\partial x}\right), \tag{7}$$

$$\frac{\partial\varphi^+(c+,\lambda)}{\partial x} = \frac{-1}{\Delta_{12}}\left(\Delta_{13}\varphi^-(c-,\lambda) + \Delta_{14}\frac{\partial\varphi^-(c-,\lambda)}{\partial x}\right) \tag{8}$$

and

$$\psi^-(c-,\lambda) = \frac{-1}{\Delta_{34}}\left(\Delta_{14}\psi^+(c+,\lambda) + \Delta_{24}\frac{\partial\psi^+(c+,\lambda)}{\partial x}\right), \tag{9}$$

$$\frac{\partial\psi^-(c-,\lambda)}{\partial x} = \frac{1}{\Delta_{34}}\left(\Delta_{13}\psi^+(c+,\lambda) + \Delta_{23}\frac{\partial\psi^+(c+,\lambda)}{\partial x}\right), \tag{10}$$

respectively, where Δ_{ij} $(1 \le i < j \le 4)$ denotes the determinant of the ith and jth columns of the matrix

$$T = \begin{bmatrix} \beta^+_{10} & \beta^+_{11} & \beta^-_{10} & \beta^-_{11} \\ \beta^+_{20} & \beta^+_{21} & \beta^-_{20} & \beta^-_{21} \end{bmatrix}.$$

The existence and uniqueness of these solutions follow from the well-known existence and uniqueness theorem of ordinary differential equation theory. Moreover, by applying the method of [12], we can prove that each of these solutions is an entire function of the parameter $\lambda \in \mathbb{C}$ for each fixed x. Taking into account (7)-(10) and the fact that the Wronskians $\omega^\pm(\lambda) := W[\varphi^\pm(x,\lambda), \psi^\pm(x,\lambda)]$ are independent of variable x, we have

$$\begin{aligned}\omega^+(\lambda) &= \varphi^+(c+,\lambda)\frac{\partial\psi^+(c+,\lambda)}{\partial x} - \frac{\partial\varphi^+(c+,\lambda)}{\partial x}\psi^+(c+,\lambda) \\ &= \frac{\Delta_{34}}{\Delta_{12}}\left(\varphi^-(c-,\lambda)\frac{\partial\psi^-(c-,\lambda)}{\partial x} - \frac{\partial\varphi^-(c-,\lambda)}{\partial x}\psi^-(c-,\lambda)\right) \\ &= \frac{\Delta_{34}}{\Delta_{12}}\omega^-(\lambda).\end{aligned}$$

It is convenient to define the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda) := \Delta_{34}\omega^-(\lambda) = \Delta_{12}\omega^+(\lambda).$$

Obviously, $\omega(\lambda)$ is an entire function. By applying the technique of [12], we can prove that there are infinitely many eigenvalues λ_n, $n = 1, 2, \ldots$, of problem (1)-(4) which coincide with zeros of the characteristic function $\omega(\lambda)$.

4 Operator treatment in a modified Hilbert space

To analyze the spectrum of BVTP (1)-(4), we shall construct an adequate Hilbert space and define a symmetric linear operator in it in such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this we assume that

$$\Delta_{12} > 0, \qquad \Delta_{34} > 0,$$
$$\theta_1 = \begin{bmatrix} \alpha_{11} & \alpha_{10} \\ \alpha'_{11} & \alpha'_{10} \end{bmatrix} > 0, \qquad \theta_2 = \begin{bmatrix} \alpha_{21} & \alpha_{20} \\ \alpha'_{21} & \alpha'_{20} \end{bmatrix} > 0$$

and introduce modified inner products on the direct sum spaces $\mathcal{H}_1 = L_2[a,c) \oplus L_2(c,b]$ and $\mathcal{H} = \mathcal{H}_1 \oplus C^2$ by

$$[f,g]_{\mathcal{H}_1} := \frac{\Delta_{34}}{p^-}\int_a^{c-} f(x)\overline{g(x)}\,dx + \frac{\Delta_{12}}{p^+}\int_{c+}^{b} f(x)\overline{g(x)}\,dx \tag{11}$$

and

$$[F,G]_{\mathcal{H}} := [f,g]_{\mathcal{H}_1} + \frac{\Delta_{34}}{\theta_1} f_1\overline{g}_1 + \frac{\Delta_{12}}{\theta_2} f_2\overline{g}_2 \tag{12}$$

for $F = (f(x), f_1, f_2), G = (g(x), g_1, g_2) \in \mathcal{H}$, respectively. Obviously, each of these inner products is equivalent to the standard inner products of the Hilbert spaces $L_2[a,c) \oplus L_2(c,b]$ and $L_2[a,c) \oplus L_2(c,b] \oplus C^2$, respectively, so $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ and $(\mathcal{H}_1, [\cdot,\cdot]_{\mathcal{H}_1})$ are also Hilbert spaces. Let us now define the boundary functionals

$$B_a[f] := \alpha_{10} f(a) - \alpha_{11} f'(a), \qquad B'_a[f] := \alpha'_{10} f(a) - \alpha'_{11} f'(a),$$
$$B_b[f] := \alpha_{20} f(b) - \alpha_{21} f'(b), \qquad B'_b[f] := \alpha'_{20} f(b) - \alpha'_{21} f'(b)$$

and construct the operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ with the domain

$$\begin{aligned} \operatorname{dom}(\mathcal{L}) := \bigl\{ & F = \bigl(f(x), f_1, f_2\bigr) : f(x), f'(x) \in AC_{\text{loc}}(a,c) \cap AC_{\text{loc}}(c,b), \\ & \text{and has a finite limits } f(c \mp 0) \text{ and } f'(c \mp 0), \ell(f) \in L_2[a,b], \\ & V_1(f) = V_2(f) = 0, f_1 = B'_a[f], f_2 = -B'_b[f] \bigr\} \end{aligned}$$

and action low

$$\mathcal{L}\bigl(f(x), B'_a[f], -B'_b[f]\bigr) = \bigl(\ell(f), B_a[f], B_b[f]\bigr).$$

Then problem (1)-(4) can be written in the operator equation form as follows:

$$\mathcal{L}F = \lambda F, \quad F = \big(f(x), B'_a[f], -B'_b[f]\big) \in \operatorname{dom}(\mathcal{L})$$

in the Hilbert space $\mathcal{H}$.

Theorem 1 *The linear operator $\mathcal{L}$ is symmetric.*

Proof By applying the method of [12] it is not difficult to show that $\operatorname{dom}(\mathcal{L})$ is dense in the Hilbert space $\mathcal{H}$. Now, let $F = (f(x), B'_a[f], -B'_b[f])$, $G = (g(x), B'_a[g], -B'_b[g]) \in \operatorname{dom}(\mathcal{L})$. By partial integration we have

$$\begin{aligned}
[\mathcal{L}F, G]_{\mathcal{H}} - [F, \mathcal{L}G]_{\mathcal{H}} = {} & \Delta_{34} W(f, \overline{g}; c-) - \Delta_{34} W(f, \overline{g}; a) \\
& + \Delta_{12} W(f, \overline{g}; b) - \Delta_{12} W(f, \overline{g}; c+) \\
& + \frac{\Delta_{34}}{\theta_1} \big(B_a[f]\overline{B'_a[g]} - B'_a[f]\overline{B_a[g]}\big) \\
& + \frac{\Delta_{12}}{\theta_2} \big(B'_b[f]\overline{B_b[g]} - B_b[f]\overline{B'_b[g]}\big),
\end{aligned} \tag{13}$$

where, as usual, $W(f, \overline{g}; x)$ denotes the Wronskians of the functions f and $\overline{g}$. From the definitions of the boundary functionals B_a and B_b we get that

$$B_a[f]\overline{B'_a[g]} - B'_a[f]\overline{B_a[g]} = \theta_1 W(f, \overline{g}; a), \tag{14}$$

$$B'_b[f]\overline{B_b[g]} - B_b[f]\overline{B'_b[g]} = -\theta_2 W(f, \overline{g}; b). \tag{15}$$

Further, taking in view the definition of $\mathcal{L}$ and applying the initial conditions (5)-(10), we derive that

$$W(f, \overline{g}; c-) = \frac{\Delta_{12}}{\Delta_{34}} W(f, \overline{g}; c+). \tag{16}$$

Finally, substituting (14), (15) and (16) in (13), we have

$$[\mathcal{L}F, G]_{\mathcal{H}} = [F, \mathcal{L}G]_{\mathcal{H}} \quad \text{for every } F, G \in \operatorname{dom}(\mathcal{L}),$$

so the operator $\mathcal{L}$ is symmetric in $\mathcal{H}$. The proof is complete. □

Corollary 1

(i) *All the eigenvalues of problem* (1)-(4) *are real.*

(ii) *If $f(x)$ and $g(x)$ are eigenfunctions corresponding to distinct eigenvalues, then they are 'orthogonal' in the sense of the equality*

$$[f, g]_{\mathcal{H}_1} + \frac{\Delta_{34}}{\theta_1} B'_a[f] B'_a[g] + \frac{\Delta_{12}}{\theta_2} B'_b[f] B'_b[g] = 0, \tag{17}$$

where $F = (f(x), B'_a[f], -B'_b[f])$, $G = (g(x), B'_a[g], -B'_b[g]) \in \operatorname{dom}(\mathcal{L})$.

5 Solvability of the corresponding inhomogeneous problem

Now, let $\lambda \in C$ not be an eigenvalue of $\mathcal{L}$. Consider the operator equation

$$(\lambda I - \mathcal{L})Y = U \tag{18}$$

for arbitrary $U = (u(x), u_1, u_2) \in \mathcal{H}$. This operator equation is equivalent to the following inhomogeneous BVTP:

$$(\lambda - \ell)y(x) = u(x), \quad x \in [a,c) \cup (c,b], \tag{19}$$

$$V_3(y) = V_4(y) = 0, \qquad \lambda B'_a[y] - B_a[y] = u_1, \qquad -\lambda B'_b[y] - B_b[y] = u_2. \tag{20}$$

We shall search the resolvent function of this BVTP in the form

$$Y(x,\lambda) = \begin{cases} d_{11}(x,\lambda)\varphi^-(x,\lambda) + d_{12}(x,\lambda)\psi^-(x,\lambda) & \text{for } x \in [a,c), \\ d_{21}(x,\lambda)\varphi^+(x,\lambda) + d_{22}(x,\lambda)\psi^+(x,\lambda) & \text{for } x \in (c,b], \end{cases} \tag{21}$$

where the functions $d_{11}(x,\lambda)$, $d_{12}(x,\lambda)$ are the solutions of the system of equations

$$\begin{cases} \frac{\partial d_{11}(x,\lambda)}{\partial x}\varphi^-(x,\lambda) + \frac{\partial d_{12}(x,\lambda)}{\partial x}\psi^-(x,\lambda) = 0, \\ \frac{\partial d_{11}(x,\lambda)}{\partial x}\frac{\partial \varphi^-(x,\lambda)}{\partial x} + \frac{\partial d_{12}(x,\lambda)}{\partial x}\frac{\partial \psi^-(x,\lambda)}{\partial x} = \frac{u(x)}{p^-} \end{cases} \tag{22}$$

for $x \in [a,c)$, and $d_{21}(x,\lambda)$, $d_{22}(x,\lambda)$ are the solutions of the system of equations

$$\begin{cases} \frac{\partial d_{21}(x,\lambda)}{\partial x}\varphi^+(x,\lambda) + \frac{\partial d_{22}(x,\lambda)}{\partial x}\psi^+(x,\lambda) = 0, \\ \frac{\partial d_{21}(x,\lambda)}{\partial x}\frac{\partial \varphi^+(x,\lambda)}{\partial x} + \frac{\partial d_{22}(x,\lambda)}{\partial x}\frac{\partial \psi^+(x,\lambda)}{\partial x} = \frac{u(x)}{p^+} \end{cases} \tag{23}$$

for $x \in (c,b]$. Since λ is not an eigenvalue, we have $\omega^{\pm}(\lambda) \neq 0$. Hence, from (22) and (23) it follows that

$$d_{11}(x,\lambda) = \frac{\Delta_{34}}{p^-\omega(\lambda)}\int_x^{c-} u(y)\psi^-(y,\lambda)\,dy + h_{11}(\lambda), \quad x \in [a,c),$$

$$d_{12}(x,\lambda) = \frac{\Delta_{34}}{p^-\omega(\lambda)}\int_a^{x} u(y)\varphi^-(y,\lambda)\,dy + h_{12}(\lambda), \quad x \in [a,c),$$

$$d_{21}(x,\lambda) = \frac{\Delta_{12}}{p^+\omega(\lambda)}\int_x^{b} u(y)\psi^+(y,\lambda)\,dy + h_{21}(\lambda), \quad x \in (c,b],$$

$$d_{22}(x,\lambda) = \frac{\Delta_{12}}{p^+\omega(\lambda)}\int_{c+}^{x} u(y)\varphi^+(y,\lambda)\,dy + h_{22}(\lambda), \quad x \in (c,b],$$

where $h_{ij}(\lambda)$ $(i,j = 1,2)$ are arbitrary functions of the parameter λ. Substituting this into (21) gives

$$Y(x,\lambda) = \begin{cases} \frac{\Delta_{34}\psi^-(x,\lambda)}{p^-\omega(\lambda)}\int_a^x \varphi^-(y,\lambda)u(y)\,dy + \frac{\Delta_{34}\varphi^-(x,\lambda)}{p^-\omega(\lambda)}\int_x^{c-} \psi^-(y,\lambda)u(y)\,dy \\ \quad + h_{11}(\lambda)\varphi^-(x,\lambda) + h_{12}(\lambda)\psi^-(x,\lambda) \quad \text{for } x \in [a,c), \\ \frac{\Delta_{12}\psi^+(x,\lambda)}{p^+\omega(\lambda)}\int_{c+}^x \varphi_2(y,\lambda)u(y)\,dy + \frac{\Delta_{12}\varphi^+(x,\lambda)}{p^+\omega(\lambda)}\int_x^{b} \psi^+(y,\lambda)u(y)\,dy \\ \quad + h_{21}(\lambda)\varphi^+(x,\lambda) + h_{22}(\lambda)\psi^+(x,\lambda) \quad \text{for } x \in (c,b]. \end{cases} \tag{24}$$

By differentiating we have

$$\frac{\partial Y(x,\lambda)}{\partial x} = \begin{cases} \frac{\Delta_{34}}{p^-\omega(\lambda)}\frac{\partial\psi^-(x,\lambda)}{\partial x}\int_a^x \varphi^-(y,\lambda)u(y)\,dy + \frac{\Delta_{34}}{p^-\omega(\lambda)}\frac{\partial\varphi^-(x,\lambda)}{\partial x}\int_x^{c-}\psi^-(y,\lambda)u(y)\,dy \\ \quad + h_{11}(\lambda)\frac{\partial\varphi^-(x,\lambda)}{\partial x} + h_{12}(\lambda)\frac{\partial\psi^-(x,\lambda)}{\partial x} \quad \text{for } x\in[a,c), \\ \frac{\Delta_{12}}{p^+\omega(\lambda)}\frac{\partial\psi^+(x,\lambda)}{\partial x}\int_{c+}^x \varphi^+(y,\lambda)u(y)\,dy + \frac{\Delta_{12}}{p^+\omega(\lambda)}\frac{\partial\varphi^+(x,\lambda)}{\partial x}\int_x^{b}\psi^+(y,\lambda)u(y)\,dy \\ \quad + h_{21}(\lambda)\frac{\partial\varphi^+(x,\lambda)}{\partial x} + h_{22}(\lambda)\frac{\partial\psi^+(x,\lambda)}{\partial x} \quad \text{for } x\in(c,b]. \end{cases} \tag{25}$$

By using equalities (24), (25) and boundary conditions (20), we can derive that

$$h_{12}(\lambda) = \frac{u_1}{\omega^-(\lambda)}, \qquad h_{21}(\lambda) = \frac{u_2}{\omega^+(\lambda)},$$
$$h_{11}(\lambda) = \frac{1}{p^+\omega^+(\lambda)}\int_{c+}^b \psi^+(y,\lambda)u(y)\,dy + \frac{u_2}{\omega^+(\lambda)}$$

and

$$h_{22}(\lambda) = \frac{1}{p^-\omega^-(\lambda)}\int_a^{c-} \varphi^-(y,\lambda)u(y)\,dy + \frac{u_1}{\omega^-(\lambda)}.$$

Putting in (24) gives

$$Y(x,\lambda) = \begin{cases} \frac{\Delta_{34}\psi^-(x,\lambda)}{p^-\omega(\lambda)}\int_a^x \varphi^-(y,\lambda)u(y)\,dy + \frac{\Delta_{34}\varphi^-(x,\lambda)}{p^-\omega(\lambda)}\int_x^{c-}\psi^-(y,\lambda)u(y)\,dy \\ \quad + \frac{\Delta_{12}\varphi^-(x,\lambda)}{\omega(\lambda)}(\frac{1}{p^+}\int_{c+}^b \psi^+(y,\lambda)u(y)\,dy + u_2) \\ \quad + \frac{\Delta_{34}u_1\psi^-(x,\lambda)}{\omega(\lambda)} \quad \text{for } x\in[a,c), \\ \frac{\Delta_{12}\psi^+(x,\lambda)}{p^+\omega(\lambda)}\int_{c+}^x \varphi^+(y,\lambda)u(y)\,dy + \frac{\Delta_{12}\varphi^+(x,\lambda)}{p^+\omega(\lambda)}\int_x^{b}\psi^+(y,\lambda)u(y)\,dy \\ \quad + \frac{\Delta_{34}\psi^+(x,\lambda)}{\omega(\lambda)}(\frac{1}{p^-}\int_a^{c-} \varphi^-(y,\lambda)u(y)\,dy + u_1) \\ \quad + \frac{\Delta_{12}u_2\varphi^+(x,\lambda)}{\omega(\lambda)} \quad \text{for } x\in(c,b]. \end{cases} \tag{26}$$

Let us introduce the Green's function as

$$G_1(x,y;\lambda) = \begin{cases} \frac{\varphi^-(x,\lambda)\psi^-(y,\lambda)}{\Delta_{34}p^-\omega^-(\lambda)} & \text{if } x\in[a,c), y\in[a,x), \\ \frac{\psi^-(x,\lambda)\varphi^-(y,\lambda)}{\Delta_{34}p^-\omega^-(\lambda)} & \text{if } x\in[a,c), y\in[x,c), \\ \frac{\psi^-(x,\lambda)\varphi^+(y,\lambda)}{\Delta_{34}p^-\omega^-(\lambda)} & \text{if } x\in[a,c), y\in(c,b], \\ \frac{\varphi^+(x,\lambda)\psi^-(y,\lambda)}{\Delta_{12}p^+\omega^+(\lambda)} & \text{if } x\in(c,b], y\in[a,c), \\ \frac{\varphi^+(x,\lambda)\psi^+(y,\lambda)}{\Delta_{12}p^+\omega^+(\lambda)} & \text{if } x\in(c,b], y\in(c,x], \\ \frac{\psi^+(x,\lambda)\varphi^+(y,\lambda)}{\Delta_{12}p^+\omega^+(\lambda)} & \text{if } x\in(c,b], y\in[x,b]. \end{cases} \tag{27}$$

Then from (26) and (27) it follows that the considered problem (19), (20) has a unique solution given by

$$\begin{aligned} Y(x,\lambda) &= \frac{\Delta_{34}}{p^-}\int_a^{c-} G_1(x,y;\lambda)u(y)\,dy + \frac{\Delta_{12}}{p^+}\int_{c+}^b G_1(x,y;\lambda)u(y)\,dy \\ &\quad + \Delta_{34}u_1\frac{\psi(x,\lambda)}{\omega(\lambda)} + \Delta_{12}u_2\frac{\varphi(x,\lambda)}{\omega(\lambda)}. \end{aligned} \tag{28}$$

In fact, we have proved the following theorem.

Theorem 2 *The resolvent operator can be represented as*

$$(\lambda I - \mathcal{L})^{-1}U(x) = \begin{pmatrix} \int_a^b G(x,y;\lambda)u(y)\,dy + \Delta_{34}u_1\frac{\psi(x,\lambda)}{\omega(\lambda)} + \Delta_{12}u_2\frac{\varphi(x,\lambda)}{\omega(\lambda)} \\ B'_a[u] \\ -B'_b[u] \end{pmatrix},$$

where

$$G(x,y;\lambda) = \begin{cases} \frac{\Delta_{34}}{p^-}G_1(x,y;\lambda) & \text{if } a < y < c, \\ \frac{\Delta_{12}}{p^+}G_1(x,y;\lambda) & \text{if } c < y < b. \end{cases} \tag{29}$$

Remark 1 Although the Green's function looks as simple as that of standard Sturm-Liouville problems, it is rather complicated because of the transmission conditions. To illustrate this situation, let us give the following example.

Example Consider the following simple case of BVTP's (1)-(4) on $[-1,1]$ with $c = 0$:

$$-y''(x) = \lambda y(x), \tag{30}$$

$$y(-1) + \lambda y'(-1) = 0, \tag{31}$$

$$\lambda y(1) + y'(1) = 0, \tag{32}$$

$$\begin{aligned} &y'(0-) = y(0+), \\ &y'(0-) = 2y'(0+), \end{aligned} \tag{33}$$

where λ is a complex spectral parameter. Putting $\lambda = \mu^2$ we find easily that

$$\begin{aligned} \varphi^-(x,\mu) &= \mu^2\cos\bigl[\mu(x+1)\bigr] - \frac{1}{\mu}\sin\bigl[\mu(x+1)\bigr], \\ \varphi^+(x,\mu) &= \left(\mu^2\cos\mu - \frac{1}{\mu}\sin\mu\right)\cos(\mu x) \\ &\quad - \frac{1}{2}\left(\mu^2\sin\mu + \frac{1}{\mu}\cos\mu\right)\sin(\mu x), \\ \psi^-(x,\mu) &= (\cos\mu - \mu\sin\mu)\cos(\mu x) \\ &\quad + 2(\sin\mu + \mu\cos\mu)\sin(\mu x), \\ \psi^+(x,\mu) &= \cos\bigl[\mu(1-x)\bigr] - \mu\sin\bigl[\mu(1-x)\bigr]. \end{aligned}$$

Using these formulas, we have

$$\begin{aligned} w(\mu) &= \bigl(2\mu^4+1\bigr)\cos^2\mu - \bigl(\mu^4+2\bigr)\sin^2\mu \\ &\quad + 3\bigl(\mu^3-\mu\bigr)\sin\mu\cos\mu. \end{aligned}$$

Consequently, the Green's function has the following form:

$$G(x,y,\mu) = \bigl\{\bigl(2\mu^4+1\bigr)\cos^2\mu - \bigl(\mu^4+2\bigr)\sin^2\mu + 3\bigl(\mu^3-\mu\bigr)\sin\mu\cos\mu\bigr\}$$

$$\times \begin{cases} \{\mu^2\cos[\mu(x+1)] - \frac{1}{\mu}\sin[\mu(x+1)]\} \times \{(\cos\mu - \mu\sin\mu)\cos(\mu y) \\ \quad + 2(\sin\mu + \mu\cos\mu)\sin(\mu y)\}, \quad -1 \le x \le y < 0, \\ \{(\cos\mu - \mu\sin\mu)\cos(\mu x) + 2(\sin\mu + \mu\cos\mu)\sin(\mu x)\} \\ \quad \times \{\mu^2\cos[\mu(y+1)] - \frac{1}{\mu}\sin[\mu(y+1)]\}, \quad -1 \le y \le x < 0, \\ \{(\cos\mu - \mu\sin\mu)\cos(\mu x) + 2(\sin\mu + \mu\cos\mu)\sin(\mu x)\} \\ \quad \times \{(\mu^2\cos\mu - \frac{1}{\mu}\sin\mu)\cos(\mu y) - \frac{1}{2}(\mu^2\sin\mu + \frac{1}{\mu}\cos\mu)\sin(\mu y)\}, \\ \quad -1 \le y < 0, 0 < x \le 1, \\ \{(\mu^2\cos\mu - \frac{1}{\mu}\sin\mu)\cos(\mu x) - \frac{1}{2}(\mu^2\sin\mu + \frac{1}{\mu}\cos\mu)\sin(\mu x)\} \\ \quad \times \{(\cos\mu - \mu\sin\mu)\cos(\mu y) + 2(\sin\mu + \mu\cos\mu)\sin(\mu y)\}, \\ \quad -1 \le x < 0, 0 < y \le 1, \\ \{(\mu^2\cos\mu - \frac{1}{\mu}\sin\mu)\cos(\mu x) - \frac{1}{2}(\mu^2\sin\mu + \frac{1}{\mu}\cos\mu)\sin(\mu x)\} \\ \quad \times \{\cos[\mu(1-y)] - \mu\sin[\mu(1-y)]\}, \quad 0 < y \le x \le 1, \\ \{\cos[\mu(1-x)] - \mu\sin[\mu(1-x)]\} \times \{(\mu^2\cos\mu - \frac{1}{\mu}\sin\mu)\cos(\mu y) \\ \quad - \frac{1}{2}(\mu^2\sin\mu + \frac{1}{\mu}\cos\mu)\sin(\mu y)\}, \quad 0 < x \le y \le 1. \end{cases}$$

6 Figures

The graph of the Green's function is displayed in Figure 1 and Figure 2 for two different values of the spectral parameter.

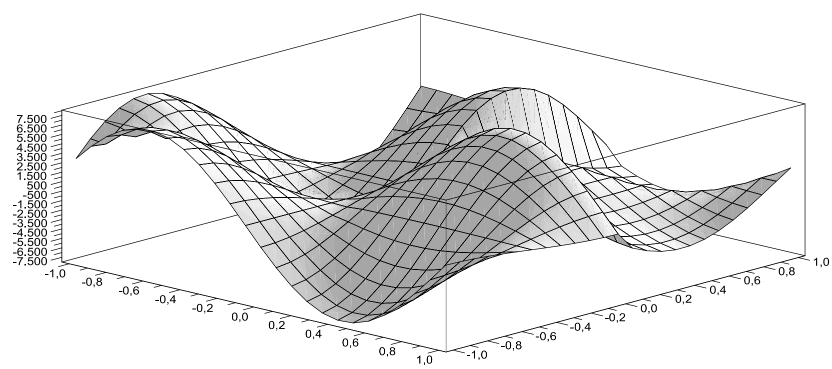

Figure 1 The graph of the Green's function $G(x,t,\mu)$ for $\mu = 3$.

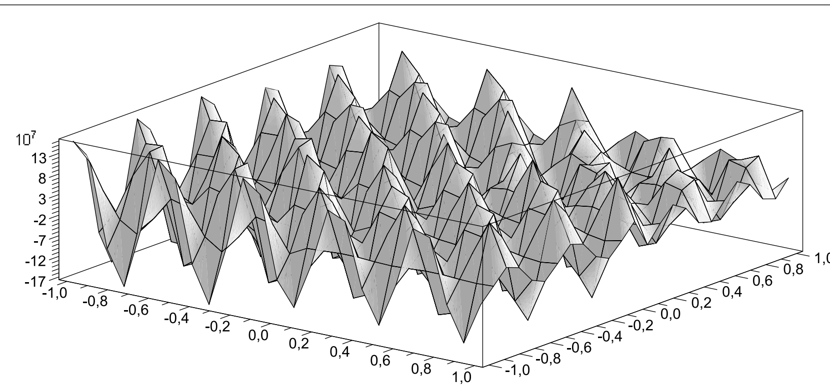

Figure 2 The graph of the Green's function $G(x,t,\mu)$ for $\mu = 15$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. The authors read and approved the final manuscript.

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