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Periodic and subharmonic solutions for a class of the second-order Hamiltonian systems with impulsive effects

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Abstract

This paper is concerned with the existence of periodic and subharmonic solutions for a class of the second-order impulsive Hamiltonian systems. It employs the linking theorem.

Keywords: critical point theorem; impulsive differential equations; periodic solution

1 Introduction and main results

In this paper, we consider the second-order impulsive differential equation

$$\begin{cases} -\ddot{q}(t) = \nabla F(t, q(t)), & t \neq t_j, t \in \mathbb{R}, \\ \Delta \dot{q}(t_j) = -g_j(q(t_j)), & j \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where $q \in \mathbb{R}^N$, $\nabla F(t,q) = \operatorname{grad}_q F(t,q)$, $g_j(q) = \operatorname{grad}_q G_j(q)$, $G_j \in (\mathbb{R}^N, \mathbb{R})$ for each $j \in \mathbb{Z}$, and the operator Δ is defined as $\Delta \dot{q}(t_j) = \dot{q}(t_j^+) - \dot{q}(t_j^-)$, where $\dot{q}(t_j^+)$ ($\dot{q}(t_j^-)$) denotes the right-hand (left-hand) limit of $\dot{q}$ at t_j. There exist an $m \in \mathbb{N}$ and a $T > 0$ such that $0 = t_0 < t_1 < \cdots < t_m = T$, $t_{j+m} = t_j + T$, and $g_{j+m} = g_j$, $j \in \mathbb{Z}$. $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in its first variable and satisfies:

(H0) $F(t,q)$ is measurable in t for each $q \in \mathbb{R}^N$ and continuously differentiable in q for a.e. $t \in [0,T]$ and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0,T]; \mathbb{R}^+)$ such that

$$\left|F(t,q)\right| \leq a\left(|q|\right)b(t), \qquad \left|\nabla F(t,q)\right| \leq a\left(|q|\right)b(t)$$

for all $q \in \mathbb{R}$ and a.e. $t \in [0,T]$.

Let

$$H_T^1 = \left\{q : \mathbb{R} \to \mathbb{R}^N \mid q, \dot{q} \in L^2\left(\left([0,T]\right), \mathbb{R}^N\right), q(t) = q(t+T), t \in \mathbb{R}\right\}.$$

Then H_T^1 is a Hilbert space with the norm defined by

$$\|q\|_{H_T^1} = \left(\int_0^T \left(\left|\dot{q}(t)\right|^2 + |q|^2\right) dt\right)^{\frac{1}{2}}, \quad q \in H_T^1.$$

For the norm in $L^2([0,T])$, we put

$$\|q\|_{L^2} = \left(\int_0^T |q(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Next we set $\Omega = \{1,2,\ldots,m-1\}$, and define a functional φ as

$$\varphi(q) = \frac{1}{2}\int_0^T |\dot{q}(t)|^2\,dt - \int_0^T F\bigl(t,q(t)\bigr)\,dt - \sum_{j\in\Omega} G_j\bigl(q(t_j)\bigr), \quad q \in H_T^1. \tag{1.2}$$

Note that φ is Fréchet differentiable at any $q \in H_T^1$ and for any $p \in H_T^1$, we have

$$\begin{aligned}\varphi'(q)(p) &= \lim_{h\to 0}\frac{\varphi(q+hp)-\varphi(q)}{h}\\ &= \int_0^T \bigl(\dot{q}(t)\dot{p}(t) - \nabla F\bigl(t,q(t)\bigr)p(t)\bigr)\,dt - \sum_{j\in\Omega} g_j\bigl(q(t_j)\bigr)p(t_j).\end{aligned}$$

It is clear that the critical points of the functional φ are classical T-periodic solutions of system (1.1).

When the impulsive function $g_j = 0$, the system (1.1) reduces to the following second-order Hamiltonian system:

$$-\ddot{q}(t) = \nabla F\bigl(t,q(t)\bigr), \quad t \in \mathbb{R}. \tag{1.3}$$

The existence of periodic solutions for system (1.3) has been discussed extensively in the literature; see [1–5].

Note that system (1.3) is called a superquadratic second-order Hamiltonian system if the potential function F satisfies

$$\lim_{q\to+\infty}\frac{F(t,q)}{|q|^2} = +\infty. \tag{1.4}$$

In 1978, Rabinowitz [6] got the nonconstant periodic solutions under the following condition: there exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(t,q) \le \nabla F(t,q)q, \quad \forall |q| \ge L, t \in [0,T], \tag{1.5}$$

which is stronger than (1.4) and is known as the Ambrosetti-Rabinowitz condition (A-R condition). From then on, many authors have devoted their work to the investigation concerning the existence of solutions of second-order systems under condition (1.5); see [7, 8] and references therein. In 2002, Fei [9] obtained the existence of solutions for system (1.3) under a kind of new superquadratic condition which is different from the A-R condition. Subsequently, Tao and Tang [10] gave the following two results, more general than Fei's.

Theorem A *Assume that F satisfies* (H0) *and the following conditions*:

(H1) $F(t,q) \ge 0$, $(t,q) \in [0,T] \times \mathbb{R}^N$,
(H2) $\lim_{|q|\to 0}\frac{F(t,q)}{|q|^2} < \frac{1}{2}\omega^2$ *uniformly for a.e.* $t \in [0,T]$,

(H3) $\liminf_{|q|\to+\infty} \frac{F(t,q)}{|q|^2} > \frac{1}{2}\omega^2$ *uniformly for a.e.* $t \in [0,T]$,
(H4) $\limsup_{|q|\to+\infty} \frac{F(t,q)}{|q|^r} \leq +\infty$ *uniformly for a.e.* $t \in [0,T]$,
(H5) $\liminf_{|q|\to+\infty} \frac{\nabla F(t,q)q-2F(t,q)}{|q|^\mu} > 0$ *uniformly for a.e.* $t \in [0,T]$,

where $\omega = \frac{2\pi}{T}$, $r > 2$, *and* $\mu > r-2$. *Then there exists a nonconstant* T*-periodic solution of system* (1.3).

Theorem B *Assume that F satisfies* (H0), (H1), (H3), (H4), (H5), *and the following condition*:

(H2′) $\lim_{|q|\to 0} \frac{F(t,q)}{|q|^2} = 0$ *uniformly for a.e.* $t \in [0,T]$.

Then there exists a sequence $\{k_n\} \subset \mathbb{N}$, $k_n \to +\infty$, *and the corresponding distinct* k_nT *are periodic solutions of system* (1.3).

It is well known that the theory of impulsive differential equations has emerged as an important area of investigation. Some classical tools such as some fixed point theorems in cones, topological degree theory, the upper and lower solutions method combined with monotone iterative technique [11–13] have been widely used to get solutions of impulsive differential equations. Recently, some researchers have studied the existence of solutions for impulsive differential equations with boundary conditions via variational methods [14–22]. For the second-order differential equation $u'' = f(t,u,u')$, we generally consider impulses in the position u and u'. However, in the motion of spacecraft instantaneous impulses depend on the position, which results in jump discontinuities in velocity, with no change in position. This motivates us to consider the second-order impulsive Hamiltonian system (1.1). By employing critical point theory and variational methods we obtain the existence of periodic and subharmonic solutions for it. The following results can be regarded as a generalization to Theorems A and B.

Theorem 1.1 *Assume that F satisfies* (H0), (H1), (H3), (H4), (H5) *and the following conditions hold*:

(H2″) $\lim_{|q|\to 0} \frac{F(t,q)}{|q|^2} < \frac{1}{4}\omega^2$ *uniformly for a.e.* $t \in [0,T]$,
(G1) $G_j(q) \geq 0$, $q \in \mathbb{R}^N$, $j = 1,2,\ldots,m$,
(G2) $\lim_{|q|\to 0} \frac{G_j(q)}{|q|^2} = 0$, $j = 1,2,\ldots,m$,
(G3) *there exists* $M_1 > 0$ *such that* $G_j(q) \leq M_1|q|^r$, $q \in \mathbb{R}^N$, $j = 1,2,\ldots,m$,
(G4) $g_j(q)q - 2G_j(q) \geq 0$, $q \in \mathbb{R}^N \setminus \{0\}$, $j = 1,2,\ldots,m$.

Then system (1.1) *has at least one non-trivial* T*-periodic solution.*

Theorem 1.2 *Assume that F satisfies* (H0), (H1), (H2′), (H3), (H4), (H5) *and* G_j *satisfies* (G1), (G2), (G3), (G4). *Then system* (1.1) *has a sequence of distinct periodic solutions with period* k_nT *satisfying* $k_n \in \mathbb{N}$ *and* $k_n \to +\infty$ *as* $n \to +\infty$.

In order to prove our theorems, we need the following result. For $u \in H_T^1$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. One has

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Sobolev's inequality)} \tag{1.6}$$

and

$$\int_0^T \left|\tilde{u}(t)\right|^2 dt \leq \frac{T^2}{4\pi^2}\int_0^T \left|\dot{u}(t)\right|^2 dt \quad \text{(Wirtinger's inequality)}. \tag{1.7}$$

Lemma 1.3 *If $u \in H_T^1$, then there exists a constant C_0 such that $\|q\|_\infty \leq C_0\|q\|_{H_T^1}$, where* $\|q\|_\infty = \max_{t\in[0,T]}|q(t)|$.

Proof The proof follows easily from the Hölder inequality. The detailed argument is similar to the proof of Lemma 2.1 in [23] and we thus omit it here. □

Lemma 1.4 [24] *Let $X = X_1 \oplus X_2$ be a real Banach space, where X_1 is a finite dimensional closed subspace of X and $X_2 = X_1^\perp$. Suppose that $\varphi \in C^1(X,R)$ satisfies the Palais-Smale condition and the following conditions*:

(i) *there exist constant $\rho > 0$ and a such that $\varphi(x) \geq a$, $\forall x \in X_2 \cap \partial B_\rho$, where* $B_\rho = \{x \in X : \|x\|_X < \rho\}$,

(ii) *there exist a constant $w < a$ and $e \in X_2$, $\|e\|_X = 1$, $s_1 > 0$, $s_2 > \rho$ such that $\varphi(x)|_{\partial Q} \leq w$, where $Q = \{x \in X \mid x = z + \lambda e, z \in X_1, |z| \leq s_1, \lambda \in (0, s_2)\}$.*

Then φ possesses a critical value.

2 Proof of Theorem 1.1

Proof of Theorem 1.1 It is well known that Lemma 1.3 holds true with the condition (C) replacing the usual Palais-Smale condition. We say the functional φ satisfies the condition (C), *i.e.*, for every sequence $\{q_n\} \subset H_T^1$, $\{q_n\}$ has a convergent subsequence if $\{\varphi(q_n)\}$ is bounded and $\lim_{n\to\infty}(1 + \|q\|_{H_T^1})\|\varphi'(q_n)\|_{H_T^1} = 0$. To this end, we prove Theorem 1.1 in the following steps.

Step 1. Pick $\{q_n\} \subset H_T^1$ such that $\{\varphi_k(q_n)\}$ is bounded and $\lim_{n\to\infty}(1 + \|q\|_{H_T^1})\|\varphi'(q_n)\|_{H_T^1} = 0$, then there exists a constant $C_1 > 0$ such that

$$\left|\varphi(q_n)\right| \leq C_1, \qquad \left(1 + \|q_n\|_{H_T^1}\right)\left\|\varphi'(q_n)\right\|_{H_T^1} \leq C_1$$

for all $n \in \mathbb{N}$. By (H4), there exist constants $C_2 > 0$ and $d_1 > 0$ such that

$$F(t,q) \leq C_2|q|^r \tag{2.1}$$

for all $|q| \geq d_1$ and a.e. $t \in [0,T]$. It follows from (H0) that $F(t,q) \leq \max_{s\in[0,d_1]} a(s)b(t)$, for all $|q| \leq d_1$ and a.e. $t \in [0,T]$. Therefore, we obtain

$$F(t,q) \leq C_2|q|^r + \max_{s\in[0,d_1]} a(s)b(t)$$

for all $q \in \mathbb{R}$ and a.e. $t \in [0,T]$. Set $C_3 = \max_{s\in[0,d_1]} a(s)\int_0^T b(t)\,dt$. By (1.2), we have

$$\begin{aligned}\frac{1}{2}\|q_n\|_{H_T^1}^2 &= \frac{1}{2}\int_0^T \left|q_n(t)\right|^2 dt + \varphi(q_n) + \int_0^T F\big(t, q_n(t)\big)\,dt + \sum_{j\in\Omega} G_j\big(q_n(t_j)\big) \\ &\leq C_1 + C_2\int_0^T \left|q_n(t)\right|^r dt + \frac{1}{2}\int_0^T \left|q_n(t)\right|^2 dt\end{aligned}$$

$$+ \max_{s\in[0,d_1]} a(s) \int_0^T b(t)\,dt + M_1 \sum_{j\in\Omega} |q_n|^r$$

$$\leq C_1 + C_3 + (C_2 + M_1) \int_0^T \left|q_n(t)\right|^r dt + \frac{T^{\frac{r-2}{r}}}{2} \left(\int_0^T \left|q_n(t)\right|^r dt \right)^{\frac{2}{r}}. \tag{2.2}$$

On the other hand, by (H5), there exist constants $C_4 > 0$ and $d_2 > 0$ such that

$$\nabla F(t,q)q - 2F(t,q) \geq C_4|q|^{\mu}, \quad |q| \geq d_2, t \in [0,T]. \tag{2.3}$$

By (H0), we have

$$\left|\nabla F(t,q)q - 2F(t,q)\right| \leq (2 + d_2) \max_{s\in[0,d_2]} a(s)b(t), \quad |q| \leq d_2, t \in [0,T]. \tag{2.4}$$

Therefore by (2.3), (2.4), and (H4), we have

$$\begin{aligned} 3C_1 &\geq 2\varphi(q_n) - \varphi'(q_n)(q_n) \\ &= \int_0^T \left[\nabla F(t,q_n)q_n - 2F(t,q_n)\right] dt + \sum_{j\in\Omega} \left[g_j\big(q_n(t_j)\big)q_n(t_j) - 2G_j\big(q_n(t_j)\big)\right] \\ &\geq C_4 \int_0^T |q_n|^{\mu}\,dt - (2 + d_2) \max_{s\in[0,d_2]} a(s) \int_0^T b(t)\,dt, \end{aligned}$$

which implies $(\int_0^T |q_n|^{\mu}\,dt)^{\frac{1}{\mu}}$ is bounded, *i.e.*, there exists a constant $C_5 > 0$ such that

$$\left(\int_0^T |q_n|^{\mu}\,dt \right)^{\frac{1}{\mu}} < C_5.$$

If $\mu > r$, then we have $\int_0^T |q_n(t)|^r\,dt \leq T^{\frac{\mu-r}{\mu}} (\int_0^T |q_n(t)|^{\mu}\,dt)^{\frac{r}{\mu}}$, which, combining with (2.2), implies that $\|q_n\|_{H_T^1}$ is bounded. If $\mu \leq r$, then we have $\int_0^T |q_n(t)|^r\,dt \leq C_0^{r-\mu} \|q_n\|_{H_T^1}^{r-\mu} \times \int_0^T |q_n(t)|^{\mu}\,dt$. Since $\mu > r - 2$, it follows from (2.2) that $\|q_n\|_{H_T^1}$ is bounded too. In a similar way to Proposition B35 in [24], we can prove that $\{q_n\}$ has a convergent subsequence. So, the functional φ satisfies the condition (C).

Step 2. We show that the functional φ satisfies the assumption (i) of Lemma 1.4. Let $X = H_T^1$, $X_1 = \mathbb{R}^n$, $X_2 = \tilde{H}_T^1 = \{q \in H_T^1 \mid \int_0^T q(t)\,dt = 0\}$. Then $H_T^1 = X_1 \oplus X_2$ and X_1 is a finite dimensional subspace of H_T^1.

By (H2″), there exists a constant $0 < d_3 < d_1$ such that $F(t,q) \leq \frac{\omega^2}{4}|q|^2$, $|q| \leq d_3$, $t \in [0,T]$. So we have

$$F(t,q) \leq \frac{\omega^2}{4}|q|^2 + \max_{s\in[d_3,d_1]} a(s)b(t)d_3^{-r}|q|^r + C_2|q|^r, \quad q \in \mathbb{R}^N, t \in [0,T]. \tag{2.5}$$

By (1.2), (1.6), (1.7), (2.5), and (G3), $\forall q \in X_2$, we have

$$\begin{aligned} \varphi(q) &= \frac{1}{2} \int_0^T \left|\dot{q}(t)\right|^2 dt - \int_0^T F\big(t,q(t)\big)\,dt - \sum_{j\in\Omega} G_j\big(q(t_j)\big) \\ &\geq \frac{1}{2} \int_0^T \left|\dot{q}(t)\right|^2 dt - \frac{\omega^2}{4} \int_0^T \left|q(t)\right|^2 dt \end{aligned}$$

$$
\begin{aligned}
&\quad - d_3^{-r} \max_{s\in[d_3,d_1]} a(s) \int_0^T b(t)\left|q(t)\right|^r dt - C_2 \int_0^T \left|q(t)\right|^r dt - M_1 \sum_{j\in\Omega}\left|q(t_j)\right|^r \\
&\geq \frac{\pi^2}{4\pi^2+T^2}\|q\|_{H_T^1}^2 - \biggl(C_0^r d_3^{-r} \max_{s\in[d_3,d_1]} a(s) \int_0^T b(t)\,dt \\
&\quad + C_2 C_0^{r-2} + M_1 C_0^{r-2}\biggr)\|q\|_{H_T^1}^r.
\end{aligned} \tag{2.6}
$$

Hence, there exist constants $a > 0$ and $\rho \in (0,1)$, such that

$$\varphi(q) \geq a > 0, \quad \forall q \in X_2 \quad \text{and} \quad \|q\|_{H_T^1} = \rho,$$

which proves (i).

Finally, we show that the functional φ satisfies the assumption (ii) of Lemma 1.4. For a given $z \in X_1 = \mathbb{R}^n$, by assumptions (H1), (G1), we have

$$\varphi(z) = -\int_0^T F(t,z)\,dt - \sum_{j\in\Omega} G_j(z) \leq 0 < a.$$

In what follows, we construct a bounded manifold $Q \subset X$ such that $\varphi(q) \leq a$, $\forall q \in \partial Q$. Pick $e = (\sqrt{\frac{2}{(1+\omega^2)T}}\cos\omega t, 0, 0, \ldots, 0) \in X_2$. By calculation, we have $\|e\|_{H_T^1} = 1$. By (H3), for

$$\delta = \inf_{t\in[0,T]} \liminf_{|q|\to+\infty} \frac{F(t,q)}{|q|^2} - \frac{\omega^2}{2} > 0,$$

there exists a constant $d_4 > 0$ such that when $|q| \geq d_4$, we have

$$F(t,q) \geq \left(\delta + \frac{\omega^2}{2}\right)|q|^2. \tag{2.7}$$

Therefore, we have

$$F(t,q) \geq \left(\delta + \frac{\omega^2}{2}\right)|q|^2 - \left(\delta + \frac{\omega^2}{2}\right)d_4^2, \quad q \in \mathbb{R}^N, t \in [0,T]. \tag{2.8}$$

Then for any given $q = z + \lambda e$, $z \in X_1$, $\lambda \in \mathbb{R}$, from (1.2), (G1) and (2.8), we get

$$
\begin{aligned}
\varphi(z+\lambda e) &= \frac{1}{2}\int_0^T \left|\lambda \dot{e}(t)\right|^2 dt - \int_0^T F\bigl(t, z+\lambda e(t)\bigr)\,dt - \sum_{j\in\Omega} G_j\bigl(z+\lambda e(t_j)\bigr) \\
&\leq \frac{1}{2}\frac{2\omega^2\lambda^2}{(1+\omega^2)T}\frac{T}{2} - \left(\delta + \frac{\omega^2}{2}\right)\int_0^T \left|z+\lambda e(t)\right|^2 dt + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 T \\
&= \frac{\omega^2\lambda^2}{2(1+\omega^2)} - \left(\delta + \frac{\omega^2}{2}\right)\frac{2\lambda^2}{(1+\omega^2)T}\frac{T}{2} - \left(\delta + \frac{\omega^2}{2}\right)Tz^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 T \\
&= -\frac{\delta\lambda^2}{1+\omega^2} - \left(\delta + \frac{\omega^2}{2}\right)Tz^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 T.
\end{aligned}
$$

Let

$$f_1(x) = -\left(\delta + \frac{\omega^2}{2}\right)Tx^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 T, \quad x \in \mathbb{R}$$

and

$$f_2(x) = -\frac{\delta x^2}{1+\omega^2}, \quad x \in \mathbb{R}.$$

Clearly, it can be seen that $f_1(x)$ and f_2 attain their maximum at zero. Therefore,

$$\varphi(z+\lambda e) \le f_1(0) + f_2(|\lambda|) = f_2(|\lambda|) + \left(\delta + \frac{\omega^2}{2}\right) d_4^2 T,$$

$$\varphi(z+\lambda e) \le f_1(|z|) + f_2(0) = f_1(|z|).$$

Since $\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} f_2(x) = -\infty$, we can choose $s_1 > 0$, $s_2 > \rho$ such that $\varphi(z+se) < 0$, for $|z| = s_1$, or $\lambda = s_2$. Let $Q = \{q \in X \mid q = z + \lambda e, z \in X_1, |z| \le s_1, \lambda \in (0, s_2)\}$, we obtain $\varphi|_{\partial Q} < 0 < a$, which proves (ii). From the above proofs, we know that the assumptions of Lemma 1.4 are satisfied. Consequently, system (1.1) admits at least one periodic solution. □

3 Proof of Theorem 1.2

Proof of Theorem 1.2 Let $k \ge 2$. Replace T by kT in the definitions of H_T^1, $\tilde{H}_T^1$, φ, and φ' in Theorem 1.1, then we obtain the corresponding spaces and functionals. We denote them by H_{kT}^1, $\tilde{H}_{kT}^1$, φ_k, and φ_k', respectively. Define

$$\|q\|_{H_{kT}^1} = \left(\int_0^{kT} \left(|\dot{q}(t)|^2 + |q|^2\right) dt\right)^{\frac{1}{2}}, \quad q \in H_{kT}^1.$$

Similar arguments to Theorem 1.1 show that the functional φ_k satisfies the condition (C). By (H2′), for $0 < \varepsilon_1 < \frac{2\pi^2}{4\pi^2+T^2}$, there exists a constant $0 < d_5 < d_1$ such that when $|q| \le d_5$ and $t \in [0,T]$, we have $|F(t,q)| \le \varepsilon_1 |q|^2$, and combining (2.1) and (H0), we obtain

$$F(t,q) \le \varepsilon_1 |q|^2 + \max_{s\in[d_5,d_1]} a(s)b(t)d_5^{-r}|q|^r + C_2|q|^r, \quad q \in \mathbb{R}^N, t \in [0,T].$$

So for any given $q \in X_2$, we have

$$\begin{aligned}
\varphi_k(q) &= \frac{1}{2}\int_0^{kT} |\dot{q}(t)|^2\,dt - \int_0^{kT} F(t,q(t))\,dt - \sum_{j=1}^{km-1} G_j(q(t_j)) \\
&\ge \frac{1}{2}\int_0^{kT} |\dot{q}(t)|^2\,dt - \varepsilon_1 \int_0^{kT} |q(t)|^2\,dt \\
&\quad - d_5^{-r} \max_{s\in[d_5,d_1]} a(s) \int_0^{kT} b(t)|q(t)|^r\,dt - C_2 \int_0^{kT} |q(t)|^r\,dt - M_1 \sum_{j=1}^{km-1} |q(t_j)|^r \\
&\ge \left(\frac{2\pi^2}{4\pi^2+T^2} - \varepsilon_1\right)\|q\|_{H_{kT}^1}^2 - \left(C_0^r d_5^{-r} C_5 + C_2 C_0^{r-2} + M_1 C_0^{r-2}\right)\|q\|_{H_{kT}^1}^r,
\end{aligned}$$

where $C_5 = \max_{s\in[d_5,d_1]} a(s) \int_0^{kT} b(t)\,dt$. Hence there exist constants $a_k > 0$ and $\rho_k \in (0,1)$ such that

$$\varphi_k(q) \ge a_k > 0, \quad \forall q \in X_2 \quad \text{and} \quad \|q\|_{H_{kT}^1} = \rho_k,$$

which proves (i) of Lemma 1.4. By the periodicity of $F(t,q)$ in t, (2.8) holds, *i.e.*:

$$F(t,q) \geq \left(\delta + \frac{\omega^2}{2}\right)|q|^2 - \left(\delta + \frac{\omega^2}{2}\right)d_4^2, \quad q \in \mathbb{R}^N, t \in [0,kT]. \tag{3.1}$$

Let $\bar{H}_{kT}^1 = \operatorname{span}\{e_k\} + \mathbb{R}$ with $e_k = (\cos k^{-1}\omega t, 0, 0, \dots, 0) \in X_2$. For any given $q = z + \lambda e_k$, $z \in X_1$, $\lambda \in \mathbb{R}$, from (1.2), (G1), and (3.1), we get

$$\begin{aligned}
\varphi_k(z+\lambda e_k) &= \frac{1}{2}\int_0^{kT} \left|\lambda \dot{e}_k(t)\right|^2 dt - \int_0^{kT} F\big(t, z+\lambda e_k(t)\big)\, dt - \sum_{j=1}^{km-1} G_j\big(z+\lambda e_k(t_j)\big) \\
&\leq \frac{1}{2}\lambda^2 k^{-2}\omega^2 \frac{kT}{2} - \left(\delta + \frac{\omega^2}{2}\right)\int_0^{kT} \left|z+\lambda e_k(t)\right|^2 dt + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT \\
&= \frac{\omega^2 T\lambda^2}{4k} - \left(\delta + \frac{\omega^2}{2}\right)\frac{kT\lambda^2}{2} - \left(\delta + \frac{\omega^2}{2}\right)kTz^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT \\
&= -\frac{((k^2-1)\omega^2 + 2\delta k^2)T\lambda^2}{4k} - \left(\delta + \frac{\omega^2}{2}\right)kTz^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT.
\end{aligned}$$

Let

$$f_1(x) = -\left(\delta + \frac{\omega^2}{2}\right)kTx^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT, \quad x \in \mathbb{R}, k \in \mathbb{N}$$

and

$$f_2(x) = -\frac{((k^2-1)\omega^2 + 2\delta k^2)Tx^2}{4k}, \quad x \in \mathbb{R}, k \in \mathbb{N}.$$

Clearly, it can be seen that $f_1(x)$ and f_2 attain their maximum at zero. Therefore, we have

$$\begin{aligned}
&\varphi(z+\lambda e) \leq f_1(0) + f_2\big(|\lambda|\big) = f_2\big(|\lambda|\big) + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT, \\
&\varphi(z+\lambda e) \leq f_1\big(|z|\big) + f_2(0) = f_1\big(|z|\big).
\end{aligned}$$

Note that $\varphi_k(z) = -\int_0^{kT} F(t,z)\,dt - \sum_{j=1}^{km-1} G_j(z) = -k\int_0^T F(t,z)\,dt - k\sum_{j\in\Omega} G_j(z) \leq 0$, for all $z \in X_1 = \mathbb{R}^N$. Since $\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} f_2(x) = -\infty$, we can choose $s_1 > 0$, $s_2 > \rho$ such that $\varphi(z + se_k) < 0$, for $|z| = s_1$, or $\lambda = s_2$. Here, we put $s_1 = 2r$, $s_2 = r$, where $r = \max\{2, 3d_4\}$. It is clear that $s_2 > 1 > \rho$. Let $Q_k = \{q \in X \mid q = z + \lambda e, z \in X_1, |z| \leq s_1, \lambda \in (0, s_2)\}$.

For any given $z + \lambda e_k \in Q_k$, we have

$$\varphi_k(z+\lambda e_k) \leq \frac{1}{2}\int_0^{kT} \left|\lambda \dot{e}_k(t)\right|^2 dt \leq \frac{T\omega^2 s_2^2}{4}.$$

For every $z + \lambda e_k \in \partial Q_k$, where $|z| = s_1$, by (3.1), we have

$$\begin{aligned}
\varphi_k(z+\lambda e_k) &= \frac{1}{2}\int_0^{kT} \left|\lambda \dot{e}_k(t)\right|^2 dt - \int_0^{kT} F\big(t, z+\lambda e_k(t)\big)\, dt - \sum_{j=1}^{km-1} G_j\big(z+\lambda e_k(t_j)\big) \\
&\leq -\frac{((k^2-1)\omega^2 + 2\delta k^2)T\lambda^2}{4k} - \left(\delta + \frac{\omega^2}{2}\right)kTz^2 + \left(\delta + \frac{\omega^2}{2}\right)d_4^2 kT \leq 0.
\end{aligned}$$

Let $E = \{t \in [0, kT] : |z + s_2 e_k| \geq \frac{\sqrt{2}r}{2}\}$, where $z = \{z_1, z_2, \dots, z_n\} \in \mathbb{R}$. We claim that $\operatorname{meas}(E) \geq \frac{kT}{2}$. We have

$$\begin{aligned} |z + s_2 e_k|^2 &= \left|z_1 + r\cos k^{-1}\omega t\right|^2 + \sum_{i=2}^{N} |z_i|^2 \\ &\geq \left|z_1 + r\cos k^{-1}\omega t\right|^2. \end{aligned}$$

If $z_1 \geq 0$, for all $t \in [0, \frac{kT}{8}] \cup [\frac{7kT}{8}, kT]$, we obtain

$$\left|z_1 + r\cos k^{-1}\omega t\right| = z_1 + r\cos k^{-1}\omega t \geq \frac{\sqrt{2}r}{2}.$$

If $z_1 < 0$, for all $t \in [\frac{3kT}{8}, \frac{5kT}{8}]$, we obtain

$$\left|z_1 + r\cos k^{-1}\omega t\right| = -z_1 - r\cos k^{-1}\omega t \geq \frac{\sqrt{2}r}{2}.$$

Therefore, the assertion is established. So, for every $z + \lambda e_k \in \partial Q_k$, where $|\lambda| = s_2$, combining with (2.7), we have

$$\begin{aligned} \varphi_k(z + \lambda e_k) &\leq \frac{1}{2}\int_0^{kT} \left|\lambda \dot{e}_k(t)\right|^2 dt - \int_0^{kT} F\bigl(t, z + \lambda e_k(t)\bigr)\, dt \\ &\leq \frac{\omega^2 Tr^2}{4k} - \int_{\{t:|z+re_k|\geq \frac{\sqrt{2}r}{2}\}} F\bigl(t, z + \lambda e_k(t)\bigr)\, dt \\ &\leq \frac{\omega^2 Tr^2}{4} - \frac{\omega^2}{2}\int_{\{t:|z+re_k|\geq \frac{\sqrt{2}r}{2}\}} \left|z + \lambda e_k(t)\right|^2 dt \\ &\leq \frac{\omega^2 Tr^2}{4} - \frac{kT\omega^2 r^2}{8} \leq 0. \end{aligned}$$

So, functional φ_k has at least one critical point q_k for every $k \in \mathbb{N}$ and

$$\varphi_k(q_k) \leq \frac{T\omega^2 s_2^2}{4} = \frac{T\omega^2 r^2}{4} = \frac{T\omega^2 \max\{4, 9d_4^2\}}{4}. \tag{3.2}$$

We claim that there exists a positive integer $k_2 > k_1$ such that $q_{kk_1} \neq q_{k_1}$ for all $kk_1 \geq k_2$. Otherwise, $\varphi_{kk_1}(q_{kk_1}) = k\varphi_{k_1}(q_{k_1}) \to +\infty$ as $k \to +\infty$, which contradicts (3.2). Repeating this process, we get a sequence $\{q_{k_n}\}$ of distinct periodic solutions of system (1.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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