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The modification of system of variational inequalities for fixed point theory in Banach spaces

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Abstract

In this paper, we use methods different from extragradient methods to prove a strong convergence theorem for the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and the set of solutions of modification of a system of variational inequalities problems in a uniformly convex and 2-uniformly smooth Banach space. Applying the main result we obtain a strong convergence theorem involving two sets of solutions of variational inequalities problems introduced by Aoyama *et al.* (Fixed Point Theory Appl. 2006:35390, 2006, doi:10.1155/FPTA/2006/35390) in a uniformly convex and 2-uniformly smooth Banach space. We also give a numerical example to support our result.

Keywords: nonexpansive mapping; strictly pseudo-contractive mapping; the modification of system of variational inequalities problems

1 Introduction

Let E be a real Banach space with its dual space E^* and let C be a nonempty closed convex subset of E. Throughout this paper, we denote the norm of E and E^* by the same symbol $\|\cdot\|$. We use the symbols '$\to$' and '$\rightharpoonup$' to denote strong and weak convergence, respectively. Recall the following definitions.

Definition 1.1 A Banach space E is said to be *uniformly convex* iff for any ϵ, $0 < \epsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ imply there exists a $\delta > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 1.2 Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *the modulus of smoothness of E* if

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

A Banach space E is said to be *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

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Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that, if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. Hilbert space, L_P (or l_p) spaces, $0 < p < \infty$ and the Sobolev spaces, W_m^p, $0 < p < \infty$ are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth, while

$$L_p \text{ (or } l_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p < \infty, \\ 2\text{-uniformly smooth} & \text{if } p \geq 2. \end{cases}$$

Definition 1.3 A mapping J from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the normalized duality mapping of E. The duality pair $\langle x, f\rangle$ represents $f(x)$ for $f \in E^*$ and $x \in E$.

It is well known that if E is smooth, then J is a single value, which we denote by j.

Definition 1.4 Let C be a nonempty subset of a Banach space E and $T : C \to C$ be a self-mapping. T is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

T is called an η-strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. It is clear that (1.1) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta \|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

Example 1.1 Let $\mathbb{R}$ be a real line endowed with Euclidean norm and let the mapping $T : (0, \frac{1}{2}) \to (0, \frac{1}{2})$ defined by

$$Tx := \frac{x^3}{1 + x^2}$$

for all $x \in (0, \frac{1}{2})$. Then T is $\frac{3}{4}$-strictly pseudo-contractive mapping.

Example 1.2 Let E be 2-uniformly smooth Banach space and let $T : E \to E$ be λ-strictly pseudo-contractive mapping. Let K be the 2-uniformly smooth constant of E and $0 \leq d \leq \frac{\lambda}{K^2}$, then $(I - d(I - T))$ is a nonexpansive mapping.

Definition 1.5 Let $C \subseteq E$ be closed convex and Q_C be a mapping of E onto C. The mapping Q_C is said to be *sunny* if $Q_C(Q_C x + t(x - Q_C x)) = Q_C x$ for all $x \in E$ and $t \geq 0$. A mapping Q_C is called *retraction* if $Q_C^2 = Q_C$. A subset C of E is called a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C.

An operator A of C into E is said to be *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be α*-inverse strongly accretive* if there exist $j(x-y) \in J(x-y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Remark 1.1 From (1.2) and (1.3), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is an η-inverse strongly accretive.

In 2000, Ansari and Yao [1] introduced the system of generalized implicit variational inequalities and proved the existence of its solution. They derived the existence results for a solution of system of generalized variational inequalities and used their results as tools to establish the existence of a solution of system of optimization problems.

Ansari *et al.* [2] introduced the system of vector equilibrium problems and prove the existence of its solution. Moreover, they also applied their result to the system of vector variational inequalities. The results of [1] and [2] were used as tools to solve Nash problem for vector-value functions and (non)convex vector valued function.

Let $A, B : C \to E$ be two nonlinear mappings. In 2010 Yao *et al.* [3] introduced the system of general variational inequalities problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.4}$$

They proved fixed points theorem by using modification of extragradient methods as follows.

Theorem 1.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X into C. Let the mappings $A, B : C \to E$ be α-inverse strongly accretive with $\alpha \geq K^2$ and β-inverse strongly accretive with $\beta \geq K^2$, respectively. Define the mapping by $Gx = Q_C(Q_C(x - Bx) - \lambda A Q_C(x - Bx))$ for all $x \in C$ and the set of fixed point of G denoted by $\Omega \neq \emptyset$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $(0,1)$. *Suppose the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *satisfy the following conditions*:

(i) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $Q'u$, *where* Q' *is the sunny nonexpansive retraction of* C *onto* Ω.

In 2013, Cai and Bu [4] introduced the system of a general variational inequalities problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.5}$$

where $\lambda, \mu > 0$. The set of solutions of (1.5) we denote by Ω'. If $\lambda = \mu = 1$, then problem (1.5) reduces to (1.4). In Hilbert space (1.5) reduces to

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.6}$$

which is introduced by Ceng *et al.* [5]. If $A = B$, then (1.6) reduces to a problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

which is introduced by Verma [6]. If $x^* = y^*$, then problem (1.7) reduces to the variational inequality for finding $x^* \in C$ such that

$$\langle A x^*, y - x^*\rangle \geq 0, \quad \forall x \in C.$$

Variational inequality theory is one of very important mathematical tools for solving many problems in economic, engineering, physical, pure and applied science *etc.*

Many authors have studied the iterative scheme for finding the solutions of a variational inequality problem; see for example [7–10].

By using the extragradient methods, Cai and Bu [4] proved a strong convergence theorem for finding the solutions of (1.5) as follows.

Theorem 1.3 *Let* C *be a nonempty closed convex subset of a* 2*-uniformly smooth and uniformly convex Banach space* E *such that* $C \pm C \subset C$. *Let* P_C *be the sunny nonexpansive retraction from* E *to* C. *Let the mapping* $A, B : C \to E$ *be* α*-inverse strongly accretive and* β*-inverse strongly accretive, respectively. Let* $\{T_i : C \to C\}_{i=0}^{\infty}$ *be an infinite family of nonexpansive mapping with* $F = \bigcap_{i=0}^{\infty} \cap \Omega' \neq \emptyset$. *Let* $S : C \to C$ *be a nonexpansive mapping and*

$D : C \to C$ be a strongly positive linear bounded operator with the coefficient $\overline{\gamma}$ such that $0 < \gamma < \overline{\gamma}$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ k_n = P_C(z_n - \lambda A z_n), \\ y_n = (1-\beta_n)x_n + \beta_n k_n, \\ x_{n+1} = \alpha_n \gamma S y_n + \gamma_n x_n + ((1-\gamma_n I - \alpha_n D))T_n y_n, \end{cases}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1;$

(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0, \quad \liminf_{n\to\infty} \beta_n > 0.$

Suppose that for any bounded subset D' of C there exists an increasing, continuous, and convex function $h_{D'}$ from $\mathbb{R}^+ \to \mathbb{R}^+$ such that $h_{D'}(0) = 0$ and $\lim_{k,l\to\infty} \sup\{h_{D'}(\|T_k z - T_l z\|) : z \in D'\} = 0$. Let T be a mapping from C into C defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \bigcap_{i=0}^{\infty} F(T_i)$. Then $\{x_n\}$ converges strongly to $z \in F$, which also solves the following variational inequality:

$$\langle \gamma S z - D z, j(p - z) \rangle \le 0, \quad \forall p \in F.$$

For the research related to the extragradient methods, some additional references are [11–13].

Motivated by (1.4) and (1.5), we introduce the problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_A A)(a x^* + (1-a) y^*), j(x - x^*) \rangle \ge 0, \\ \langle y^* - (I - \lambda_B B) x^*, j(x - y^*) \rangle \ge 0 \end{cases} \tag{1.8}$$

for all $x \in C$, $\lambda_A, \lambda_B > 0$ and $a \in [0,1]$. This problem is called *the modification of a system of variational inequalities problems* in Banach space. If $a = 0$, then (1.8) reduces to (1.5).

Motivated by Theorems 1.2 and 1.3, we use the methods different from extragradient methods to prove a strong convergence theorem for finding the solutions of (1.8) and an element of the set of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings in a uniformly convex and 2-uniformly smooth Banach space. Applying the main result, we obtain a strong convergence theorem involving two sets of solutions of variational inequalities problems introduced by Aoyama *et al.* [14] in a uniformly convex and 2-uniformly smooth Banach space. Moreover, we also give a numerical example to support our main results in the last section.

2 Preliminaries

The following lemmas and definitions are important tools to prove the results in the next sections.

Definition 2.1 ([15]) Let C be a nonempty convex subset of a Banach space. Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of mappings of C into itself. For each $j = 1,2,\dots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S^A : C \to C$ as follows:

$$
\begin{aligned}
U_0 &= T_1 = I,\\
U_1 &= T_1\bigl(\alpha_1^1 S_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\bigr),\\
U_2 &= T_2\bigl(\alpha_1^2 S_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\bigr),\\
U_3 &= T_3\bigl(\alpha_1^3 S_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\bigr),\\
&\vdots\\
U_{N-1} &= T_{N-1}\bigl(\alpha_1^{N-1} S_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\bigr),\\
S^A = U_N &= T_N\bigl(\alpha_1^N S_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\bigr).
\end{aligned}
\tag{2.1}
$$

This mapping is called the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$, and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.1 ([15]) *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \min\{\kappa_i : i = 1,2,\dots,N\}$ with $K^2 \leq \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$, and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S^A) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$ and S^A is a nonexpansive mapping.*

Lemma 2.2 ([16]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([17]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$$

for any $x, y \in E$.

Lemma 2.4 ([18]) *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then $F(T)$ is a sunny nonexpansive retract of C.*

Lemma 2.5 ([19]) *Let C be a nonempty closed convex subset of a smooth Banach space and Q_C be a retraction from E onto C. Then the following are equivalent:*
(i) *Q_C is both sunny and nonexpansive;*
(ii) *$\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$ for all $x \in E$ and $y \in C$.*

It is obvious that if E is a Hilbert space, we find that a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C. From Lemma 2.5, let $x \in E$ and $x_0 \in C$. Then we have $x_0 = Q_C x$ if and only if $\langle x - x_0, J(y - x_0)\rangle \leq 0$, for all $y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

Lemma 2.6 ([20]) *Let E be a uniformly convex Banach space and $B_r = \{x \in E : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty] \to [0,\infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g\big(\|x - y\|\big)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.7 ([21]) *Let C be a closed and convex subset of a real uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty}\big\langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u)\big\rangle \leq 0$$

for any given $u \in C$.

Lemma 2.8 ([17]) *Let $r > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and for any $\alpha \in [0,1]$, we have $\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x - y\|)$.*

Lemma 2.9 ([22]) *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.10 *Let C be a nonempty closed convex subset of a smooth Banach space E and let $A, B : C \to E$ be mappings. Let Q_C be a sunny nonexpansive retraction of E onto C. For every $\lambda_A, \lambda_B > 0$ and $a \in [0,1]$. The following are equivalent:*

(a) (x^*, z^*) *is a solution of* (1.8);
(b) x^* *is a fixed point of mapping* $G : C \to C$, *i.e.*, $x^* \in F(G)$, *defined by*

$$Gx = Q_C(I - \lambda_A A)\big(aI + (1-a)Q_C(I - \lambda_B B)\big)x, \quad \forall x \in C,$$

where $z^* = Q_C(I - \lambda_B B)x^*$.

Proof First we show that (a) $\Rightarrow$ (b). Let (x^*, z^*) is a solution of (1.8), and we have

$$\begin{cases} \langle x^* - (I - \lambda_A A)(ax^* + (1-a)z^*), j(x - x^*)\rangle \geq 0, \\ \langle z^* - (I - \lambda_B B)x^*, j(x - z^*)\rangle \geq 0 \end{cases}$$

for all $x \in C$. From Lemma 2.5, we have

$$x^* = Q_C(I - \lambda_A A)\big(ax^* + (1-a)z^*\big)$$

and $z^* = Q_C(I - \lambda_B B)x^*$.

It follows that

$$x^* = Q_C(I - \lambda_A A)\big(ax^* + (1-a)Q_C(I - \lambda_B B)x^*\big) = Gx^*.$$

Then $x^* \in F(G)$, where $z^* = Q_C(I - \lambda_B B)x^*$.

Next we claim that (b) $\Rightarrow$ (a). Let $x^* \in F(G)$ and $z^* = Q_C(I - \lambda_B B)x^*$. Then

$$x^* = Gx^* = Q_C(I - \lambda_A A)\big(ax^* + (1-a)Q_C(I - \lambda_B B)x^*\big) = Q_C(I - \lambda_A A)\big(ax^* + (1-a)z^*\big).$$

From Lemma 2.5, we have

$$\begin{cases} \langle x^* - (I - \lambda_A A)(ax^* + (1-a)z^*), j(x - x^*)\rangle \geq 0, \\ \langle z^* - (I - \lambda_B B)x^*, j(x - z^*)\rangle \geq 0 \end{cases}$$

for all $x \in C$. Then we find that (x^*, z^*) is a solution of (1.8). □

Example 2.1 Let $\mathbb{R}$ be a real line with the Euclidean norm and let $A, B : \mathbb{R} \to \mathbb{R}$ defined by $Ax = \frac{x-1}{4}$ and $Bx = \frac{x-1}{2}$ for all $x \in \mathbb{R}$. The mapping $G : \mathbb{R} \to \mathbb{R}$ defined by

$$Gx = (I - 2A)\left(\frac{1}{2}I + \frac{1}{2}(I - 3B)\right)x$$

for all $x \in \mathbb{R}$. Then $1 \in F(G)$ and $(1, 1)$ is a solution of (1.8).

3 Main results

Theorem 3.1 *Let* C *be a nonempty closed convex subset of a uniformly convex and* 2-*uniformly smooth Banach space* E *and let* Q_C *be a sunny nonexpansive retraction of* E *onto* C. *Let* $A, B : C \to E$ *be* α- *and* β-*inverse strongly accretive operators, respectively. Define the mapping* $G : C \to C$ *by* $Gx = Q_C(I - \lambda_A A)(aI + (1-a)Q_C(I - \lambda_B B))x$ *for all* $x \in C$, $\lambda_A \in (0, \frac{\alpha}{K^2})$, $\lambda_B \in (0, \frac{\beta}{K^2})$ *and* $a \in [0, 1]$, *where* K *is the* 2-*uniformly smooth constant of* E.

Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\kappa = \min\{\kappa_i : i = 1,2,\dots,N\}$ with $K^2 \le \kappa$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$, and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$, and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that $\mathcal{F} = F(G) \cap \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by $u, x_1 \in C$ and

$$x_{n+1} = G(\alpha_n u + \beta_n x_n + \gamma_n S^A x_n), \quad \forall n \ge 1, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \le \beta_n \le d < 1$ *for some $c, d > 0$ and for all $n \ge 1$*;

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}} u$ and (x_0, z_0) is a solution of (1.8), *where $z_0 = Q_C(I - \lambda_B B)x_0$.*

Proof First, we show that $Q_C(I - \lambda_A A)$ and $Q_C(I - \lambda_B B)$ are nonexpansive mappings. Let $x, y \in C$; we have

$$\begin{aligned}
&\|Q_C(I - \lambda_A A)x - Q_C(I - \lambda_A A)y\|^2 \\
&\quad \le \|x - y - \lambda_A(Ax - Ay)\|^2 \\
&\quad \le \|x - y\|^2 - 2\lambda_A\langle Ax - Ay, j(x - y)\rangle + 2K^2\lambda_A^2 \|Ax - Ay\|^2 \\
&\quad \le \|x - y\|^2 - 2\lambda_A \alpha \|Ax - Ay\|^2 + 2K^2\lambda_A^2 \|Ax - Ay\|^2 \\
&\quad \le \|x - y\|^2 - 2\lambda_A(\alpha - K^2\lambda_A)\|Ax - Ay\|^2 \\
&\quad \le \|x - y\|^2.
\end{aligned}$$

Then $Q_C(I - \lambda_A A)$ is a nonexpansive mapping. By using the same method we find that $Q_C(I - \lambda_B B)$ is a nonexpansive mapping. From the definition of G, we see that G is a nonexpansive mapping. Let $x^* \in \mathcal{F}$. Put $y_n = \alpha_n u + \beta_n x_n + \gamma_n S^A x_n$ for all $n \ge 1$. From the definition of x_n and Lemma 2.10, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|Gy_n - x^*\| \\
&\le \|y_n - x^*\| \\
&= \|\alpha_n(u - x^*) + \beta_n(x_n - x^*) + \gamma_n(S^A x_n - x^*)\| \\
&\le \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|S^A x_n - x^*\| \\
&\le \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\le \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

Applying mathematical induction, we can conclude that the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$.

From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|Gy_n - Gy_{n-1}\| \\ &\le \left\|\alpha_n u + \beta_n x_n + \gamma_n S^A x_n - \alpha_{n-1}u - \beta_{n-1}x_{n-1} - \gamma_{n-1}S^A x_{n-1}\right\| \\ &\le |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\ &\quad + \gamma_n\left\|S^A x_n - S^A x_{n-1}\right\| + |\gamma_n - \gamma_{n-1}|\left\|S^A x_{n-1}\right\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\ &\quad + |\gamma_n - \gamma_{n-1}|\left\|S^A x_{n-1}\right\|. \end{aligned} \tag{3.2}$$

Applying (3.2), the condition (iii), and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.3}$$

From the definition of x_n, we have

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &\le \left\|y_n - x^*\right\|^2 \\ &\le \alpha_n\left\|u-x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\left\|S^A x_n - x^*\right\|^2 \\ &\quad - \beta_n\gamma_n g\left(\left\|S^A x_n - x_n\right\|\right) \\ &\le \alpha_n\left\|u-x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \beta_n\gamma_n g\left(\left\|S^A x_n - x_n\right\|\right).\end{aligned}$$

It follows that

$$\begin{aligned}\beta_n\gamma_n g\left(\left\|S^A x_n - x_n\right\|\right) &\le \alpha_n\left\|u-x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \left\|x_{n+1}-x^*\right\|^2 \\ &\le \alpha_n\left\|u-x^*\right\|^2 + \left(\left\|x_n - x^*\right\| + \left\|x_{n+1}-x^*\right\|\right)\|x_{n+1}-x_n\|.\end{aligned}$$

From (3.3) and the conditions (i) and (ii), we have

$$\lim_{n\to\infty} g\left(\left\|S^A x_n - x_n\right\|\right) = 0.$$

From the property of g, we have

$$\lim_{n\to\infty}\left\|S^A x_n - x_n\right\| = 0. \tag{3.4}$$

From the definition of y_n, we have

$$y_n - x_n = \alpha_n(u - x_n) + \gamma_n\left(S^A x_n - x_n\right).$$

From the condition (i) and (3.4), we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.5}$$

From the definition of y_n, we have

$$y_n - S^A x_n = \alpha_n\bigl(u - S^A x_n\bigr) + \beta_n\bigl(x_n - S^A x_n\bigr).$$

From the condition (i) and (3.4), we obtain

$$\lim_{n\to\infty}\bigl\|y_n - S^A x_n\bigr\| = 0. \tag{3.6}$$

From the nonexpansiveness of S^A, we have

$$\begin{aligned}\bigl\|S^A y_n - y_n\bigr\| &\le \bigl\|S^A y_n - S^A x_n\bigr\| + \bigl\|S^A x_n - y_n\bigr\| \\ &\le \bigl\|y_n - S^A x_n\bigr\| + \|x_n - y_n\|.\end{aligned}$$

From (3.5) and (3.6), we have

$$\lim_{n\to\infty}\bigl\|S^A y_n - y_n\bigr\| = 0. \tag{3.7}$$

From the definition of x_n, we have

$$\begin{aligned}\|Gy_n - y_n\| &\le \|Gy_n - x_n\| + \|x_n - y_n\| \\ &= \|x_{n+1} - x_n\| + \|x_n - y_n\|.\end{aligned}$$

From (3.3) and (3.5), we have

$$\lim_{n\to\infty}\|Gy_n - y_n\| = 0. \tag{3.8}$$

Define the mapping $B : C \to C$ by $Bx = \epsilon Gx + (1-\epsilon)S^A x$ for all $x \in C$ and $\epsilon \in (0,1)$. From Lemmas 2.1 and 2.9, we have $F(B) = F(G) \cap F(S^A) = F(G) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) = \mathcal{F}$. From the definition of B, (3.7) and (3.8), we have

$$\lim_{n\to\infty}\|y_n - By_n\| = 0. \tag{3.9}$$

Since G and S^A are nonexpansive mappings, we have B is a nonexpansive mapping. From Lemma 2.7, we have

$$\limsup_{n\to\infty}\bigl\langle u - x_0, j(y_n - x_0)\bigr\rangle \le 0, \tag{3.10}$$

where $x_0 = Q_{\mathcal{F}}u$.

Finally, we show that the sequence $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1} - x_0\|^2 &\le \|y_n - x_0\|^2 \\ &= \bigl\|\alpha_n(u - x_0) + \beta_n(x_n - x_0) + \gamma_n\bigl(S^A x_n - x_0\bigr)\bigr\|^2 \\ &\le \bigl\|\beta_n(x_n - x_0) + \gamma_n\bigl(S^A x_n - x_0\bigr)\bigr\|^2 + 2\alpha_n\bigl\langle u - x_0, j(y_n - x_0)\bigr\rangle \\ &\le (1-\alpha_n)\|x_n - x_0\|^2 + 2\alpha_n\bigl\langle u - x_0, j(y_n - x_0)\bigr\rangle.\end{aligned}$$

Applying Lemma 2.2, the condition (i) and (3.10), we can conclude that the sequence $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}}u$ and (x_0, z_0) is a solution of (1.8), where $z_0 = Q_C(I - \lambda_B B)x_0$. This completes the proof. □

The following corollary is a strong convergence theorem involving problem (1.5). This result is a direct proof from Theorem 3.1. We, therefore, omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction of E onto C. Let $A, B : C \to E$ be α- and β-inverse strongly accretive operators, respectively. Define the mapping $G : C \to C$ by $Gx = Q_C(I - \lambda_A A)(Q_C(I - \lambda_B B))x$ for all $x \in C$, $\lambda_A \in (0, \frac{\alpha}{K^2})$, $\lambda_B \in (0, \frac{\beta}{K^2})$, where K is the 2-uniformly smooth constant of E. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\kappa = \min\{\kappa_i : i = 1, 2, \ldots, N\}$ with $K^2 \leq \kappa$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$, and $\alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$, and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Assume that $\mathcal{F} = F(G) \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by $u, x_1 \in C$ and*

$$x_{n+1} = G\bigl(\alpha_n u + \beta_n x_n + \gamma_n S^A x_n\bigr), \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n \leq d < 1$ *for some $c, d > 0$ and for all $n \geq 1$*;

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}}u$ and (x_0, z_0) is a solution of (1.5), *where $z_0 = Q_C(I - \lambda_B B)x_0$.*

4 Applications

In this section, we prove a strong convergence theorem involving two sets of solutions of variational inequalities in Banach space. We give some useful lemmas and definitions to prove Theorem 4.4.

Let $A : C \to E$ be a mapping. The variational inequality problem in a Banach space is to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\bigl\langle Ax^*, j\bigl(x - x^*\bigr)\bigr\rangle \geq 0, \quad \forall x \in C. \tag{4.1}$$

This problem was considered by Aoyama *et al.* [14]. The set of solutions of the variational inequality in a Banach space is denoted by $S(C, A)$, that is,

$$S(C, A) = \bigl\{u \in C : \bigl\langle Au, J(v - u)\bigr\rangle \geq 0, \forall v \in C\bigr\}. \tag{4.2}$$

The variational inequalities problems have been studied by many authors; see, for example, [11, 23].

Lemma 4.1 ([14]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$S(C,A) = F\bigl(Q_C(I-\lambda A)\bigr).$$

Lemma 4.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T,S : C \to C$ be nonexpansive mappings with $F(T) \cap F(S) \neq \emptyset$. Define the mapping $T_a : C \to C$ by $T_a x = S(ax + (1-a)Tx)$ for all $x \in C$ and $a \in (0,1)$. Then $F(T_a) = F(T) \cap F(S)$ and T_a is a nonexpansive mapping.*

Proof It is easy to see that $F(T) \cap F(S) \subseteq F(T_a)$. Let $x_0 \in F(T_a)$ and $x^* \in F(S) \cap F(T)$. From the definition of T_a, we have

$$\begin{aligned}\|x_0 - x^*\|^2 &\leq \|a(x_0 - x^*) + (1-a)(Tx_0 - x^*)\|^2 \\ &\leq a\|x_0 - x^*\|^2 + (1-a)\|Tx_0 - x^*\|^2 - a(1-a)g\bigl(\|x_0 - Tx_0\|\bigr) \\ &\leq \|x_0 - x^*\|^2 - a(1-a)g\bigl(\|x_0 - Tx_0\|\bigr).\end{aligned} \tag{4.3}$$

It follows that

$$g\bigl(\|x_0 - Tx_0\|\bigr) = 0.$$

Applying the property of g, we have $x_0 = Tx_0$, that is, $x_0 \in F(T)$. Since $x_0 \in F(T_a)$ and $x_0 \in F(T)$, we have

$$x_0 = S\bigl(ax_0 + (1-a)Tx_0\bigr) = Sx_0.$$

It follows that $x_0 \in F(S)$. Hence $F(T_a) \subseteq F(T) \cap F(S)$. Applying (4.3), we have T_a is a nonexpansive mapping. □

Lemma 4.3 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction from E onto C. Let $A,B : C \to E$ be α- and β-inverse strongly accretive operators, respectively. Define a mapping G as in Lemma 2.10 and for every $\lambda_A \in (0, \frac{\alpha}{K^2})$, $\lambda_B \in (0, \frac{\beta}{K^2})$ and $a \in (0,1)$ where K is 2-uniformly smooth constant. If $S(C,A) \cap S(C,B) \neq \emptyset$, then $F(G) = S(C,A) \cap S(C,B)$.*

Proof From Lemma 4.1, we have

$$S(C,A) = F\bigl(Q_C(I-\lambda_A A)\bigr) \quad \text{and} \quad S(C,B) = F\bigl(Q_C(I-\lambda_B B)\bigr).$$

Using the same method as Theorem 3.1, we find that $Q_C(I-\lambda_A A)$ and $Q_C(I-\lambda_B B)$ are nonexpansive mappings.

From the definition of G and Lemma 4.2, we have

$$F(G) = F\big(Q_C(I-\lambda_A A)\big) \cap F\big(Q_C(I-\lambda_B B)\big) = S(C,A) \cap S(C,B). \qquad \square$$

From Theorem 3.1 and Lemma 4.3, we have the following theorem.

Theorem 4.4 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction of E onto C. Let $A, B : C \to E$ be α- and β-inverse strongly accretive operators, respectively. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\kappa = \min\{\kappa_i : i = 1,2,\dots,N\}$ with $K^2 \le \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$, and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$, and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that $\mathcal{F} = S(C,A) \cap S(C,B) \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by $u, x_1 \in C$, and*

$$\begin{cases} y_n = \alpha_n u + \beta_n x_n + \gamma_n S^A x_n, \\ x_{n+1} = Q_C(I-\lambda_A A)(aI + (1-a)Q_C(I-\lambda_B B))y_n, \quad \forall n \ge 1, \end{cases} \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ and $a \in (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\lambda_A \in (0, \frac{\alpha}{K^2})$, $\lambda_B \in (0, \frac{\beta}{K^2})$. Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $0 < c \le \beta_n \le d < 1$ *for some* $c, d > 0$ *and for all* $n \ge 1$;

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Then the sequence $\{x_n\}$ converges strongly to $x_0 = Q_{\mathcal{F}} u$ and (x_0, z_0) is a solution of (1.8), *where $z_0 = Q_C(I-\lambda_B B)x_0$.*

From Theorem 4.4, we have the following result.

Example 4.1 Let $l^2 = \{x = (x_i)_{i=1}^\infty : \sum_{i=1}^\infty |x_i|^2 < \infty\}$ with norm define by $\|x\| = (\sum_{i=1}^\infty |x_i|)^{\frac{1}{2}}$. Define the mappings $A, B : l^2 \to l^2$ by $Ax = 2x$ and $Bx = 3x$ for all $x = (x_i)_{i=1}^\infty \in l^2$.

For every $i = 1,2,\dots,5$, define the mappings $S_i, T_i : l^2 \to l^2$ by $S_i x = \frac{x}{2^i}$ and $T_i x = \frac{x}{3^i}$ $x = (x_i)_{i=1}^\infty \in l^2$. Let S^A be S^A-mapping generated by $S_1, S_2, \dots, S_5$, $T_1, T_2, \dots, T_5$, and $\alpha_1, \alpha_2, \dots, \alpha_5$ where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ for all $j = 1,2,\dots,5$ and $\alpha_1^j = \alpha_2^j = \alpha_3^j = \frac{1}{3}$. Let the sequence $\{x_n\} \subseteq l^2$ be generated by $u, x_1 = (x_i^1)_{i=1}^\infty \in l^2$ and

$$\begin{cases} y_n = \frac{1}{9n} u + \frac{10n-1}{18n} x_n + \frac{8n-1}{18n} S^A x_n, \\ x_{n+1} = Q_{l^2}(I-4A)(\frac{1}{4} I + \frac{3}{4} Q_{l^2}(I-3B))y_n, \quad \forall n \ge 1, \end{cases}$$

where Q_{l^2} is a sunny nonexpansive retraction of l^2 onto l^2. Then the sequence $\{x_n\}$ converges strongly to 0 and $(0,0)$ is a solution of (1.8).

Remark 4.5 If $E = l_p$ $(p \geq 2)$, then Theorem 4.4 also holds.

5 Example and numerical results

In this section, we give a numerical example to support the main result.

Example 5.1 Let $\mathbb{R}$ be the real line with Euclidean norm and let $C = [0, \frac{\pi}{2}]$ and $A, B : C \to \mathbb{R}$ be mappings defined by $Ax = \frac{x}{2}$ and $Bx = \frac{x}{4}$ for all $x \in C$. For every $i = 1, 2, \ldots, N$, define the mapping $S_i, T_i : C \to C$ by $T_i x = \frac{\sin x}{i}$ and $S_i x = \frac{x^2}{x+i}$ for all $x \in C$ and $\frac{1}{(N+1)^2} \leq \frac{1}{N^2}$.

Suppose that S^A is the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$, and $\alpha_1, \alpha_2, \ldots, \alpha_N$ where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ and $\alpha_1^j = \alpha_2^j = \alpha_3^j = \frac{1}{3}$ for all $j = 1, 2, \ldots, N$. Define the mapping $G : C \to C$ by $Gx = Q_C(I - \frac{1}{5}A)(\frac{1}{2}I + \frac{1}{2}Q_C(I - \frac{1}{17}B))x$ for all $x \in C$. Let the sequence $\{x_n\}$ be generated by (3.1), where $\alpha_n = \frac{1}{7n}$, $\beta_n = \frac{6n-1}{14n}$, and $\gamma_n = \frac{8n-1}{14n}$ for all $n \geq 1$. Then $\{x_n\}$ converges strongly to 0 and $(0, 0)$ is a solution of (1.8).

Solution. For every $i = 1, 2, \ldots, N$, it is easy to see that T_i is a nonexpansive mapping and S_i is $\frac{1}{i^2}$-strictly pseudo-contractive mappings with $\bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) = \{0\}$. Then A is $\frac{1}{4}$-inverse strongly accretive and B is $\frac{1}{16}$-inverse strongly accretive. From the definition of G, we have $F(G) = \{0\}$ and $(0, 0)$ is a solution of (1.8). Then $\mathcal{F} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap F(G) = \{0\}$.

For every $n \geq 1$ and $i = 1, 2, \ldots, N$, the mappings T_i, S_i, G, A, B and sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy all conditions in Theorem 3.1. Since the sequence $\{x_n\}$ is generated by (3.1), from Theorem 3.1, we find that the sequence $\{x_n\}$ converges strongly to 0 and $(0, 0)$ is a solution in (1.8).

Next, we will divide our iterations into two cases as follows:

(i) $x_1 = \frac{\pi}{2}$, $u = \frac{\pi}{4}$ and $n = N = 20$,
(ii) $x_1 = \frac{\pi}{4}$, $u = \frac{\pi}{6}$ and $n = N = 20$.

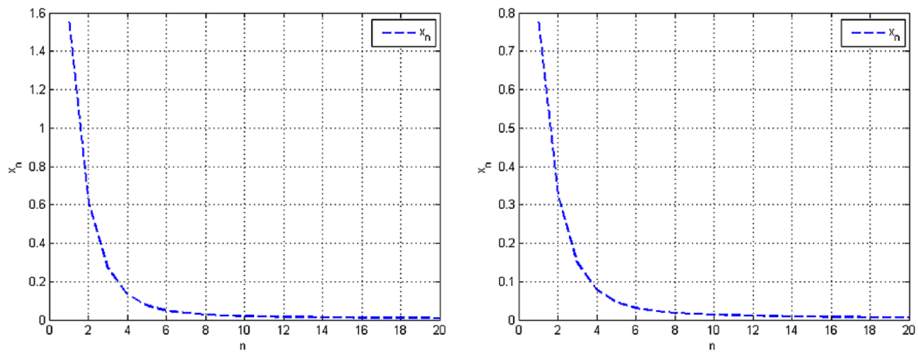
Table 1 and Figure 1 show the values of sequence $\{x_n\}$ for both cases.

Conclusion

(i) Table 1 and Figure 1 show that the sequences $\{x_n\}$ converge to 0, where $\{0\} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap F(G)$.
(ii) Theorem 3.1 guarantees the convergence of $\{x_n\}$ in Example 5.1.

Table 1 The values of $\{x_n\}$ with $x_1 = \frac{\pi}{2}$, $u = \frac{\pi}{4}$, and $x_1 = \frac{\pi}{4}$, $u = \frac{\pi}{6}$

n	$x_1 = \frac{\pi}{2}, u = \frac{\pi}{4}$	$x_1 = \frac{\pi}{4}, u = \frac{\pi}{6}$
	x_n	x_n
1	1.5707963268	0.7853981634
2	0.6127630899	0.3232983687
3	0.2701199079	0.1495005671
4	0.1333284242	0.0775756830
5	0.0750990544	0.0458214332
⋮	⋮	⋮
10	0.0200438855	0.0133281318
⋮	⋮	⋮
16	0.0114504755	0.0076335361
17	0.0107016897	0.0071344156
18	0.0100455821	0.0066970377
19	0.0094657530	0.0063104954
20	0.0089495227	0.0059663460



(a) $x_1 = \frac{\pi}{2}, u = \frac{\pi}{4}$ (b) $x_1 = \frac{\pi}{4}, u = \frac{\pi}{6}$

Figure 1 The convergence of $\{x_n\}$ with different initial values.

(iii) If the sequence $\{x_n\}$ is generated by (4.4), from Theorem 4.4 and Example 5.1, we also see that the sequence $\{x_n\}$ converges to 0, where $\{0\} = S(C,A) \cap (C,B) \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$.

Competing interests
The author declares that they have no competing interests.

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