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Common fixed points and best approximations in locally convex spaces

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Abstract

We extend the main results of Aamri and El Moutawakil and Pant to the weakly compatible or R-weakly commuting pair (T, f) of maps, where T is multivalued. As applications, common fixed point theorems are obtained for new class of maps called R-subcommuting maps in the setup of locally convex topological vector spaces. We also study some results on best approximation via common fixed point theorems.

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1. Introduction and preliminaries

The study of common fixed points of compatible mappings has emerged as an area of vigorous research activity ever since Jungck [1] introduced the notion of compatible mappings. The concept of compatible mappings was introduced as a generalization of commuting mappings. In 1994, Pant [2] introduced the concept of R-weakly commuting maps which is more general than compatibility of two maps. Several authors discussed various results on coincidence and common fixed point theorem for compatible single-valued and multivalued maps. Among others Kaneko [3] extended well-known result of Nadler [4] to multivalued f-contraction maps as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a continuous map. Let T be closed bounded valued f-contraction map on X which commutes with f and $T(X) \subseteq f(X)$. Then, f and T have a coincidence point in X. Suppose moreover that one of the following holds: either (i) $fx \neq f^2x$ implies $fx \notin Tx$ or (ii) $fx \in Tx$ implies* lim *$f^n x$ exists. Then, f and T have a common fixed point.*

It is pointed out in [5] that condition (i) in the above result implies condition (ii). A great deal of work has been done on common fixed points for commutative, weakly commutative, R-weakly commutative and compatible maps (see [1,2,6-11]). The following more general common fixed point theorem for 1-subcommutative maps was proved in [12].

Theorem 1.2. *Let M be a nonempty τ-bounded, τ-sequentially complete and q-star-shaped subset of a Hausdorff locally convex space (E, τ). Let T and I be selfmaps of M. Suppose that T is I-nonexpansive, $I(M) = M$, $Iq = q$, I is nonexpansive and affine. If T and I are* 1*-subcommutative maps, then T and I have a common fixed point provided* *one of the following conditions holds:*

(i) *M is τ-sequentially compact.*
(ii) *T is a compact map.*
(iii) *M is weakly compact in (E, τ), I is weakly continuous and $I - T$ is demiclosed at* 0.
(iv) *M is weakly compact in an Opial space (E, τ) and I is weakly continuous.*

In this article, we begin with a common fixed point result for a pair (T, f) of weakly compatible as well as R-weakly commuting maps in the setting of a Hausdorff locally convex space. This result provides a nonmetrizable analogue of Theorem 1.2 for weakly compatible as well as R-weakly commutative pair of maps and improves main results of Davies [13] and Jungck [14]. As applications, we establish some theorems concerning common fixed points of a new class, R-subcommuting maps, which in turn generalize and strengthen Theorem 1.2 and the results due to Dotson [15], Jungck and Sessa [16], Lami Dozo [17] and Latif and Tweddle [18]. We also extend and unify well-known results on fixed points and common fixed points of best approximation for R-subcommutative maps.

Throughout this article, X will denote a complete Hausdorff locally convex topological vector space unless stated otherwise, P the family of continuous seminorms generating the topology of X and $K(X)$ the family of nonempty compact subsets of X. For each $p \in P$ and $A, B \in K(X)$, we define

$$D_p(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}[p(a-b)]\ ,\ \sup_{b\in B}\inf_{a\in A}[p(a-b)]\right\}.$$

Although p is only a seminorm, D_p is a Hausdorff metric on $K(X)$ (cf. [19]). For any $u \in X$, $M \subset X$ and $p \in P$, let

$$d_p(u,M) = \inf\{p(u-y)\ :\ y \in M\}$$

and let $P_M(u) = \{y \in M : p(y - u) = d_p(u, M)$, for all $p \in P\}$ be the set of best M-approximations to $u \in X$. For any mapping $f : M \to X$, we define (cf. [6])

$$C_M^f(u) = \{y \in M\ :\ fy \in P_M(u)\}\ \text{and}\ D_M^f(u) = P_M(u) \cap C_M^f(u).$$

Let M be a nonempty subset of X. A mapping $T : M \to K(M)$ is called multivalued contraction if for each $p \in P$, there exists a constant k_p, $0 < k_p < 1$ such that for each $x, y \in M$, we have

$$D_p(Tx,\ Ty) \le k_p p(x-y).$$

The map T is called nonexpansive if for each $x, y \in M$ and $p \in P$,

$$D_p(Tx,\ Ty) \le p(x-y).$$

Let $f : M \to M$ be a single-valued map. Then, $T : M \to K(M)$ is called an f-contraction if there exists k_p, $0 < k_p < 1$ such that for each $x, y \in M$ and for each $p \in P$, we have

$$D_p(Tx,\ Ty) \leq k_p p(fx - fy).$$

If we have the Lipschitz constant $k_p = 1$ for all $p \in P$, then T is called an f-nonexpansive mapping. The pair (T, f) is said to be compatible if, whenever there is a sequence $\{x_n\}$ in M satisfying $\lim_{n\to\infty} fx_n \in \lim_{n\to\infty} Tx_n$ (provided $\lim_{n\to\infty} fx_n$ exists in M and $\lim_{n\to\infty} Tx_n$ exists in $K(M)$), then $\lim_{n\to\infty} D_p(fTx_n,\ Tfx_n) = 0$, for all $p \in P$. The pair (T, f) is called R-weakly commuting, if for each $x \in M$, $fTx \in K(M)$ and

$$D_p(fTx, Tfx) \leq R\ d_p(fx, Tx)$$

for some positive real R and for each $p \in P$. If $R = 1$, then the pair (T, f) is called weakly commuting [10]. For $M = X$ and T a single-valued, the definitions of compatibility and R-weak commutativity reduce to those given by Jungck [1] and Pant [2], respectively.

A point x in M is said to be a common fixed point (coincidence point) of f and T if $x = fx \in Tx$. ($fx \in Tx$). We denote by $F(f)$ and $F(T)$ the set of fixed points of f and T, respectively. A subset M of X is said to be q-starshaped if there exists a $q \in M$, called the starcenter of M, such that for any $x \in M$ and $0 \leq \alpha \leq 1$, $\alpha q + (1 - \alpha)\, x \in M$.

Shahzad [20] introduced the notion of R-subcommuting maps and proved that this class of maps contains properly the class of commuting maps.

We extend this notion to the pair (T, f) of maps when T is not necessarily single-valued. Suppose $q \in F(I)$, M is q-starshaped with $T(M) \subset M$ and $f(M) \subset M$. Then, f and T are R-subcommutative if for each $x \in M$, $fTx \in K(M)$ and there exists some positive real number R such that

$$D_p(fTx, Tfx) \leq \frac{R}{h} d_p(hTx + (1-h)q, fx)$$

for each $p \in P$, $h \in (0, 1)$ and $x \in M$.

Obviously, commutativity implies R-subcommutativity (which in turn implies R-weak commutativity) but the converse does not hold as the following example shows.

Example 1.1. *Consider* $M = [1, \infty)$ *with the usual metric of reals. Define*

$$Tx = \{4x - 3\}, fx = 2x^2 - 1 \text{ for all } x \in M. \text{ Then,}$$
$$|\ Tfx - fTx\ | = 24(x-1)^2.$$

Further $|Tfx - ftx| \leq (R/h)|(hTx + (1 - h)q) - fx|$ *for all* x *in* M, $h \in (0, 1)$ *with* $R = 12$ *and* $q = 1 \in F(f)$. *Thus,* f *and* T *are R-subcommuting but not commuting.*

The mapping T from M into 2^X (the family of all nonempty subsets of X) is said to be demiclosed if for every net $\{x_\alpha\}$ in M and any $y_\alpha \in Tx_\alpha$ such that x_α converges strongly to x and y_α converges weakly to y, we have $x \in M$ and $y \in Tx$. We say X satisfies Opial's condition if for each $x \in X$ and every net $\{x_\alpha\}$ converging weakly to x, we have

$$\liminf p(x_\alpha - x) < \liminf p(x_\alpha - y) \text{ for any } y \neq x \text{ and } p \in P.$$

The Hilbert spaces and Banach spaces having a weakly continuous duality mapping satisfy Opial's condition [17].

2. Main results

We use a technique due to Latif and Tweddle [18], based on the images of the composition of a pair of maps, to obtain common fixed point results for a new class of maps in the context of a metric space.

Theorem 2.1. *Let X be a metric space and $f : X \to X$ be a map. Suppose that $T : X \to CB(X)$ is an f-contraction such that the pair (T, f) is weakly compatible (or R-weakly commuting) and $TX \subset fX$ such that fX is complete. Then, f and T have a common fixed point provided one of the following conditions holds for all $x \in X$:*

(i) $fx \neq f^2x$ *implies* $fx \notin Tx$

(ii) $fx \in Tx$ *implies*

$$d(fx, f^2x) < \max\{d(fx, Tfx), d(f^2x, Tfx)\}$$

whenever right-hand side is nonzero.

(iii) $fx \in Tx$ *implies*

$$d(fx, f^2x) < \max\{d(Tx, Tfx), d(fx, Tfx), d(f^2x, Tfx), d(Tx, f2x)\}$$

whenever right-hand side is nonzero.

(iv) $fx \in Tx$ *implies*

$$d(x, fx) < \max\{d(x, Tx), d(fx, Tx)\}$$

whenever the right-hand side is nonzero.

(v) $fx \in Tx$ *implies*

$$d(fx, f^2x) < \max\{d(Tx, Tfx), [d(Tx, fx) + d(f^2x, Tfx)]/2, [d(fx, Tfx) + d(f^2x, Tx)]/2\}$$

whenever the right-hand side is nonzero.

Proof. Define $Jz = Tf^{-1}z$ for all $z \in fX = G$. Note that for each $z \in G$ and x, yf^1z, the f-contractiveness of T implies that

$$H(Tx, Ty) \leq kd(fx, fy) = 0.$$

Hence, $Jz = Ta$ for all $a \in f^1z$ and J is multivalued map from G into $CB(G)$. For any $w, z \in G$, we have

$$H(Jw, Jz) = H(Tx, Ty)$$

for any $x \in f^{1}w$ and $yf^{1}z$. But T is an f-contraction so there is $k \in (0, 1)$ such that

$$\begin{aligned} H(Jw, Jz) &= H(Tx, Ty) \\ &\leq kd(fx, fy) = kd(w, z) \end{aligned}$$

which implies that J is a contraction. It follows from Nadler's fixed point theorem [4] that there exists $z_0 \in G$ such that $z_0 \in Jz_0$. Since $Jz_0 = Tx_0$ for any $x_0 \in f^{1}z_0$, so $fx_0 = z_0 \in Jz_0 = Tx_0$.

Thus, by the weak compatibility of f and T,

$$fTx_0 = Tfx_0 \quad \text{and} \quad f^2x_0 = f\,fx_0 \in fTx_0 = Tfx_0. \tag{2.1}$$

If the pair (T, f) is R-weakly commuting, then

$$H(fTx_0, Tfx_0) \leq Rd(fx_0, Tx_0) = 0,$$

implies that (2.1) holds.

(i) As $fx_0 \in Tx_0$ so we get by (2.1)

$$fx_0 = f^2x_0 \in fTx_0 = Tfx_0.$$

That is, fx_0 is the required common fixed point of f and T.

(ii) Suppose that $fx_0 \neq f^2x_0$. Then,

$$\begin{aligned} d(fx_0, f^2x_0) &< \max\{d(fx_0, Tfx_0), d(f^2x_0, Tfx_0)\} \\ &= d(fx_0, Tfx_0) \leq d(fx_0, f^2x_0) \end{aligned}$$

which is a contradiction. Thus, $fx_0 = f^2x_0$ and result follows from (2.1).

The conditions (iii) and (iv) imply (ii) (see [2] for details).

(v) Suppose that $fx_0 \neq f^2x_0$. Then,

$$\begin{aligned} d(fx_0, f^2x_0) &< \max\left\{d(Tx_0, Tfx_0)\,, [d(fx_0, Tx_0) + d(f^2x_0, Tfx_0)]/2,\right. \\ &\left.[d(f^2x_0, Tx_0) + d(fx_0, Tfx_0)]/2\right\} \\ &\leq \max\left\{d(fx_0, f^2x_0), [d(f^2x_0, fx_0) + d(fx_0, f^2x_0)]/2\right\} \\ &= d(fx_0, f^2x_0) \end{aligned}$$

which is a contradiction. Hence, $fx_0 = f^2x_0$ and so fx_0 is the required common fixed point of f and T.

Theorem 2.2. *Let X be a metric space and $f : X \to X$ be a map. Suppose that $T : X \to C(X)$ is an f-Lipschitz map such that the pair (T, f) is weakly compatible (or R-weakly commuting) and $cl(TX) \subset fX$ where fX is complete. If the pair (T, f) satisfies the property $(E.\ A)$, then f and T have a common fixed point provided one of the conditions (i)-(v) in Theorem 2.1 holds.*

Proof. As the pair (T, f) satisfies property $(E.\ A)$, there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $t \in \lim Tx_n$ for some t in X. Since $t \in cl(TX) \subset fX$ so $t = fx_0$ for some x_0 in X. Further as T is f-Lipschitz, we obtain

$$H(Tx_n, Tx_0) \leq kd(fx_n, fx_0).$$

Taking limit as $n \to \infty$, we get $\lim Tx_n = Tx_0$ and hence $fx_0 \in Tx_0$. The weak compatibility or R-weak commutativity of the pair (T, f) implies that (2.1) holds. The result now follows as in Theorem 1.2.

Theorem 2.3. *Assume that X, f and T are as in Theorem 2.2 with the exception that T being f-Lipschitz, T satisfies the following inequality;*

$$H(Tx, Ty) < \max\left\{d(fx, fy)\ , [d(Tx, fx) + d(fy, Ty)]/2, \right.$$
$$\left. [d(fx, Ty) + d(fy, Tx)]/2\right\}.$$

Then, conclusion of Theorem 2.2 holds.

Proof. As the pair (T, f) satisfies property $(E.\ A)$, there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $t \in \lim Tx_n$ for some t in X. Since $t \in cl(TX) \subset fX$ so $t = fx_0$ for some x_0 in X. We claim that $fx_0 \in Tx_0$. Assume that $fx_0 \notin Tx_0$, then we obtain

$$H(Tx_n, Tx_0) < \max\left\{d(fx_n, fx_0), [d(Tx_n, fx_n) + d(fx_0, Tx_0)]/2, \right.$$
$$\left. [d(fx_n, Tx_0) + d(fx_0, Tx_n)]/2\right\}.$$

Letting $n \to \infty$ yields,

$$\begin{aligned} H(A, Tx_0) &< \max\ \left\{[d(A, fx_0) + d(fx_0, Tx_0)]/2,\ [d(fx_0, Tx_0) + d(fx_0, A)]/2\right\} \\ &= \max\ \left\{d(fx_0, Tx_0)/2,\ d(fx_0, Tx_0)/2\right\} \\ &= d(fx_0, Tx_0)/2. \end{aligned}$$

As $fx_0 \in A$, so $d(fx_0, Tx_0) \le H(A, Tx_0)$ and hence $d(fx_0, Tx_0) < d(fx_0, Tx_0)/2$ which is a contradiction. Thus, $fx_0 \in Tx_0$. The weak compatibility or R-weak commutativity of the pair (T, f) implies that (2.1) holds. The result now follows as in Theorem 1.2.

3. Applications

There are plenty of spaces which are not normable (see [[21], p. 113]). So it is natural to consider fixed point and approximation results in the context of a locally convex space. In this section, we show that the problem concerning the existence of common fixed points of R-subcommuting maps on sets not necessarily convex or compact in locally convex spaces has a solution.

Remark 3.1. *Theorem 2.1 (i) holds in the setup of a Hausdorff complete locally convex space X (the same proof holds with the exception that we take $T : X \to K(X)$ and apply Theorem 1* [22]*instead of Nadler's fixed point theorem to obtain a fixed point of the multivalued contraction J).*

Theorem 3.1. *Let M be a weakly compact subset of a Hausdorff complete locally convex space X which is starshaped with respect to $q \in M$. Let $f : M \to M$ be an affine weakly continuous map with $f(M) = M$, $f(q) = q$, $T : M \to K(M)$ be an f-nonexpansive map and the pair (T, f) is R-subcommutative. Suppose the following conditions hold:*

(a) *$fx \neq f^2x$ implies $\lambda fx + (1 - \lambda)q \notin Tx$, $\lambda \ge 1$ (cf.* [23]),
(b) *either $f - T$ is demiclosed at* 0 *or X is an Opial's space.*

Then, f and T have a common fixed point.

Proof. For each real number h_n with $0 < h_n < 1$ and $h_n \to 1$ as $n \to \infty$, we define

$$T_n : M \to K(M) \text{ by } T_n x = h_n Tx + (1 - h_n)q$$

Obviously each T_n is f-contraction map. Note that

$$\begin{aligned} D_p(T_n fx, fT_n x) &\le h_n D_p(Tfx, fTx) \\ &\le h_n (R/h_n) d_p(h_n Tx + (1 - h_n)q, fx) \\ &= R d_p(T_n x, fx), \end{aligned}$$

which implies that (T_n, f) is R-weakly commutative pair for each n. Next, we show that if $fx \neq f^2x$, then $fx \notin T_n x$ for all $n \geq 1$. Suppose that $fx \in T_n x = h_n Tx + (1 - h_n)q$. Then, $fx = h_n u + (1 - h_n)q$ for some $u \in Tx$ which implies that $(h_n)^{-1}[fx - (1 - h_n)q] \in Tx$ and this contradicts hypothesis (a). By Remark 3.1 each pair (T_n, f) has a common fixed point. That is, there is $x_n \in M$ such that

$$x_n = fx_n \in T_n x_n \quad \text{for all } n \geq 1.$$

The set M is weakly compact, we can find a subsequence still denoted by $\{x_n\}$ such that x_n converges weakly to $x_0 \in M$. Since f is weakly continuous so fx_n converges weakly to fx_0. Since X is Hausdorff so $x_0 = fx_0$. As $fx_n \in T_n x_n = h_n Tx_n + (1 - h_n)q$ so there is some $u_n \in Tx_n$ such that $fx_n = h_n u_n + (1 - h_n)q$ which implies that $fx_n - u_n = ((1 - h_n)/h_n)(q - fx_n)$ converges to 0 as $n \to \infty$. Hence, by the demiclosedness of $f - T$ at 0, we get that $0 \in (f - T)x_0$. Thus, $x_0 = fx_0 \in Tx_0$ as required.

In case X is an Opial's space, Lemma 2.5 [24] or Lemma 3.2 [25] implies that $f - T$ is demiclosed at 0. The result now follows from the above argument.

If $T : M \to M$ is single-valued in Theorems 3.1, we get the following analogue of Theorem 6 [16] for a pair of maps which are not necessarily commutative in the set up of Hausdorff locally convex spaces.

Theorem 3.2. *Let M be a weakly compact subset of a Hausdorff complete locally convex space X which is starshaped with respect to $q \in M$. Suppose f and T are R-subcommutative selfmaps of M. Assume that f is continuous in the weak topology on M, f is affine, $f(M) = M$, $f(q) = q$, T is f-nonexpansive map and $fx \neq f^2x$ implies $\lambda fx + (1 - \lambda)q \neq Tx$ for $x \in M$ and $\lambda \geq 1$. Then, there exists $a \in M$ such that $a = fa = Ta$ provided that either (i) $f - T$ is demiclosed at 0, or (ii) $\times$ satisfies Opial's condition.*

If f is the identity on M, then Theorem 3.2 (i) gives the conclusion of Theorem 2 of Dotson [15] for Hausdorff locally convex spaces. A result similar to Theorem 3.2 (ii) for closed balls of reflexive Banach spaces appeared in [8].

Finally, we consider an application of Theorem 3.2 to best approximation theory; our result sets an analogue of Theorem 3.2 [6] for the maps which are not necessarily commuting in the setup of locally convex spaces and extends the corresponding results of Shahzad [20] to locally convex spaces.

Theorem 3.3. *Let T and f be selfmaps of a Hausdorff complete locally convex space X and $M \subset X$ such that $T(\partial M) \subset M$, where ∂M is the boundary of M in X. Let $u \in F(T) \cap F(f)$, $D = D_M^f(u)$ be nonempty weakly compact and starshaped with respect to $q \in F(f)$, f is affine and weakly continuous, $f(D) = D$, and $fx \neq f^2x$ implies $\lambda fx + (1 - \lambda)q \neq Tx$ for $\times \in D$ and $\lambda \geq 1$. Suppose that T is f-nonexpansive on $D \cup \{u\}$ and f is* *nonexpansive on* $P_M(u) \cup \{u\}$. *If f and T are R-subcommutative on D, then T, f have a common fixed point in* $P_M(u)$ *under each one of the conditions (i)-(ii) of Theorem 3.2.*

Proof. Let $y \in D$. Then, $fy \in D$ because $f(D) = D$ and hence $f(y) \in PM(u)$. By the definition of D, $y \in \partial M$ and since $T(\partial M) \subset M$, it follows that $Ty \in M$. By f-nonexpansiveness of T we get

$$p(Ty - u) = p(Ty - Tu) \leq p(fy - fu) \quad \text{for each } p \in P.$$

As $fu = u$ and $fy \in P_M(u)$ so for each $p \in P$, $p(Ty - u) \leq p(fy - u) = d_p(u, M)$ and hence $Ty \in P_M(u)$. Further as f is nonexpansive on $P_M(u) \cup \{u\}$, so for every $p \in P$, we obtain

$$\begin{aligned} p(fTy - u) &= p(fTy - fu) \leq p(Ty - u) = p(Ty - Tu) \leq p(fy - fu) \\ &= p(fy - u) = d_p(u, M). \end{aligned}$$

Thus, $fTy \in P_M(u)$ and hence $Ty \in C_M^f(u)$. Consequently, $Ty \in D$ and so $T, f : D \to D$ satisfy the hypotheses of Theorem 3.2. Thus, there exists $a \in P_M(u)$ such that $a = fa = Ta$.

Remark 3.2. (i) *Theorem 3.2 extends Theorem 1.2 to multivalued f-nonexpansive map T where the pair* (T, f) *is assumed to be R-subcommutative. Here we have also relaxed the nonexpansiveness of the map f.*

(ii) *Theorem 3.3 extends Theorem 3.3* [12], *which is itself a generalization of several approximation results.*

(iii) *If* $f(P_M(u)) \subseteq P_M(u)$, *then* $PM(u)C_M^f(u)$ *and so* $D_M^f(u) = P_M(u)$ (*cf.* [1]). *Thus, Theorem 3.3 holds for* $D = P_M(u)$. *Hence, Theorem 3.1* [12], *Theorem 7* [16], *Theorem 2.6* [26], *Theorem 3* [27], *Corollaries 3.1, 3.3, 3.4, 3.6 (i), 3.7 and 3.8 of*[28]*and many other results are special cases of Theorem 3.3 (see also Remarks 3.2* [12]).

Competing interests
The author declares that they have no competing interests.

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