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Some results on an infinite family of accretive operators in a reflexive Banach space

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Abstract

The purpose of this article is to investigate a Halpern-like proximal point algorithm for common zero points of an infinite family of accretive operators. Possible computational errors are taken into account. Strong convergence theorems are established in a reflexive Banach space.

Keywords: accretive operator; fixed point; resolvent; nonexpansive mapping; zero

1 Introduction

The class of accretive operators is an important class of nonlinear operators. Interest in accretive operators stems mainly from their firm connection with equations of evolutions. It is well known that many physically significant problems can be modeled by initial value problems of the following form: $x'(t) + Ax(t) = 0, x(0) = x_0$ where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. If $x(t)$ is dependent on t, then the above problem is reduced to $Au = 0$ whose solutions correspond to the equilibrium points of the initial value problem. An early fundamental result in the theory of accretive operators, due to Browder [1], states that the initial value problem is solvable if A is locally Lipschitz and accretive on E. One of the most popular techniques for solving zero points of accretive operators is the proximal point algorithm, which was proposed by Martinet [2, 3] and generalized by Rockafellar [4, 5].

Halpern algorithm is efficient to study fixed points of nonexpansive mappings. The advantage of Halpern algorithm for nonexpansive mappings is that strong convergence is guaranteed without any compact assumptions or projections involved. Recently Halpern-like proximal point algorithms have been extensively studied by many authors; see [6–22] and the references therein.

In this article, we investigate common zeros of an infinite family of accretive operators based on a Halpern-like proximal point algorithm. Strong convergence theorems are established in a reflexive and strictly convex Banach space which has a weakly continuous duality mapping.

2 Preliminaries

Let R^+ be the positive real number set. Let $\varphi : [0,\infty] := R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a gauge function. Let E be a Banach space with the dual E^*. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\| \varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality mapping.

Let $U_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. The norm of E is said to be Fréchet differentiable if for each $x \in U_E$, the limit is attained uniformly for all $y \in U_E$. The norm of E is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in U_E$. It is well known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E. It is well known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single-valued and uniformly norm to weak* continuous on each bounded subset of E.

Following Browder [23], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to J_φ). It is well known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial \Phi(\|x\|), \quad \forall x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\left\{ \frac{\|x+y\| - \|x-y\|}{2} - 1 : x \in U_E, \|y\| \leq t \right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable.

A Banach space E is said to be strictly convex if and only if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$.

E is said to be uniformly convex if for any $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex.

Let D be a nonempty subset of C. Let $Q_D : C \to D$. Q is said to be

(1) contraction if $Q_D^2 = Q_D$;

(2) sunny if for each $x \in C$ and $t \in (0,1)$, we have $Q_D(tx + (1-t)Q_D x) = Q_D x$;

(3) sunny nonexpansive retraction if Q_D is sunny, nonexpansive, and it is a contraction.

D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. The following result, which was established in [24], describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q_C : E \to C$ be a retraction and J be the normalized duality mapping on E. Then the following are equivalent:

(1) Q_C is sunny and nonexpansive;

(2) $\|Q_C x - Q_C y\|^2 \leq \langle x-y, J(Q_C x - Q_C y)\rangle$, $\forall x, y \in E$;

(3) $\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$, $\forall x \in E$, $y \in C$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C. Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$ and let $x_0 \in C$. Then we have from the above that $x_0 = Q_C x$ if and only if $\langle x - x_0, J(y - x_0)\rangle \leq 0$ for all $y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

Let C be a nonempty, closed, and convex subset of E. Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to denote the set of fixed points of S. Recall that S is said to be α-contractive iff there exists a constant $\alpha \in [0,1)$ such that $\|Sx - Sy\| \leq \alpha\|x-y\|$, $\forall x, y \in C$. S is said to be nonexpansive iff $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$. It is well known that many nonlinear problems can be reduced to the search for fixed points of nonexpansive mappings, for example, equilibrium problems, saddle point problems, and variational inequalities. Let K be a nonempty closed and convex subset of a smooth Banach space E. Recall the following variational inequality. Find a point $u \in C$ such that $\langle Au, J(v-u)\rangle \geq 0$, $\forall v \in C$. This problem is connected with fixed point problems of nonexpansvie mappings. From [25], we know that this variational inequality problem is equivalent to fixed point problems of nonlinear mapping $Q_K(I - rA)$, where I is the identity mapping and r is a positive real number.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. For an accretive operator A, we can define a single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Next, we give lemmas which play important roles in the article.

Lemma 2.1 [26] *Let E be a reflexive Banach space which has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be nonempty, closed, and convex subset of E. Let $f: C \to C$ be an α-contractive mapping and let $T: C \to C$ be a nonexpansive mapping. Let $x_t \in C$ be the unique fixed point of the mapping $tf + (1-t)T$, where $t \in (0,1)$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point $\bar{x}$ of T, where $\bar{x}$ is the unique solution to the following variational inequality*: $\langle f(\bar{x}) - \bar{x}, J_\varphi(p - \bar{x})\rangle \leq 0$, $\forall p \in \bigcap_{m=1}^N A_m^{-1}(0)$.

Lemma 2.2 [27] *Let C be a closed convex subset of a strictly convex Banach space E. Let $S_m: C \to C$ be a nonexpansive mapping for each $m \geq 1$. Let $\{\delta_m\}$ be a real number sequence in $(0,1)$ such that $\sum_{m=1}^\infty \delta_m = 1$. Suppose that $\bigcap_{m=1}^\infty F(S_m)$ is nonempty. Then the mapping $\sum_{m=1}^\infty \delta_m S_m$ is nonexpansive with $F(\sum_{m=1}^\infty \delta_m S_m) = \bigcap_{m=1}^\infty F(S_m)$.*

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [28].

Lemma 2.3 *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds*:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*
Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.4 [29] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5 [30] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{e_n\}$ be three nonnegative real sequences satisfying $b_{n+1} \leq (1-a_n)b_n + a_n c_n + e_n$, $\forall n \geq n_0$, where n_0 is some positive integer, $\{a_n\}$ is a number sequence in $(0,1)$ such that $\sum_{n=n_0}^\infty a_n = \infty$, $\{c_n\}$ is a number sequence such that $\limsup_{n\to\infty} c_n \leq 0$ and $\sum_{n=n_0}^\infty e_n = \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3 Main results

Theorem 3.1 *Let E be a reflexive and strictly convex Banach space which has a weakly continuous duality mapping J_φ. Let A_i be an m-accretive operator in E with zeros for each $i \geq 1$. Assume that $\bigcap_{i=1}^\infty \overline{D(A_i)}$ is convex and $\bigcap_{i=1}^\infty A_i^{-1}(0)$ is not empty. Let f be an α-contraction* *on $\bigcap_{i=1}^{\infty}\overline{D(A_i)}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ and $\{\delta_{n,i}\}$ be real number sequences in $(0,1)$. Let $\{e_n\}$ be a bounded computational error in $\bigcap_{i=1}^{\infty}\overline{D(A_i)}$. Let $\{x_n\}$ be a sequence in $\bigcap_{i=1}^{\infty}\overline{D(A_i)}$ generated by the following process:*

$$\begin{cases} x_1 \in \bigcap_{i=1}^{\infty}\overline{D(A_i)}, \quad \text{chosen arbitrarily,} \\ y_n = \alpha'_n x_n + \beta'_n \sum_{i=1}^{\infty} \delta_{n,i} J_{r_i} x_n + \gamma'_n e_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $J_{r_i} = (I + r_i A_i)^{-1}$. Assume that the following conditions are satisfied:

(1) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \sum_{i=1}^{\infty} \delta_{n,i} = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then $\{x_n\}$ converges strongly to $x = P_{\bigcap_{i=1}^{\infty}\overline{D(A_i)}} f(x)$, where $P_{\bigcap_{i=1}^{\infty}\overline{D(A_i)}}$ is the sunny nonexpansive contraction onto $\bigcap_{i=1}^{\infty}\overline{D(A_i)}$.

Proof The proof is split into four steps.

Step 1. Show that $\{x_n\}$ is bounded.

Fixing $p \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$, we get

$$\begin{aligned} \|y_n - p\| &\leq \alpha'_n \|x_n - p\| + \beta'_n \left\| \sum_{i=1}^{\infty} \delta_{n,i} J_{r_i} x_n - p \right\| + \gamma'_n \|e_n - p\| \\ &\leq \alpha'_n \|x_n - p\| + \beta'_n \sum_{i=1}^{\infty} \delta_{n,i} \|J_{r_i} x_n - p\| + \gamma'_n \|e_n - p\| \\ &\leq \left(\alpha'_n + \beta'_n\right) \|x_n - p\| + \gamma'_n \|e_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \left\|f(x_n) - p\right\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \left\|f(p) - p\right\| + \beta_n \|x_n - p\| \\ &\quad + \gamma_n \|x_n - p\| + \gamma'_n \gamma_n \|e_n - p\| \\ &\leq \left(1 - \alpha_n(1-\alpha)\right) \|x_n - p\| + \alpha_n (1-\alpha) \frac{\|f(p) - p\|}{1-\alpha} + \gamma'_n M_1, \end{aligned}$$

where M_1 is some appropriate constant. This implies that

$$\|x_{n+1} - p\| \leq \max\left\{ \frac{\|f(p) - p\|}{1-\alpha}, \|x_1 - p\| \right\} + \sum_{n=1}^{\infty} \gamma'_n M < \infty.$$

We find that $\{x_n\}$ is bounded. It follows that $\{y_n\}$ is also bounded. This completes Step 1.

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Putting $z_n=\sum_{i=1}^{\infty}\delta_{n,i}J_{r_i}x_n$, we see that

$$\begin{aligned}\|y_n-y_{n-1}\| &\le \left\|\sum_{i=1}^{\infty}\delta_{n,i}J_{r_i}x_{n-1}-\sum_{i=1}^{\infty}\delta_{n-1,i}J_{r_i}x_{n-1}\right\| \\ &\quad+\left\|\sum_{i=1}^{\infty}\delta_{n,i}J_{r_i}x_n-\sum_{i=1}^{\infty}\delta_{n,i}J_{r_n}x_{n-1}\right\| \\ &\le \sum_{i=1}^{\infty}|\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|+\|x_n-x_{n-1}\|.\end{aligned}$$

Define $\lambda_n:=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. This yields

$$\begin{aligned}\|\lambda_n-\lambda_{n-1}\| &\le \frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|+\frac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1})-y_{n-1}\|+\|y_n-y_{n-1}\| \\ &\le \frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|+\frac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1})-y_{n-1}\|+\|x_n-x_{n-1}\| \\ &\quad+\sum_{i=1}^{\infty}|\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|.\end{aligned}$$

Hence, we find that

$$\begin{aligned}\|\lambda_n-\lambda_{n-1}\|-\|x_n-x_{n-1}\| &\le \frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|+\frac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1})-y_{n-1}\| \\ &\quad+\sum_{i=1}^{\infty}|\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|.\end{aligned}$$

Using restrictions (1), (2), and (3), we get

$$\limsup_{n\to\infty}\bigl(\|\lambda_n-\lambda_{n-1}\|-\|x_n-x_{n-1}\|\bigr)\le 0.$$

Using Lemma 2.5, we obtain $\lim_{n\to\infty}\|\lambda_n-x_n\|=0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.1}$$

This completes Step 2.

Step 3. Show that $\limsup_{n\to\infty}\langle f(x)-x,J_\varphi(x_n-x)\rangle\le 0$.

Define a mapping T by $T:=\sum_{i=1}^{\infty}\delta_i J_{r_i}$. Using Lemma 2.1, we find that T is nonexpansive with $F(T)=\bigcap_{i=1}^{\infty}F(J_{r_i})=\bigcap_{i=1}^{N}A_i^{-1}(0)$. Note that

$$\begin{aligned}&\|x_n-Tx_n\| \\ &\quad\le \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\| \\ &\quad\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-Tx_n\|+\beta_n\|x_n-Tx_n\|+\gamma_n\sum_{i=1}^{\infty}|\delta_{n,i}-\delta_i|\|J_{r_i}x_n\|.\end{aligned}$$

Using (3.1), we find from restrictions (2), (3), and (4) that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.2}$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(x) - x, J_\varphi(x_n - x)\rangle = \lim_{j\to\infty} \langle f(x) - x, J_\varphi(x_{n_j} - \bar{x})\rangle. \tag{3.3}$$

Since E is reflexive, we may further assume that $x_{n_j} \rightharpoonup \hat{x}$ for some $\hat{x} \in \bigcap_{i=1}^{\infty} \overline{D(A_i)}$. Since J_φ is weakly continuous, we find from Lemma 2.1 that

$$\limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - \hat{x}\|) + \Phi(\|x - \hat{x}\|), \quad \forall x \in E.$$

Putting $f(x) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|)$, $\forall x \in E$, we have

$$f(x) = f(\hat{x}) + \Phi(\|x - \hat{x}\|), \quad \forall x \in E. \tag{3.4}$$

It follows from (3.2) that

$$\begin{aligned} f(T\hat{x}) &= \limsup_{j\to\infty} \Phi(\|x_{n_j} - T\hat{x}\|) \\ &\le \limsup_{j\to\infty} \Phi(\|Tx_{n_j} - T\hat{x}\|) \\ &\le \limsup_{j\to\infty} \Phi(\|x_{n_j} - \hat{x}\|) = f(\bar{x}). \end{aligned} \tag{3.5}$$

Note that $f(T\hat{x}) = f(\hat{x}) + \Phi(\|T\hat{x} - \hat{x}\|)$. This yields from (3.5) $\Phi(\|T\hat{x} - \hat{x}\|) \le 0$. This implies that $\hat{x} \in F(T) = \bigcap_{i=1}^{N} A_i^{-1}(0)$. It follows that

$$\limsup_{n\to\infty} \langle f(x) - x, J_\varphi(x_n - x)\rangle \le 0. \tag{3.6}$$

This completes Step 3.

Step 4. Show that $x_n \to x$ as $n \to \infty$.

Using Lemma 2.1, we find that

$$\begin{aligned} \Phi(\|x_{n+1} - x\|) &= \Phi(\|\alpha_n(f(x_n) - f(x)) + \alpha_n(f(x) - x) + \beta_n(x_n - x) + \gamma_n(y_n - x)\|) \\ &\le \Phi(\|\alpha_n(f(x_n) - f(x)) + \beta_n(x_n - x) + \gamma_n(y_n - x)\|) \\ &\quad + \alpha_n\langle f(x) - x, J_\varphi(x_{n+1} - x)\rangle \\ &\le (1 - \alpha_n(1-\alpha))\Phi(\|x_n - \bar{x}\|) + \alpha_n\langle f(\bar{x}) - \bar{x}, J_\varphi(x_{n+1} - \bar{x})\rangle + M_2\gamma_n', \end{aligned}$$

where M_2 is some appropriate constant. It follows from Lemma 2.4 that $\Phi(\|x_n - x\|) \to 0$. This implies that $\lim_{n\to\infty} \|x_n - x\| = 0$. This completes the proof. □

In the framework of Hilbert spaces, Theorem 3.1 is reduced to the following result.

Corollary 3.2 *Let E be a Hilbert space. Let A_i be a maximal monotone operator in E with zeros for each $i \geq 1$. Assume that $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$ is convex and $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$ is not empty. Let f be an α-contraction on $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, and $\{\delta_{n,i}\}$ be real number sequences in $(0,1)$. Let $\{e_n\}$ be a bounded computational error in $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$. Let $\{x_n\}$ be a sequence in $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$ generated by the following process*:

$$\begin{cases} x_1 \in \bigcap_{i=1}^{\infty} \overline{D(A_i)}, & \textit{chosen arbitrarily}, \\ y_n = \alpha'_n x_n + \beta'_n \sum_{i=1}^{\infty} \delta_{n,i} J_{r_i} x_n + \gamma'_n e_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, & \forall n \geq 1, \end{cases}$$

where $J_{r_i} = (I + r_i A_i)^{-1}$. Assume that the following conditions are satisfied:

(1) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \sum_{i=1}^{\infty} \delta_{n,i} = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then $\{x_n\}$ converges strongly to $x = P_{\bigcap_{i=1}^{\infty} \overline{D(A_i)}} f(x)$, where $P_{\bigcap_{i=1}^{\infty} \overline{D(A_i)}}$ is the metric contraction onto $\bigcap_{i=1}^{\infty} \overline{D(A_i)}$.

For a single accretive operator, Theorem 3.1 is reduced to the following result.

Corollary 3.3 *Let E be a reflexive and strictly convex Banach space which has a weakly continuous duality mapping J_φ. Let A be an m-accretive operator in E with zeros. Assume that $\overline{D(A)}$ is convex and $A^{-1}(0)$ is not empty. Let f be an α-contraction on $\overline{D(A)}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ be real number sequences in $(0,1)$. Let $\{e_n\}$ be a bounded computational error in $\overline{D(A)}$. Let $\{x_n\}$ be a sequence in $\overline{D(A_i)}$ generated by the following process*:

$$\begin{cases} x_1 \in \overline{D(A)}, & \textit{chosen arbitrarily}, \\ y_n = \alpha'_n x_n + \beta'_n J_r x_n + \gamma'_n e_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, & \forall n \geq 1, \end{cases}$$

where $J_r = (I + rA)^{-1}$. Assume that the following conditions are satisfied:

(1) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

Then $\{x_n\}$ converges strongly to $x = P_{\overline{D(A)}} f(x)$, where $P_{\overline{D(A)}}$ is the sunny nonexpansive contraction onto $\overline{D(A)}$.

Finally, we investigate the Ky Fan inequality, which is also known as the equilibrium problem [31].

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{3.7}$$

$EP(F)$ stands for the solution set of the equilibrium problem.

To study equilibrium problem (3.7), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Lemma 3.4 [31] *Let C be a nonempty, closed, and convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$J_r x := \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *J_r is single-valued*;

(b) *J_r is firmly nonexpansive*;

(c) *$F(J_r) = EP(F)$*;

(d) *$EP(F)$ is closed and convex.*

Theorem 3.5 *Let C be a nonempty, closed, and convex subset of a Hilbert space E and let F_i be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *for each $i \geq 1$. Assume that $\bigcap_{i=1}^{\infty} EP(F_i)$ is not empty. Let f be an α-contraction on C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, and $\{\delta_{n,i}\}$ be real number sequences in $(0,1)$. Let $\{e_n\}$ be the bounded computational error in C. Let $\{x_n\}$ be a sequence in C generated by the following process*:

$$\begin{cases} x_1 \in H, \quad \textit{chosen arbitrarily}, \\ F_i(z_{n,i}, z) + \frac{1}{r_i}\langle z - z_{n,i}, z_{n,i} - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = \alpha'_n x_n + \beta'_n \sum_{i=1}^{\infty} \delta_{n,i} z_{n,i} + \gamma'_n e_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 1. \end{cases}$$

Assume that the following conditions are satisfied:

(1) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \sum_{i=1}^{\infty} \delta_{n,i} = 1$;

(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(4) $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then $\{x_n\}$ converges strongly to $x = P_{\bigcap_{i=1}^{\infty} EP(F_i)} f(x)$, where $P_{\bigcap_{i=1}^{\infty} EP(F_i)}$ is the metric projection onto $\bigcap_{i=1}^{\infty} EP(F_i)$.

Table 1 The framework of the ILA

ILA:	**Ishikawa-like algorithm (for equilibrium problem (3.7))**
Step 0:	Choose $x_1 \in C, \alpha_1, \beta_1, \gamma_1, \alpha_1', \beta_1', \gamma_1' \in [0,1]$. Set $n := 1$.
Step 1:	Given $x_n \in C$. Choose $\alpha_n, \beta_n, \gamma_n, \alpha_n', \beta_n', \gamma_n' \in [0,1]$ and compute $x_{n+1} \in C$ as
	$F(z_n, z) + \frac{1}{r}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C,$
	$y_n = \alpha_n' x_n + \beta_n' z_n + \gamma_n' e_n,$
	$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n.$
Update $n := n+1$ and go to Step 1.	

Proof From Lemma 3.4, we find that $z_{n,i} = J_{r_{n,i}} x_n$, where $J_{r_{n,i}}$ is defined as follows:

$$J_{r_{n,i}} x := \left\{ z \in C : F_i(z,y) + \frac{1}{r_{n,i}}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad \forall x \in H.$$

From Theorem 3.1, we find the desired conclusion immediately. □

Remark Let $F = x^2 - xy - 2x + 2y$ be a bifunction from $[0,1] \times [0,1]$ to $\mathbb{R}$. It is easy to see that F satisfies conditions (A1)-(A4). Let $f(x) = \frac{x}{2}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{2n}$, $\gamma_n = \frac{n-2}{2n}$, $\alpha_n = \frac{n}{n^2}$, $\beta_n = \frac{n^2-n-1}{n^2}$, and $\gamma_n' = e_n = \frac{1}{n^2}$. Let $\{x_n\}$ be a sequence in C generated in the ILA (see Table 1). Then $\{x_n\}$ converges to zero.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by SHW. PZ performed some steps of the proofs. Both authors read and approved the final manuscript.

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