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Existence of positive solutions of nonlinear fractional q-difference equation with parameter

Xinhui Li, Zhenlai Han* and Shurong Sun

*Correspondence:
hanzhenlai@163.com
School of Mathematical Sciences,
University of Jinan, Jinan, Shandong
250022, P.R. China

Abstract
In this paper, we study the boundary value problem of a class of nonlinear fractional q-difference equations with parameter involving the Riemann-Liouville fractional derivative. By means of a fixed point theorem in cones, some positive solutions are obtained. As applications, some examples are presented to illustrate our main results.
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1 Introduction

The q-difference calculus is an interesting and old subject that many researchers devote their time to studying. The q-difference calculus or quantum calculus were first developed by Jackson [1, 2], while basic definitions and properties can be found in the papers [3, 4]. The q-difference calculus describes many phenomena in various fields of science and engineering [1].

The origin of the fractional q-difference calculus can be traced back to the works in [5, 6] by Al-Salam and by Agarwal.

The q-difference calculus is a necessary part of discrete mathematics. More recently, there has been much research activity concerning the fractional q-difference calculus [7–15]. Relevant theory about fractional q-difference calculus has been established [16], such as q-analogues of integral and difference fractional operators properties as Mittag-Leffler function [17], q-Laplace transform, q-Taylor's formula [18, 19], just to mention some. It is not only the requirements of the fractional q-difference calculus theory but also its the broad application.

Apart from this old history of q-difference equations, the subject has received a considerable interest of many mathematicians and from many aspects, theoretical and practical. Specifically, q-difference equations have been widely used in mathematical physical problems, dynamical system and quantum models [20], q-analogues of mathematical physical problems including heat and wave equations [21], sampling theory of signal analysis [22, 23]. What is more, the fractional q-difference calculus plays an important role in quantum calculus.

As generalizations of integer order q-difference, fractional q-difference can describe physical phenomena much better and more accurately. Perhaps due to the development of fractional differential equations [24–26], an interest has been observed in studying boundary value problems of fractional q-difference equations, especially about the existence of solutions for boundary value problems [3, 4, 27, 28].

In 2010, Ferreita [3] considered the existence of nontrivial solutions to the fractional q-difference equation

$$\left(D_q^\alpha y\right)(x) = -f\left(x, y(x)\right), \quad 0 < x < 1,$$

subjected to the boundary conditions

$$y(0) = 0, \qquad y(1) = 0,$$

where $1 < \alpha \leq 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function.

In 2011, Ferreita [4] went on studying the existence of positive solutions to the fractional q-difference equation

$$\left(D_q^\alpha y\right)(x) = -f\left(x, y(x)\right), \quad 0 < x < 1,$$

subjected to the boundary conditions

$$y(0) = (D_q y)(0) = 0, \qquad (D_q y)(1) = \beta \geq 0,$$

where $2 < \alpha \leq 3$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. By constructing a special cone and using Krasnosel'skii fixed point theorem, some existence results of positive solutions were obtained.

In 2011, El-Shahed and Al-Askar [27] studied the existence of a positive solution for a boundary value problem of the nonlinear factional q-difference equation

$${}_cD_q^\alpha u + a(t)f(t) = 0, \quad 0 \leq t \leq 1, 2 < \alpha \leq 3,$$

with the boundary conditions

$$\begin{aligned} &u(0) = D_q^2 u(0) = 0, \\ &\gamma D_q u(1) + \beta D_q^2 u(1) = 0, \end{aligned}$$

where $\gamma, \beta \leq 0$ and ${}_cD_q^\alpha$ is fractional q-derivative of Caputo type.

In 2012, Liang and Zhang [28] studied the existence and uniqueness of positive solutions for the three-point boundary problem of fractional q-differences

$$\begin{aligned} &\left(D_q^\alpha u\right)(t) + f\left(t, u(t)\right) = 0, \quad 0 < t < 1, 2 < \alpha < 3, \\ &u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = \beta(D_q u)(\eta), \end{aligned}$$

where $0 < \beta\eta^{\alpha-2} < 1$. By using a fixed-point theorem in partially ordered sets, they got some sufficient conditions for the existence and uniqueness of positive solutions to the above boundary problem.

To the best of our knowledge, there are few papers that consider the boundary value of nonlinear fractional q-difference equations with parameters. Theories and applications seem to be just being initiated. In this paper we investigate the existence of solutions for the following two-point boundary value problem of nonlinear fractional q-difference equations

$$\left(D_q^\alpha u\right)(x) + \lambda f\left(u(x)\right) = 0, \quad 0 < x < 1, \tag{1.1}$$

subject to the boundary conditions

$$u(0) = D_q u(0) = D_q u(1) = 0, \tag{1.2}$$

where $0 < q < 1$, $2 < \alpha < 3$, $f : C((0,1),(0,\infty))$. We prove the existence of positive solutions for boundary value problem (1.1)-(1.2) by utilizing a fixed point theorem in cones. Several existence results for positive solutions in terms of different values of the parameter λ are obtained. This work is motivated by papers [25, 28].

The paper is organized as follows. In Section 2, we introduce some definitions of q-fractional integral and differential operator together with some basic properties and lemmas to prove our main results. In Section 3, we investigate the existence of positive solutions for boundary value problem (1.1)-(1.2) by a fixed point theorem in cones. Moreover, some examples are given to illustrate our main results.

2 Preliminaries

In the following section, we collect some definitions and lemmas about fractional q-integral and fractional q-derivative which are referred to in [3].

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^0 = 1, \qquad (a-b)^n = \prod_{k=0}^{n-1}\left(a - bq^k\right), \quad n \in \mathbb{N}, a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

It is easy to see that $[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}$. And note that if $b = 0$ then $a^{(\alpha)} = a^\alpha$.

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{for } x \neq 0, \qquad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$\left(D_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(D_q^n f\right)(x) = D_q\left(D_q^{n-1} f\right)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined on the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} f\left(xq^n\right)q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined on the interval $[0, b]$, its q-integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined as

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), \quad n \in \mathbb{N}.$$

From the definition of q-integral and the properties of series, we can get the following results concerning q-integral, which are helpful in the proofs of our main results.

Lemma 2.1 (1) *If f and g are q-integral on the interval $[a, b]$, $\alpha \in \mathbb{R}$, $c \in [a, b]$, then*

(i) $\int_a^b (f(t) + g(t))\, d_q t = \int_a^b f(t)\, d_q t + \int_a^b g(t)\, d_q t$;

(ii) $\int_a^b \alpha f(t)\, d_q t = \alpha \int_b^a f(t)\, d_q t$;

(iii) $\int_a^b f(t)\, d_q t = \int_a^c f(t)\, d_q t + \int_c^b f(t)\, d_q t$;

(2) *If $|f|$ is q-integral on the interval $[0, x]$, then* $|\int_0^x f(t)\, d_q t| \leq \int_0^x |f(t)|\, d_q t$;

(3) *If f and g are q-integral on the interval $[0, x]$, $f(t) \leq g(t)$ for all $t \in [0, x]$, then* $\int_0^x f(t)\, d_q t \leq \int_0^x g(t)\, d_q t$.

The fundamental theorem of calculus applies to these operators I_q and D_q, *i.e.*,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of q-integral operator and q-differential operator can be found in the book [16].

We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$
\begin{aligned}
&{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)}, \\
&\left({}_xD_q \int_0^x f(x,t)\,d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_qt + f(qx,x).
\end{aligned}
$$

Remark 2.1 We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Definition 2.1 [6] Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_qt, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 [18] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Next, we list some properties about q-derivative and q-integral that are already known in the literature.

Lemma 2.2 [6, 18] *Let* $\alpha, \beta \geq 0$ *and* f *be a function defined on* $[0,1]$. *Then the following formulas hold*:

(i) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$;
(ii) $(D_q^\alpha I_q^\alpha f) = f(x)$.

Lemma 2.3 [3] *Let* $\alpha > 0$ *and* p *be a positive integer. Then the following equality holds*:

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^k f\right)(0).$$

Lemma 2.4 [29] *Let* X *be a Banach space and* $P \subseteq X$ *be a cone. Suppose that* Ω_1 *and* Ω_2 *are bounded open sets contained in* X *such that* $0 \in \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega_2$. *Suppose further that* $S : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *is a completely continuous operator. If either*

(i) $\|Su\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Su\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Su\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Su\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *then* S *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The next result is important in the sequel.

Lemma 2.5 [4] *Let* $f(u(x)) \in C[0,1]$ *be a given function. Then the boundary value problem*

$$\left(D_q^\alpha u\right)(x) + f\left(u(x)\right) = 0, \quad 0 < x < 1, \tag{2.1}$$

$$u(0) = D_q u(0) = D_q u(1) = 0, \tag{2.2}$$

has a unique solution

$$u(x) = \int_0^1 G(x,qt)f\bigl(u(t)\bigr)\,d_qt,$$

where

$$G(x,qt) = \frac{1}{\Gamma_q(\alpha)}\begin{cases}(1-qt)^{(\alpha-2)}x^{\alpha-1}-(x-qt)^{\alpha-1}, & 0\le qt\le x\le 1,\\ (1-qt)^{(\alpha-2)}x^{\alpha-1}, & 0\le x\le qt\le 1,\end{cases}$$

is the Green function of boundary value problem (2.1)-(2.2).

The following properties of the Green function play important roles in this paper.

Lemma 2.6 [4] *Function G defined above satisfies the following conditions*:

$$\begin{aligned}&(1)\quad G(x,qt)\ge 0 \text{ and } G(x,qt)\le G(1,qt) \text{ for all } 0\le x,t\le 1.\\ &(2)\quad G(x,qt)\ge g(x)G(1,qt) \text{ for all } 0\le x,t\le 1 \text{ with } g(x)=x^{\alpha-1}.\end{aligned} \tag{2.3}$$

3 Main results

We are now in a position to state and prove our main results in this paper.

Let the Banach space $B = C[0,1]$ be endowed with the norm $\|u\| = \sup_{x\in[0,1]}|u(x)|$. Let τ be a real constant with $0<\tau<1$ and define the cone $P\subset B$ by $P = \{u\in C[0,1] \mid u(x)\ge 0, \min_{x\in[\tau,1]}u(x)\ge \tau^{\alpha-1}\|u\|\}$.

Suppose that u is a solution of boundary value problem (1.1)-(1.2). Then

$$u(x) = \lambda\int_0^1 G(x,qt)f\bigl(u(t)\bigr)\,d_qt,\quad t\in[0,1]. \tag{3.1}$$

Define the operator $A_\lambda : P\to B$ by

$$A_\lambda u(x) = \lambda\int_0^1 G(x,qt)f\bigl(u(t)\bigr)\,d_qt.$$

Then we have the following results.

Lemma 3.1 $A_\lambda : P\to P$ *is completely continuous.*

Proof It is easy to see that the operator $A_\lambda : P\to P$ is continuous in view of continuity of G and f.

By Lemmas 2.1 and 2.6, we have

$$\begin{aligned}\min_{x\in[\tau,1]}A_\lambda\bigl(u(x)\bigr) &= \min_{x\in[\tau,1]}\lambda\int_0^1 G(x,qt)f\bigl(u(t)\bigr)\,d_qt\\ &\ge \tau^{\alpha-1}\left(\lambda\int_\tau^1 G(1,qt)f\bigl(u(t)\bigr)\,d_qt\right)\\ &= \tau^{\alpha-1}\|A_\lambda u\|.\end{aligned}$$

Thus, $A_\lambda(P)\subset P$.

Now, let $\Omega \subset P$ be bounded, *i.e.*, there exists a positive constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$. Let $L = \max_{\|u\|\leq M} |f(u(x))| + 1$. Then, for $u \in \Omega$, from Lemmas 2.1 and 2.6, we have

$$\left|A_\lambda u(x)\right| \leq \lambda \int_0^1 \left|G(x,qt)f\bigl(u(t)\bigr)\right| dt \leq \lambda L \int_0^1 G(1,qt)\, d_qt.$$

Hence, $A_\lambda(\Omega)$ is bounded.

On the other hand, for any given $\varepsilon > 0$, setting

$$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon\Gamma_q(\alpha)}{2L\lambda}\right\},$$

then for each $u \in \Omega$, $0 \leq x_1 \leq x_2 \leq 1$ and $|x_2 - x_1| < \delta$, one has $|A_\lambda u(x_2) - A_\lambda u(x_1)| < \varepsilon$, that is to say, $A_\lambda(\Omega)$ is equicontinuous. In fact,

$$\begin{aligned}
&\left|A_\lambda u(x_2) - A_\lambda u(x_1)\right| \\
&\quad= \left|\lambda \int_0^1 G(x_2,qt)f\bigl(u(t)\bigr)\, d_qt - \lambda \int_0^1 G(x_1,qt)f\bigl(u(t)\bigr)\, d_qt\right| \\
&\quad\leq \lambda \int_0^1 \left|G(x_2,qt) - G(x_1,qt)f\bigl(u(t)\bigr)\right| d_qt\lambda L \int_0^1 \left|G(x_2,qt) - G(x_1,qt)\right| d_qt \\
&\quad= \lambda L\left(\int_0^{x_1} \left|G(x_2,qt) - G(x_1,qt)\right| d_qt + \int_{x_1}^{x_2} \left|G(x_2,qt) - G(x_1,qt)\right| d_qt\right. \\
&\qquad\left. + \int_{x_2}^1 \left|G(x_2,qt) - G(x_1,qt)\right| d_qt\right) \\
&\quad= \lambda L\left(\int_0^{x_1} \frac{1}{\Gamma_q(\alpha)}\left[(1-qt)^{(\alpha-2)}\bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr) - (x_2-qt)^{(\alpha-1)} + (x_1-qt)^{(\alpha-1)}\right] d_qt\right. \\
&\qquad + \int_{x_1}^{x_2} \frac{1}{\Gamma_q(\alpha)}\left[(1-qt)^{(\alpha-2)}x_2^{\alpha-1} - (x_2-qt)^{(\alpha-1)} - (1-qt)^{(\alpha-2)}x_1^{\alpha-1}\right] d_qt \\
&\qquad\left. + \int_{x_2}^1 \frac{1}{\Gamma_q(\alpha)}\left[(1-qt)^{(\alpha-2)}\bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr)\right] d_qt\right).
\end{aligned}$$

Now we rearrange the above equation as follows, and from the properties of *q-integral*, we get

$$\begin{aligned}
&\left|A_\lambda u(x_2) - A_\lambda u(x_1)\right| \\
&\quad= \lambda L\frac{1}{\Gamma_q(\alpha)}\left\{\int_0^1 (1-qt)^{(\alpha-2)}\bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr)\, d_qt\right. \\
&\qquad\left. + \int_0^{x_1} (x_1-qt)^{(\alpha-1)}\, d_qt - \int_0^{x_2} (x_2-qt)^{(\alpha-1)}\, d_qt\right\} \\
&\quad\leq \lambda L\frac{1}{\Gamma_q(\alpha)}\left\{\int_0^1 \bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr)\, d_qt + \int_0^{x_1} (x_1-qt)^{(\alpha-1)}\, d_qt - \int_0^{x_2} (x_2-qt)^{(\alpha-1)}\, d_qt\right\} \\
&\quad\leq \lambda L\frac{1}{\Gamma_q(\alpha)}\left\{\int_0^1 \bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr)\, d_qt + x_1^\alpha \int_0^1 (1-qt)^{(\alpha-1)}\, d_qt - x_2^\alpha \int_0^1 (1-qt)^{(\alpha-1)}\, d_qt\right\} \\
&\quad= \lambda L\frac{1}{\Gamma_q(\alpha)}\left\{\bigl(x_2^{\alpha-1} - x_1^{\alpha-1}\bigr)\int_0^1 d_qt + \bigl(x_1^\alpha - x_2^\alpha\bigr)\int_0^1 (1-qt)^{(\alpha-1)}\, d_qt\right\}
\end{aligned}$$

$$= \lambda L \frac{1}{\Gamma_q(\alpha)} \left\{ \left(x_2^{\alpha-1} - x_1^{\alpha-1}\right)(1-q) \sum_{n=0}^{\infty} q^n + \left(x_1^{\alpha} - x_2^{\alpha}\right)(1-q) \sum_{n=0}^{\infty} \left(1-q^{n+1}\right)^{(\alpha-1)} q^n \right\}$$

$$\leq \lambda L \frac{1}{\Gamma_q(\alpha)} \left(\left(x_2^{\alpha-1} - x_1^{\alpha-1}\right) + \left(x_1^{\alpha} - x_2^{\alpha}\right)\right)(1-q) \sum_{n=0}^{\infty} q^n$$

$$= \lambda L \frac{1}{\Gamma_q(\alpha)} \left\{ \left(x_2^{\alpha-1} - x_1^{\alpha-1}\right) + \left(x_1^{\alpha} - x_2^{\alpha}\right)\right\}$$

$$\leq \lambda L \frac{1}{\Gamma_q(\alpha)} \left(x_2^{\alpha-1} - x_1^{\alpha-1}\right).$$

Now, we estimate $x_2^{\alpha-1} - x_1^{\alpha-1}$:

(1) for $0 \leq x_1 < \delta$, $\delta \leq x_2 < 2\delta$, $x_2^{\alpha-1} - x_1^{\alpha-1} \leq x_2^{\alpha-1} < (2\delta)^{\alpha-1} \leq 2\delta$;

(2) for $0 \leq x_1 < x_2 \leq \delta$, $x_2^{\alpha-1} - x_1^{\alpha-1} \leq x_2^{\alpha-1} < \delta^{\alpha-1} \leq 2\delta$;

(3) for $\delta \leq x_1 < x_2 \leq 1$, from the mean value theorem of differentiation, we have $x_2^{\alpha-1} - x_1^{\alpha-1} \leq (\alpha-1)(x_2 - x_1) \leq 2\delta$.

Thus, we have that

$$\left|A_\lambda u(x_2) - A_\lambda u(x_1)\right| < \frac{2\lambda L\delta}{\Gamma_q(\alpha)} < \varepsilon.$$

By means of the Arzela-Ascoli theorem, $A_\lambda : P \to P$ is completely continuous. The proof is completed. □

For convenience, we define

$$F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u}, \qquad F_\infty = \limsup_{u \to +\infty} \frac{f(u)}{u},$$

$$f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \liminf_{u \to +\infty} \frac{f(u)}{u},$$

$$C_1 = \int_0^1 G(1, qt)\, d_q t, \qquad C_2 = \int_\tau^1 \tau^{2\alpha-2} G(1, qt)\, d_q t.$$

The main results of the paper are as follows.

Theorem 3.1 *If $f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((f_\infty C_2)^{-1}, (F_0 C_1)^{-1}\right), \tag{3.2}$$

boundary value problem (1.1)-(1.2) *has at least one positive solution. Here we impose $(f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0 C_1)^{-1} = +\infty$ if $F_0 = 0$.*

Proof Let λ satisfy (3.2) and $\varepsilon > 0$ be such that

$$\left((f_\infty - \varepsilon) C_2\right)^{-1} \leq \lambda \leq \left((F_0 + \varepsilon) C_1\right)^{-1}. \tag{3.3}$$

By the definition of F_0, we can know that there exists $r_1 > 0$ such that

$$f(u) \leq (F_0 + \varepsilon) u \quad \text{for } 0 \leq u \leq r_1, \tag{3.4}$$

so if $u \in P$ with $\|u\| = r_1$, then by (2.3) and (3.4), we have

$$\begin{aligned}\|A_\lambda u\| &\le \lambda \int_0^1 G(1, qt) f\bigl(u(t)\bigr)\, d_q t \\ &\le \lambda \int_0^1 G(1, qt)(F_0 + \varepsilon) r_1 \, d_q t \\ &\le \lambda (F_0 + \varepsilon) r_1 C_1 \le r_1 \\ &= \|u\|.\end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in B : \|u\| < r_1\}$, then

$$\|A_\lambda\| \le \|u\| \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.5}$$

Let $r_3 > 0$ be such that

$$f(u) \ge (f_\infty - \varepsilon) u \quad \text{for } u \ge r_3. \tag{3.6}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1, \tau^{1-\alpha} r_3\}$, then by (2.3) and (3.6) we have

$$\begin{aligned}\|A_\lambda u\| &\ge A_\lambda\bigl(u(t)\bigr) = \lambda \int_0^1 G(x, qt) f\bigl(u(t)\bigr)\, d_q t \\ &\ge \lambda \int_\tau^1 G(x, qt) f\bigl(u(t)\bigr)\, d_q t \\ &\ge \lambda \int_\tau^1 \tau^{\alpha-1} G(1, qt)(f_\infty - \varepsilon) u(t)\, d_q t \\ &\ge \lambda \int_\tau^1 \tau^{2\alpha-2} G(1, qt)(f_\infty - \varepsilon)\|u\|\, d_q t \\ &= \lambda C_2 (f_\infty - \varepsilon)\|u\| \\ &\ge \|u\|.\end{aligned} \tag{3.7}$$

Thus, if we set

$$\Omega_2 = \bigl\{u \in B : \|u\| < r_2\bigr\}, \tag{3.8}$$

then

$$\|A_\lambda u\| \ge \|u\| \quad \text{for } u \in P \in \partial\Omega_2. \tag{3.9}$$

Now, from (3.5), (3.9) and Lemma 2.4, we conclude that A_λ has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$, and it is clear that u is a positive solution of (1.1)-(1.2). The proof is completed. □

Theorem 3.2 *If $f_0 C_2 > F_\infty C_1$ holds, then for each*

$$\lambda \in \bigl((f_0 C_2)^{-1}, (F_\infty C_1)^{-1}\bigr), \tag{3.10}$$

boundary value problem (1.1)-(1.2) *has at least one positive solution. Here we impose* $(f_0 C_2)^{-1} = 0$ *if* $f_0 = +\infty$ *and* $(F_\infty C_1)^{-1} = +\infty$ *if* $F_\infty = 0$.

Proof Let λ satisfy (3.10) and $\varepsilon > 0$ be given such that

$$\bigl((f_0 - \varepsilon)C_2\bigr)^{-1} \le \lambda \le \bigl((F_\infty + \varepsilon)C_1\bigr)^{-1}. \tag{3.11}$$

From the definition of f_0, we can see that there exists $r_1 > 0$ such that

$$f(u) \ge (f_0 - \varepsilon)u, \quad 0 < u \le r_1. \tag{3.12}$$

Further, if $u \in P$, $\|u\| = r_1$, then the flowing is similar to the second part of Theorem 3.1:

$$\begin{aligned} \|A_\lambda u\| &\ge \lambda \int_0^1 G(x,qt) f\bigl(u(t)\bigr)\, d_q t \\ &\ge \lambda \int_\tau^1 G(x,qt) f\bigl(u(t)\bigr)\, dt \\ &\ge \lambda \int_\tau^1 \tau^{\alpha-1} G(1,qt) f\bigl(u(t)\bigr)\, d_q t \\ &\ge \lambda \int_\tau^1 \tau^{\alpha-1} G(1,qt)(f_0 - \varepsilon) u(t)\, d_q t \\ &\ge \lambda \int_\tau^1 \tau^{2\alpha-2} G(1,qt)(f_0 - \varepsilon)\|u\|\, d_q t \\ &= \lambda C_2 (f_0 - \varepsilon)\|u\| \ge \|u\|. \end{aligned}$$

We can obtain that $\|A_\lambda u\| \ge \|u\|$. Thus, if we choose $\Omega_1 = \{u \in B : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \ge \|u\| \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.13}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \le (F_\infty + \varepsilon)u \quad \text{for } u \ge R_1. \tag{3.14}$$

We consider two cases.

Case 1. Suppose that f is bounded. Then there exists some $M > 0$ such that $f(u) \le M$ for $u \in (0, +\infty)$. Define $r_3 = \max\{2r_1, \lambda M C_1\}$. Then if $u \in P$ with $\|u\| = r_3$, we have

$$\|A_\lambda u\| \le \lambda \int_0^1 G(1,qt) f\bigl(u(t)\bigr)\, d_q t \le \lambda M \int_0^1 G(1,qt)\, d_q t \le \lambda M C_1 \le r_3 = \|u\|.$$

Hence,

$$\|A_\lambda u\| \le \|u\| \quad \text{for } u \in (0, +\infty). \tag{3.15}$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, \tau^{1-\alpha} R_1\}$ such that

$$f(u) \le f(r_4) \quad \text{for } 0 < u \le r_4. \tag{3.16}$$

Let $u \in P$ with $\|u\| = r_4$. Then by (2.3) and (3.14) we get

$$\begin{aligned}\|A_\lambda u\| &\leq \lambda \int_0^1 G(1,qt)f\bigl(u(t)\bigr)\,d_qt \leq \lambda \int_0^1 G(1,qt)(F_\infty + \varepsilon)u\,d_qt \\ &= \lambda C_1(F_\infty + \varepsilon)\|u\| \leq \|u\|.\end{aligned}$$

Thus, (3.15) is also true.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in B : \|u\| < r_2\}$, where $r_2 = \max\{r_3, r_4\}$, then

$$\|A_\lambda\| \leq \|u\| \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.17}$$

Now that we have obtained (3.13) and (3.17), it follows from Lemma 2.4 that A_λ has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear that u is a positive solution of (1.1)-(1.2). The proof is completed. □

Theorem 3.3 *If there exist* $k_1 > k_2 > 0$ *such that*

$$\max_{0\leq u\leq k_1} f(u) \leq \frac{k_1}{\lambda C_1}, \qquad \min_{\tau\leq u\leq k_2} f(u) \geq \frac{k_2}{\lambda C_2},$$

then boundary value problem (1.1)-(1.2) *has a positive solution* $u \in P$ *with* $k_2 \leq \|u\| \leq k_1$.

Proof Choose $\Omega_1 = \{u \in B : \|u\| < k_2\}$. Then, for $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}\|A_\lambda u\| \geq A_\lambda u(t) &= \lambda \int_0^1 G(x,qt)f\bigl(u(t)\bigr)\,d_qt \\ &\geq \lambda \int_\tau^1 G(1,qt)f\bigl(u(t)\bigr)\,d_qt \\ &\geq \lambda \int_\tau^1 \tau^{\alpha-1}G(1,qt)\min_{\tau\leq u\leq k_2} f\bigl(u(t)\bigr)\,d_qt \\ &\geq \lambda \int_\tau^1 \tau^{2\alpha-2}G(1,qt)\frac{k_2}{\lambda C_2}\,d_qt \\ &= \lambda C_2\frac{k_2}{\lambda C_2} = k_2 = \|u\|.\end{aligned} \tag{3.18}$$

For another thing, choose $\Omega_2 = \{u \in B : \|u\| < k_1\}$, then, for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}\|A_\lambda u\| &\leq \lambda \int_0^1 G(1,qt)f\bigl(u(t)\bigr)\,d_qt \leq \lambda \int_0^1 G(1,qt)\max_{0\leq u\leq k_1} f\bigl(u(t)\bigr)\,d_qt \\ &\leq \lambda \int_0^1 G(1,qt)\frac{k_1}{\lambda C_1}\,d_qt = k_1 = \|u\|.\end{aligned} \tag{3.19}$$

Now that we have obtained (3.18) and (3.19), it follows from Lemma 2.4 that A_λ has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $k_2 \leq \|u\| \leq k_1$. It is clear that u is a positive solution of (1.1)-(1.2). The proof is completed. □

4 Examples

In this section, we present some examples to illustrate our main results.

Example 4.1 Consider the following boundary value problem:

$$\bigl(D_{\frac{1}{2}}^{\frac{5}{2}} u\bigr)(x) + \lambda u^2 = 0, \quad 0 < t < 1, \tag{4.1}$$

$$u(0) = D_{\frac{1}{2}}(0) = D_{\frac{1}{2}}(1) = 0. \tag{4.2}$$

Let $q = \tau = \frac{1}{2}$, $\alpha = \frac{5}{2}$ and $f(u) = u^2$. Then

$$f_\infty = +\infty, \qquad F_0 = 0, \qquad C_1 = \int_0^1 G(1, qt)\, d_q t, \qquad C_2 = \int_\tau^1 \tau^{2\alpha-2} G(1, qt)\, d_q t,$$

and so $f_\infty C_2 > F_0 C_1$. By Theorem 3.1, boundary value problem (4.1)-(4.2) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 4.2 Consider the following boundary value problem:

$$\bigl(D_{\frac{1}{2}}^{\frac{5}{2}} u\bigr)(x) + \lambda(2 + \sin u) = 0, \quad 0 < t < 1, \tag{4.3}$$

$$u(0) = D_{\frac{1}{2}}(0) = D_{\frac{1}{2}}(1) = 0. \tag{4.4}$$

Let $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$ and $f(u) = 2 + \sin u$. Then $f_0 = \infty$, $F_\infty = 0$,

$$C_1 = \int_0^1 G(1, qt)\, d_q t, \qquad C_2 = \int_\tau^1 \tau^{2\alpha-2} G(1, qt)\, d_q t.$$

It is clear that $F_\infty C_1 < f_0 C_2$. By Theorem 3.2, boundary value problem (4.3)-(4.4) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 4.3 We can still consider the example that has been given in Example 4.2,

$$\bigl(D_{\frac{1}{2}}^{\frac{5}{2}} u\bigr)(x) + \lambda(2 + \sin u) = 0, \quad 0 < t < 1, \tag{4.5}$$

$$u(0) = D_{\frac{1}{2}}(0) = D_{\frac{1}{2}}(1) = 0. \tag{4.6}$$

Here $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $f(u) = 2 + \sin u$. Take $0 < \tau < 1$. Then

$$C_1 = \int_0^1 G(1, qt)\, d_q t, \qquad C_2 = \int_\tau^1 \tau^{2\alpha-2} G(1, qt)\, d_q t.$$

Set $k_1 = 3\lambda C_1$, $k_2 = \lambda C_2$ with $\lambda > \frac{\tau}{c_2}$. Then $k_1 > k_2$, and

$$\max_{0 \le u \le k_1} f(u) \le 3 \le \frac{k_1}{\lambda C_1}, \qquad \min_{\tau \le u \le k_2} f(u) \ge 1 \ge \frac{k_2}{\lambda C_2}.$$

Thus all the conditions in Theorem 3.3 hold. Hence, by Theorem 3.3, boundary value problem (4.5)-(4.6) has a positive solution with $k_2 \le \|u\| \le k_1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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