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Existence of periodic solutions for first-order delay differential equations via critical point theory

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Abstract

By the weak linking theorem and the linking theorem, we study the existence of periodic solutions for the following system of delay differential equations:

$$u'(t) = -f\big(u(t-r)\big), \tag{1}$$

where $f \in C(R^n, R^n)$, and $r > 0$ is a given constant. Two existence theorems of $4r$-periodic solutions of (1) are obtained.

Keywords: weak linking; linking; periodic solution; critical point

1 Introduction and preliminaries

Consider the following system of delay differential equations:

$$u'(t) = -f\big(u(t-r)\big), \tag{1.1}$$

where $f \in C(R^n, R^n)$, and $r > 0$ is a given constant.

As $n \equiv 1$, the existence of the periodic solutions for (1.1) has been extensively studied in the past years (for example, see [1–5]). However, their methods are not variational. Few results of the existence of periodic solutions for delay differential equations have been obtained by the variational method. In 2005, Guo and Yu [6] took the lead in using the variational approaches to study the existence of multiple periodic solutions for (1.1), and a multiplicity result was given. Recently, using the variational approaches, the multiplicity of the periodic solutions for the following system:

$$\begin{cases} u'(t) = -\Lambda u(t+r) - f(t, u(t-r)), \\ u(0) = -u(2r), \qquad u(0) = u(4r) \end{cases}$$

was studied by Wu and Wu in [7]. In the present paper, our main purpose is to study the existence of the periodic orbits for system (1.1) via the linking and weak linking theory.

Throughout this paper, we always assume that

(f_1) f is odd, *i.e.*, for any $x \in R^n$, $f(-x) = -f(x)$;

(f_2) there exists a continuously differentiable function F such that $\nabla F(x) = f(x)$ for all $x \in R^n$, and $F(0) = 0$;

(f_3) $f(x) = Ax + o(|x|)$ as $|x| \to \infty$, where $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix with

$$\|A\| := \max_{1\le i,j\le n} |a_{ij}| < \lambda^{-1}$$

and $\lambda^{-1}(-1)^{j+1}(2j-1) \notin \sigma(A)$ (the set of all eigenvalues of A) for any $j \in Z^+$, where $\lambda = \frac{2r}{\pi}$ and Z^+ is the set of all positive integers.

In the following, we give some preliminaries.

Definition 1.1 ([8]) Let E be a Hilbert space and $I \in C^1(E, R)$. The function I' is called weak-to-weak continuous if

$$u_k \rightharpoonup u \quad \text{in } E \quad \Rightarrow \quad I'(u_k) \rightharpoonup I'(u). \tag{1.2}$$

Definition 1.2 ([8]) A subset A of a Banach space E links a subset B of E weakly if for every $I \in C^1(E, R)$ satisfying (1.2) and

$$a_0 := \sup_A I \le b_0 := \inf_B I, \tag{1.3}$$

there are a sequence $\{u_k\} \subset E$ and a constant c such that

$$b_0 \le c < \infty \tag{1.4}$$

and

$$I(u_k) \to c, \qquad I'(u_k) \to 0. \tag{1.5}$$

The following lemma is Example 1 in [8].

Lemma 1.1 *Let E be a separable Hilbert space, and let M, N be a closed subspace such that $E = M \oplus N$. Let*

$$B_R = \{u \in E : \|u\| < R\} \tag{1.6}$$

and take $A = \partial B_R \cap N$, $B = M$. Then A links B weakly.

One can easily find that (1.1) can be changed to the equation

$$u'(t) = -\lambda f\left(u\left(t - \frac{\pi}{2}\right)\right) \tag{1.7}$$

by making the change of variable $t \mapsto \frac{\pi}{2r} t = \lambda^{-1} t$. Thus, a $4r$-periodic solution of (1.1) corresponds to a 2π-periodic solution of (1.7).

Similar to the treatment in Guo and Yu [6], we introduce the following spaces. Let $L^2(S^1, R^n)$ denote the set of n-tuples of 2π periodic functions which are square integrable.

Let $C^\infty(S^1, R^n)$ be the space of 2π-periodic C^∞ vector-valued functions with dimension n. For any $u \in C^\infty(S^1, R^n)$, it has the following Fourier expansion in the sense that it is convergent in the space $L^2(S^1, R^n)$:

$$u(t) = \frac{a_0^u}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left(a_j^u \cos jt + b_j^u \sin jt\right),$$

where $a_0^u, a_j^u, b_j^u \in R^n$. Set $H^{\frac{1}{2}}(S^1, R^n)$ is the closure of $C^\infty(S^1, R^n)$ with respect to the Hilbert norm

$$\|u\|_{H^{\frac{1}{2}}} = \left[\left|a_0^u\right|^2 + \sum_{j=1}^{+\infty} (1+j)\left(\left|a_j^u\right|^2 + \left|b_j^u\right|^2\right) \right]^{\frac{1}{2}}.$$

More specifically, $H^{\frac{1}{2}}(S^1, R^n) = \{u \in L^2(S^1, R^n) : \|u\|_{H^{\frac{1}{2}}} < +\infty\}$ with the inner product

$$\langle u, v \rangle = \left(a_0^u, a_0^v\right) + \sum_{j=1}^{+\infty} (1+j)\left[\left(a_j^u, a_j^v\right) + \left(b_j^u, b_j^v\right)\right]$$

for any $u, v \in H^{\frac{1}{2}}(S^1, R^n)$, where $(\cdot,\cdot)$ denotes the usual inner product in R^n. In the sequel, we denote by H the Hilbert space $H^{\frac{1}{2}}(S^1, R^n)$. The norm on H is defined by

$$\|u\|_H = \left[\left|a_0^u\right|^2 + \sum_{j=1}^{+\infty} (1+j)\left(\left|a_j^u\right|^2 + \left|b_j^u\right|^2\right) \right]^{\frac{1}{2}}.$$

Now consider a functional I defined on H

$$I(u) = \int_0^{2\pi} \left[\frac{1}{2}\left(\dot{u}\left(t + \frac{\pi}{2}\right), u(t) \right) + \lambda F\big(u(t)\big) \right] dt, \quad \forall u \in H,$$

where $\dot{u}(t)$ denotes the weak derivative of u.

We define an operator $L : H \to H^*$ as follows: for any $u \in H$, Lu is defined by

$$Lu(v) = \int_0^{2\pi} \left(\dot{u}\left(t + \frac{\pi}{2}\right), v(t) \right) dt, \quad \forall v \in H, \tag{1.8}$$

where H^* denotes the dual space of H. By the Riesz representation theorem, we can identify H^* with H. Thus, Lu can also be viewed as a function belonging to H such that $\langle Lu, v \rangle = Lu(v)$ for any $u, v \in H$. Define

$$\Phi(u) = \lambda \int_0^{2\pi} F\big(u(t)\big)\, dt, \quad \forall u \in H. \tag{1.9}$$

Then $I(u)$ can be rewritten as

$$I(u) = \frac{1}{2}\langle Lu, u \rangle + \Phi(u), \quad \forall u \in H. \tag{1.10}$$

Define the bounded linear operator $\zeta : H \to H$ as follows: for any $u \in H$, $\zeta u(\cdot) = u(\cdot + \frac{\pi}{2})$. Next, we set $E = \{u \in H : \zeta^2 u = -u\}$. Then E is a closed subspace of H and is invariant with respect to L. It is easy to check that L is a bounded linear operator on H, $L|_E$ is self-adjoint, and E is also invariant with respect to Φ' under condition (f_1) (see Guo and Yu [6]). By Proposition B.37 in [9] and Lemma 2.2 in [6], we have the following two lemmas.

Lemma 1.2 *Assume that f satisfies (f_2) and the following condition*:

(f_0) *there are constants $a_1, a_2 > 0$ and $\alpha \geq 1$ such that*

$$\left|f(x)\right| \leq a_1 + a_2|x|^\alpha$$

for all $x \in R^n$.

Then the functional I is continuously differentiable on H and $I'(u)$ is defined by

$$\left\langle I'(u), v\right\rangle = \int_0^{2\pi} \left[\frac{1}{2}\left(\dot{u}\left(t + \frac{\pi}{2}\right) - \dot{u}\left(t - \frac{\pi}{2}\right), v(t)\right) + \lambda\left(f\left(u(t)\right), v(t)\right)\right] dt, \quad \forall v \in H.$$

Moreover, $\Phi' : H \to H^*$ is a compact mapping defined as follows:

$$\left\langle \Phi'(u), v\right\rangle = \lambda \int_0^{2\pi} \left(f\left(u(t)\right), v(t)\right) dt, \quad \forall v \in H.$$

By the Riesz theorem, we can view $\Phi'(u)$ as an element of H for any $u \in H$. As usual, we identify $u \in H$ and its continuous representative.

We have the following fact.

Lemma 1.3 *Assume that f satisfies* (f_0), (f_1), (f_2). *Then critical points of functional I restricted to E are 2π-periodic solutions of system* (1.7).

Remark 1.1 It is pointed in [6] that a critical point u of I in H will be a weak solution of (1.7). However, a simple regularity argument shows that $u \in C^1(S^1, R^n)$.

Remark 1.2 As usual, we should deal with (1.10) in the space H. But, according to Lemma 1.3, we only need to treat the functional I in the subspace E of H.

Lemma 1.4 ([9]) *For each $s \in [1, \infty)$, $H^{\frac{1}{2}}(S^1, R^n)$ is compactly embedded in $L^s(S^1, R^n)$. In particular there is $\alpha_s > 0$ such that*

$$\|u\|_{L^s} \leq \alpha_s \|u\|$$

for all $u \in H^{\frac{1}{2}}(S^1, R^n)$.

2 Main results

Theorem 2.1 *Assume that f satisfies* (f_1), (f_2) *and* (f_3). *Then* (1.1) *possesses at least one $4r$-periodic solution.*

Proof Let $e_1, e_2, \ldots, e_n$ denote the usual normal orthogonal bases in R^n and set

$$M = \overline{\operatorname{span}\{e_k \cos(4j-1)t, e_k \sin(4j-1)t : j \in Z^+, k = 1, 2, \ldots, n\}},$$
$$N = \overline{\operatorname{span}\{e_k \cos(4j-3)t, e_k \sin(4j-3)t : j \in Z^+, k = 1, 2, \ldots, n\}},$$

where Z^+ is the set of all positive integers. Then $E = M \oplus N$. For any $u \in E$, it has a Fourier expansion as follows:

$$u(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[a^u_{2j-1} \cos(2j-1)t + b^u_{2j-1} \sin(2j-1)t\right] = x(t) + y(t), \tag{2.1}$$

where all $a^u_{2j-1}, a^u_{4j-1}, a^u_{4j-3}, b^u_{2j-1}, b^u_{4j-1}, b^u_{4j-3} \in R^n$,

$$x(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[a^u_{4j-1} \cos(4j-1)t + b^u_{4j-1} \sin(4j-1)t\right],$$
$$y(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} \left[a^u_{4j-3} \cos(4j-3)t + b^u_{4j-3} \sin(4j-3)t\right].$$

Consequently, we have

$$\frac{1}{2}\|u\|_H^2 \le \langle Lx, x\rangle - \langle Ly, y\rangle \le \|u\|_H^2. \tag{2.2}$$

Let

$$\|u\|^2 = \langle Lx, x\rangle - \langle Ly, y\rangle. \tag{2.3}$$

Then (2.2) and (2.3) show that $\|\cdot\|$ and $\|\cdot\|_H$ are two equivalent norms on E. Henceforth we use the norm $\|\cdot\|$ as the norm for E. And the spaces M, N are mutually orthogonal with respect to the associated inner product.

First, we prove that $I(u)$ satisfies (1.2) in E.

Let $\{u_k\}$ be any sequence which converges to some u weakly in E. By the compactness of the embedding $E \hookrightarrow L^2(S^1, R^n)$, we assume that

$$u_k \to u \quad \text{in } L^2(S^1, R^n),$$
$$u_k \to u \quad \text{a.e. in } S^1.$$

Thus, $(f(u_k), v) \to (f(u), v)$ a.e. for all $v \in E$. Since $f(x) \in C(R^n, R^n)$ satisfies (f_3), there exist positive constants M_1 and M_2 such that

$$\left|(f(u_k), v)\right| \le M_1|v| + M_2|u_k||v|, \quad \forall v \in E. \tag{2.4}$$

Note that the right-hand side of (2.4) converges to $M_1|v| + M_2|u||v|$ in $L^1(S^1, R^n)$. Hence $\{(f(u_k), v)\} \subset L^1(S^1, R)$ is uniformly absolutely continuous. Hence, by Vitali's theorem,

$$\int_0^{2\pi} \left(f(u_k(t)), v(t)\right) dt \to \int_0^{2\pi} \left(f(u(t)), v(t)\right) dt, \quad \forall v \in E. \tag{2.5}$$

Moreover, since L is a bounded self-adjoint linear operator on E,

$$Lu_k \rightharpoonup Lu \quad \text{in } E. \tag{2.6}$$

According to (2.5) and (2.6), we get that I' is weak-to-weak continuous.

Next, we prove

$$I(u) \to +\infty \quad \text{as } u \in M, \|u\| \to \infty \tag{2.7}$$

and

$$I(v) \to -\infty \quad \text{as } v \in N, \|v\| \to \infty. \tag{2.8}$$

Indeed, by (f_3), we know that there exists a positive constant c such that

$$\left|f(x)\right| \le c + \left(\|A\| + \frac{1-\lambda\|A\|}{4\lambda}\right)|x|, \quad \forall x \in R^n. \tag{2.9}$$

Thus, for $u \in M$, by (2.9), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\langle Lu, u\rangle + \lambda \int_0^{2\pi} F\bigl(u(t)\bigr)\,dt \\ &= \frac{1}{2}\|u\|^2 + \lambda \int_0^{2\pi} \left(\int_0^1 \bigl(f\bigl(su(t)\bigr), u(t)\bigr)\,ds\right) dt \\ &\ge \frac{1}{2}\|u\|^2 - \lambda \int_0^{2\pi} \left[\left(\frac{1}{2}\|A\| + \frac{1-\lambda\|A\|}{8\lambda}\right)\bigl|u(t)\bigr|^2 + c\bigl|u(t)\bigr|\right] dt \\ &= \frac{1}{2}\|u\|^2 - \frac{1+3\lambda\|A\|}{8}\|u\|_{L^2}^2 - \lambda c\|u\|_{L^1} \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1+3\lambda\|A\|}{8}\|u\|^2 - c_1\|u\| \\ &= \frac{3-3\lambda\|A\|}{8}\|u\|^2 - c_1\|u\|, \end{aligned} \tag{2.10}$$

where $c_1 > 0$ is a given constant. Since $\frac{3-3\lambda\|A\|}{8} > 0$, (2.10) implies (2.7). The proof of (2.8) is similar. In fact, when $v \in N$, by (2.9), we get that

$$\begin{aligned} I(v) &= \frac{1}{2}\langle Lv, v\rangle + \lambda \int_0^{2\pi} F\bigl(v(t)\bigr)\,dt \\ &= -\frac{1}{2}\|v\|^2 + \lambda \int_0^{2\pi} F\bigl(v(t)\bigr)\,dt \\ &\le -\frac{1}{2}\|v\|^2 + \frac{1+3\lambda\|A\|}{8}\|v\|_{L^2}^2 + \lambda c\|v\|_{L^1} \\ &\le -\left(\frac{3-3\lambda\|A\|}{8}\right)\|v\|^2 + c_1\|v\|. \end{aligned}$$

This implies (2.8).

Note that (2.10) implies

$$b_0 = \inf_M I > -\infty. \tag{2.11}$$

The combination of (2.8) and (2.11) implies that there is $R > 0$ such that (1.3) holds with $A = \partial B_R \cap N$, $B = M$. By Lemma 1.1 we know that there are a sequence $\{u_k\} \subset E$ and a constant c such that

$$I(u_k) \to c, \quad b_0 \le c < \infty, \qquad I'(u_k) \to 0. \tag{2.12}$$

Finally, we show that the sequence $\{u_k\}$ is bounded in E. To do this, assume that $\rho_k = \|u_k\| \to \infty$, and write $\tilde{u}_k = \frac{1}{\rho_k} u_k$. Then $\|\tilde{u}_k\| = 1$. From Lemma 1.4, there is a renamed subsequence such that

$$\begin{aligned} &\tilde{u}_k \rightharpoonup \tilde{u} \quad \text{in } E, \\ &\tilde{u}_k \to \tilde{u} \quad \text{in } L^2(S^1, R^n). \end{aligned}$$

By (f_3), for any $\epsilon > 0$, there exists a constant $\bar{r} > 0$ such that

$$\left|f(x) - Ax\right| < \epsilon |x| \quad \text{for all } |x| > \bar{r}. \tag{2.13}$$

Moreover, by the continuity of f, there is a constant $c_2 > 0$ such that

$$\left|f(x)\right| \le c_2 \quad \text{if } |x| \le \bar{r}. \tag{2.14}$$

By (2.13) and (2.14), for any $v \in E$, we have

$$\begin{aligned} &\left| \frac{1}{\rho_k} \int_0^{2\pi} \left(f\left(u_k(t)\right), v(t)\right) dt - \int_0^{2\pi} \left(A\tilde{u}(t), v(t)\right) dt \right| \\ &\quad = \left| \frac{1}{\rho_k} \int_0^{2\pi} \left(f\left(u_k(t)\right) - Au_k(t), v(t)\right) dt + \int_0^{2\pi} \left(A\tilde{u}_k(t) - A\tilde{u}(t), v(t)\right) dt \right| \\ &\quad \le \frac{1}{\rho_k} \left[\int_{|u_k| > \bar{r}} \epsilon \left|u_k(t)\right| \left|v(t)\right| dt + \int_{|u_k| \le \bar{r}} \left(c_2 + \|A\| \bar{r}\right) \left|v(t)\right| dt \right] \\ &\qquad + \int_0^{2\pi} \|A\| \left|\tilde{u}_k(t) - \tilde{u}(t)\right| \left|v(t)\right| dt \\ &\quad \le \epsilon \int_0^{2\pi} \left|\tilde{u}_k(t)\right| \left|v(t)\right| dt + \frac{1}{\rho_k} \int_0^{2\pi} \left(c_2 + \|A\| \bar{r}\right) \left|v(t)\right| dt \\ &\qquad + \int_0^{2\pi} \|A\| \left|\tilde{u}_k(t) - \tilde{u}(t)\right| \left|v(t)\right| dt \\ &\quad \to 0 \end{aligned} \tag{2.15}$$

as $\epsilon \to 0$ and $k \to \infty$. This shows

$$\frac{1}{\rho_k} \int_0^{2\pi} \left(f\left(u_k(t)\right), v(t)\right) dt \to \int_0^{2\pi} \left(A\tilde{u}(t), v(t)\right) dt. \tag{2.16}$$

Hence

$$\begin{aligned}\frac{\langle I'(u_k),v\rangle}{\rho_k} &= \langle L\tilde{u}_k,v\rangle + \frac{\lambda}{\rho_k}\int_0^{2\pi}\bigl(f\bigl(u_k(t)\bigr),v(t)\bigr)\,dt \\ &\to \langle L\tilde{u},v\rangle + \lambda\int_0^{2\pi}\bigl(A\tilde{u}(t),v(t)\bigr)\,dt. \end{aligned}\tag{2.17}$$

By (2.12) and (2.17), we see that

$$\langle L\tilde{u},v\rangle + \lambda\int_0^{2\pi}\bigl(A\tilde{u}(t),v(t)\bigr)\,dt = 0$$

for all $v \in E$, *i.e.*,

$$\int_0^{2\pi}\left(\dot{\tilde{u}}\left(t+\frac{\pi}{2}\right) + \lambda A\tilde{u}(t),v(t)\right)dt = 0 \tag{2.18}$$

for all $v \in E$.

Set

$$\tilde{u}(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}\bigl[a_{2j-1}^{\tilde{u}}\cos(2j-1)t + b_{2j-1}^{\tilde{u}}\sin(2j-1)t\bigr],$$

$$v(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}\bigl[a_{2j-1}^{v}\cos(2j-1)t + b_{2j-1}^{v}\sin(2j-1)t\bigr].$$

Then, by (2.18), one can obtain

$$\sum_{j=1}^{+\infty}\bigl[\bigl(\bigl(\lambda A + (-1)^j(2j-1)I\bigr)a_{2j-1}^{\tilde{u}},a_{2j-1}^{v}\bigr) + \bigl(\bigl(\lambda A + (-1)^j(2j-1)I\bigr)b_{2j-1}^{\tilde{u}},b_{2j-1}^{v}\bigr)\bigr] = 0,$$

where I is the $n \times n$ unit matrix. For any j, take $v(t) = \frac{1}{\sqrt{\pi}}e_i\cos(2j-1)t$ and $v(t) = \frac{1}{\sqrt{\pi}}e_i\sin(2j-1)t$, where $i = 1,2,\dots,n$. An easy computation shows that

$$\bigl(\lambda A + (-1)^j(2j-1)I\bigr)a_{2j-1}^{\tilde{u}} = 0 \tag{2.19}$$

and

$$\bigl(\lambda A + (-1)^j(2j-1)I\bigr)b_{2j-1}^{\tilde{u}} = 0. \tag{2.20}$$

Hence, by $\lambda^{-1}(-1)^{j+1}(2j-1) \notin \sigma(A)$, we get that $\tilde{u} \equiv 0$.

Let $\tilde{u}_k = \tilde{x}_k + \tilde{y}_k$, where $\tilde{x}_k \in M$, $\tilde{y}_k \in N$. A proof similar to (2.16) shows that

$$\frac{1}{\rho_k}\int_0^{2\pi}\bigl(f\bigl(u_k(t)\bigr),\tilde{x}_k(t)\bigr)\,dt \to \int_0^{2\pi}\bigl(A\tilde{u}(t),\tilde{x}(t)\bigr)\,dt = 0 \tag{2.21}$$

and

$$\frac{1}{\rho_k}\int_0^{2\pi}\bigl(f\bigl(u_k(t)\bigr),\tilde{y}_k(t)\bigr)\,dt \to \int_0^{2\pi}\bigl(A\tilde{u}(t),\tilde{y}(t)\bigr)\,dt = 0, \tag{2.22}$$

where $\tilde{u} = \tilde{x} + \tilde{y}$, $\tilde{x} \in M$, $\tilde{y} \in N$. Thus

$$\begin{aligned}
&\frac{\langle I'(u_k), \tilde{x}_k\rangle}{\rho_k} - \frac{\langle I'(u_k), \tilde{y}_k\rangle}{\rho_k} \\
&\quad = \|\tilde{x}_k\|^2 + \|\tilde{y}_k\|^2 + \frac{\lambda}{\rho_k}\int_0^{2\pi} \bigl(f\bigl(u_k(t)\bigr), \tilde{x}_k(t)\bigr)\,dt - \frac{\lambda}{\rho_k}\int_0^{2\pi} \bigl(f\bigl(u_k(t)\bigr), \tilde{y}_k(t)\bigr)\,dt \\
&\quad = 1 + \frac{\lambda}{\rho_k}\int_0^{2\pi} \bigl(f\bigl(u_k(t)\bigr), \tilde{x}_k(t)\bigr)\,dt - \frac{\lambda}{\rho_k}\int_0^{2\pi} \bigl(f\bigl(u_k(t)\bigr), \tilde{y}_k(t)\bigr)\,dt \\
&\quad \to 1.
\end{aligned} \tag{2.23}$$

On the other hand, (2.12) implies

$$\frac{\langle I'(u_k), \tilde{x}_k\rangle}{\rho_k} - \frac{\langle I'(u_k), \tilde{y}_k\rangle}{\rho_k} \to 0,$$

which contradicts (2.23). Thus ρ_k must be bounded. Consequently, there is a renamed subsequence of $\{u_k\}$ such that $u_k \rightharpoonup u$ in E. Hence, by the weak-to-weak continuity of I', we have

$$\bigl\langle I'(u_k), v\bigr\rangle \to \bigl\langle I'(u), v\bigr\rangle, \quad \forall v \in E. \tag{2.24}$$

Now, the combination of (2.12) and (2.24) implies that $I'(u) = 0$. This completes the proof. □

Remark 2.1 Let $n = 1$, $r = \frac{\pi}{3}$ and $f(x) = x$. Then $f(x)$ satisfies all the conditions of Theorem 2.1.

In order to give our another result, we still need the following preliminaries.

Let P_M, P_N be the projectors of E onto M, N associated with the given splitting of E. Set

$$\begin{aligned}
\mathcal{H} = \bigl\{&\Psi \in C\bigl([0,1] \times E, E\bigr) : \Psi(0,u) = u \text{ and} \\
&P_N\Psi(t,u) = P_N u - K(t,u), \text{where } K : [0,1] \times E \to N \text{ is compact}\bigr\}.
\end{aligned}$$

Recall that K is continuous and maps bounded sets to relatively compact sets since K is compact. Let $S, Q \subset E$ with $Q \subset \widetilde{E}$, a given subspace of E. Then ∂Q will refer to the boundary of Q in $\widetilde{E}$.

Definition 2.1 We say S and ∂Q link if whenever $\Psi \in \mathcal{H}$ and $\Psi(t, \partial Q) \cap S = \emptyset$ for all $t \in [0,1]$, then $\Psi(t, Q) \cap S \neq \emptyset$ for all $t \in [0,1]$.

Lemma 2.1 ([9]) *Let $\rho > 0$, $S \equiv \partial B_\rho \cap M$, $e \in M \cap \partial B_1$, $r_1 > \rho$, $r_2 > 0$, $Q \equiv \{re : r \in (0, r_1)\} \oplus (B_{r_2} \cap N)$, and $\widetilde{E} \equiv \operatorname{span}\{e\} \oplus N$. Then S and ∂Q link.*

Lemma 2.2 ([9]) *Suppose $I \in C^1(E, R)$ satisfies the (PS) condition and*

(I_1) *$I(u) = \frac{1}{2}\langle Lu, u\rangle + \Phi(u)$, where $Lu = L_1P_Mu + L_2P_Nu$ and $L_1 : M \to M$, $L_2 : N \to N$ are bounded self-adjoint,*

(I_2) *Φ' is compact,*

(I_3) *there exist a subspace $\widetilde{E} \subset E$ and sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\alpha > \beta$ such that*

(i) *$S \subset M$ and $I|_S \geq \alpha$,*

(ii) *Q is bounded and $I|_{\partial Q} \leq \beta$,*

(iii) *S and ∂Q link.*

Then I possesses a critical value $c \geq \alpha$.

The following is our another main result.

Theorem 2.2 *Assume that f satisfies* (f_1), (f_2) *and the following conditions*:

(f_4) *$F(x) \leq 0$ for all $x \in R^n$,*

(f_5) *$f(x) = o(|x|)$ as $|x| \to 0$,*

(f_6) *there exist constants $c > 0$, $p > 2$ and $\tilde{r} > 0$ such that*

$$\left|f(x)\right| \leq c\left(1 + |x|^{p-1}\right), \quad \forall x \in R^n$$

and

$$\left(f(x), x\right) \leq pF(x) < 0, \quad \forall |x| \geq \tilde{r}.$$

Then (1.1) *possesses at least one nonconstant $4r$-periodic solution.*

Proof We will show that I satisfies the hypotheses of Lemma 2.2. This will lead to a nonconstant $4r$-periodic solution of (1.1). We divide the proof of Theorem 2.2 into the following three parts.

First, we prove that I satisfies (I_1) and (I_2) of Lemma 2.2.

Note that $L(M) \subset M$, $L(N) \subset N$ and L is bounded self-adjoint on E. We see that I satisfies (I_1) of Lemma 2.2 with $L_1 = L|_M$, $L_2 = L|_N$ and

$$\Phi(u) = \lambda \int_0^{2\pi} F\left(u(t)\right) dt.$$

By Proposition B.37 in [9], (f_6) implies that Φ' is compact. Hence (I_2) holds.

Next, we show that I satisfies (I_3) of Lemma 2.2.

By (f_5), for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\left|F(x)\right| \leq \epsilon |x|^2$$

whenever $|x| \leq \delta$. By (f_6), there is a constant $c = c(\epsilon)$ such that

$$\left|F(x)\right| \leq c|x|^p \quad \text{for } |x| \geq \delta.$$

Hence

$$\left|F(x)\right| \leq \epsilon |x|^2 + c|x|^p, \quad \forall x \in R^n. \tag{2.25}$$

By (2.25) and Lemma 1.4, for any $u \in M$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\langle Lu, u\rangle + \Phi(u) \\ &= \frac{1}{2}\langle Lu, u\rangle + \lambda \int_0^{2\pi} F\big(u(t)\big)\,dt \\ &= \frac{1}{2}\|u\|^2 + \lambda \int_0^{2\pi} F\big(u(t)\big)\,dt \\ &\ge \frac{1}{2}\|u\|^2 - \lambda \int_0^{2\pi} \epsilon\big|u(t)\big|^2 + c\big|u(t)\big|^p\,dt \\ &\ge \frac{1}{2}\|u\|^2 - \lambda\epsilon\alpha_2^2\|u\|^2 - \lambda c\alpha_p^p\|u\|^p. \end{aligned} \tag{2.26}$$

Choose $\epsilon = (4\lambda\alpha_2^2)^{-1}$. Since $p > 2$, there is small $\rho > 0$ such that $\frac{1}{8}\rho^2 \ge \lambda c\alpha_p^p\rho^p$. Then, for $u \in \partial B_\rho \cap M$, (2.26) implies that $I(u) \ge \frac{1}{8}\rho^2 := \alpha > 0$. Consequently, I satisfies (I_3)(i) with $S = \partial B_\rho \cap M$.

Set $e \in \partial B_1 \cap M$ and

$$Q = \{re : 0 < r < r_1\} \oplus (B_{r_2} \cap N), \tag{2.27}$$

where $r_1 > \rho$ and $r_2 > 0$ are free constants for the moment. Define $\widetilde{E} = \operatorname{span}\{e\} \oplus N$. Then $Q \subset \widetilde{E}$ and S and ∂Q link by Lemma 2.1.

By (f_6), there are constants $c_1, c_2 > 0$ such that

$$F(x) \le -c_1|x|^p + c_2 \tag{2.28}$$

for all $x \in R^n$. Thus, for $v \in B_{r_2} \cap N$, by the Hölder inequality (note that $p > 2$) and orthogonality, we get that

$$\begin{aligned} I(re + v) &= \frac{1}{2}r^2 - \frac{1}{2}\|v\|^2 + \lambda \int_0^{2\pi} F\big(re(t) + v(t)\big)\,dt \\ &\le r^2 - \frac{1}{2}\|v\|^2 - \lambda \int_0^{2\pi} c_1\big|re(t) + v(t)\big|^p\,dt + 2\lambda\pi c_2 \\ &\le r^2 - \frac{1}{2}\|v\|^2 - c_3\left(\int_0^{2\pi} \big|re(t) + v(t)\big|^2\,dt\right)^{\frac{p}{2}} + 2\lambda\pi c_2 \\ &= r^2 - \frac{1}{2}\|v\|^2 - c_3\left(\int_0^{2\pi} \big[r^2\big|e(t)\big|^2 + \big|v(t)\big|^2\big]\,dt\right)^{\frac{p}{2}} + 2\lambda\pi c_2 \\ &\le r^2 - \frac{1}{2}\|v\|^2 - c_4 r^p + 2\lambda\pi c_2, \end{aligned} \tag{2.29}$$

where $c_3, c_4 > 0$ are constants. Now, choose large $r_1 > \rho$ and $r_2 > r_1$ such that

$$r^2 - c_4 r^p + 2\lambda\pi c_2 \le 0, \quad \forall r \ge r_1 \tag{2.30}$$

and

$$r_1^2 - \frac{1}{2}r_2^2 + 2\lambda\pi c_2 \le 0. \tag{2.31}$$

By (2.27), (2.29), (2.30), (2.31) and (f_4), one can easily check that $I|_{\partial Q} \leq 0 := \beta$. Hence I satisfies (I_3)(ii).

To sum up, I satisfies (I_3) of Lemma 2.2.

Finally, we check that I satisfies the (PS) condition. Let $\{u_k\} \subset E$ be a sequence such that $|I(u_k)| \leq c_0$ and $I'(u_k) \to 0$ as $k \to \infty$. Then, for large k, by (f_6) and (2.28), we have

$$\begin{aligned} 2c_0 + \|u_k\| &\geq 2I(u_k) - \langle I'(u_k), u_k \rangle \\ &= \lambda \int_0^{2\pi} \left[2F(u_k(t)) - \left(f(u_k(t)), u_k(t)\right)\right] dt \\ &= \lambda \int_0^{2\pi} (2-p)F(u_k(t))\, dt + \lambda \int_0^{2\pi} \left[pF(u_k(t)) - \left(f(u_k(t)), u_k(t)\right)\right] dt \\ &\geq \lambda \int_0^{2\pi} (2-p)F(u_k(t))\, dt - c_5 \\ &\geq \lambda(p-2) \int_0^{2\pi} \left(c_1 |u_k(t)|^p - c_2\right) dt - c_5 \\ &\geq c_6 \|u_k\|_{L^p}^p - c_7, \end{aligned} \tag{2.32}$$

where $c_5, c_6, c_7 > 0$ are constants.

Let $u_k = x_k + y_k$, where $x_k \in M$, $y_k \in N$. Then, for large k, by (f_6), Lemma 1.4 and the Hölder inequality, we get that

$$\begin{aligned} \|x_k\| &\geq \left|\langle I'(u_k), x_k \rangle\right| \\ &= \left| \|x_k\|^2 + \lambda \int_0^{2\pi} \left(f(u_k(t)), x_k(t)\right) dt \right| \\ &\geq \|x_k\|^2 - \lambda \int_0^{2\pi} \left|f(u_k(t))\right| \left|x_k(t)\right| dt \\ &\geq \|x_k\|^2 - \lambda \int_0^{2\pi} c\left(1 + |u_k(t)|^{p-1}\right) |x_k(t)|\, dt \\ &\geq \|x_k\|^2 - \lambda c \|x_k\|_{L^1} - \lambda c \|u_k\|_{L^p}^{p-1} \|x_k\|_{L^p} \\ &\geq \|x_k\|^2 - \lambda c \alpha_1 \|x_k\| - \lambda c \alpha_p \|u_k\|_{L^p}^{p-1} \|x_k\|. \end{aligned} \tag{2.33}$$

This implies that

$$\|x_k\| \leq 1 + \lambda c \alpha_1 + \lambda c \alpha_p \|u_k\|_{L^p}^{p-1}. \tag{2.34}$$

Similarly, one can easily see that

$$\|y_k\| \leq 1 + \lambda c \alpha_1 + \lambda c \alpha_p \|u_k\|_{L^p}^{p-1}. \tag{2.35}$$

By (2.32), (2.34) and (2.35), there is a constant $c_8 > 0$ such that

$$\begin{aligned} \|u_k\| &\leq \|x_k\| + \|y_k\| \\ &\leq 2\left(1 + \lambda c \alpha_1 + \lambda c \alpha_p \|u_k\|_{L^p}^{p-1}\right) \end{aligned}$$

$$\leq 2\left[1 + \lambda c\alpha_1 + \lambda c\alpha_p c_6^{\frac{1-p}{p}} \left(2c_0 + c_7 + \|u_k\|\right)^{\frac{p-1}{p}}\right]$$
$$\leq c_8\left(1 + \|u_k\|^{\frac{p-1}{p}}\right),$$

which implies that $\{u_k\}$ is bounded in E.

By the compactness of Φ', going if necessary to a subsequence, we can assume that

$$u_k \rightharpoonup u \quad \text{in } E$$

and

$$\Phi'(u_k) \to \Phi'(u) \quad \text{in } E.$$

Let $u = x + y$, where $x \in M$ and $y \in N$. Then

$$\|x_k - x\|^2 = \langle I'(u_k) - I'(u), x_k - x\rangle - \langle \Phi'(u_k) - \Phi'(u), x_k - x\rangle \to 0$$

as $k \to \infty$. Similarly, we have $\|y_k - y\|^2 \to 0$ as $k \to \infty$. Hence $u_k \to u$ in E. Hence I satisfies the (PS) condition.

Therefore, Theorem 2.2 follows from Lemma 2.2. □

Remark 2.2 Let $n = 1$ and $f(x) = -x^{\frac{5}{3}}$. Then $f(x)$ satisfies all the conditions of Theorem 2.2 with $p = \frac{8}{3}$ and $\tilde{r} > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Browder, F: A further generalization of the Schauder fixed-point theorem. Duke Math. J. **32**, 575-578 (1965)
2. Grafton, RB: A periodicity theorem for autonomous functional differential equations. J. Differ. Equ. **6**, 87-109 (1969)
3. Jones, GS: The existence of periodic solutions of $f'(x) = -\alpha f(x-1)[1 + f(x)]$. J. Math. Anal. Appl. **5**, 435-450 (1962)
4. Nussbaum, RD: Periodic solutions of some nonlinear autonomous functional differential equations, II. J. Differ. Equ. **14**, 368-394 (1973)
5. Nussbaum, RD: Periodic solutions of some nonlinear autonomous functional differential equations. Ann. Mat. Pura Appl. **10**, 263-306 (1974)
6. Guo, ZM, Yu, JS: Multiplicity results for periodic solutions to delay differential equations via critical point theory. J. Differ. Equ. **218**, 15-35 (2005)
7. Wu, K, Wu, X: Multiplicity results of periodic solutions for a class of first order delay differential equations. J. Math. Anal. Appl. **390**, 427-438 (2012)
8. Schechter, M, Zou, WM: Weak linking. Nonlinear Anal. **55**, 695-706 (2003)
9. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 65. Am. Math. Soc., Providence (1986)

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