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Mild solutions to fractional differential inclusions with nonlocal conditions

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Abstract

This article is concerned with the existence of mild solutions for fractional differential inclusions with nonlocal conditions in Banach spaces. The results are obtained by using fractional calculus, Hausdorff measure of noncompactness, and the multivalued fixed point theorem. The results obtained in the present paper extend some related results on this topic.

Keywords: fractional differential inclusions; nonlocal conditions; Hausdorff measure of noncompactness; equicontinuous semigroup; mild solution

1 Introduction

Fractional differential equations and inclusions have gained considerable interest due to their applications in various fields, such as physics, mechanics, and engineering, in part because they have been found to be more realistic and practical to describe many natural phenomena [1–8]. For more details about fractional calculus and fractional differential equations, we refer the reader to the books by Podlubny [9], Sabatier et al. [10], Kilbas et al. [11], and the papers by Eidelman and Kochubei [12], Lakshmikantham and Vatsala [13], and Agarwal et al. [14].

The study of abstract nonlocal differential problems was initiated by Byszewski and Lakshmikantham [15], who gave three theorems on the existence and uniqueness of the mild, strong, and classical solutions of a semilinear evolution nonlocal Cauchy problem by using the method of semigroups and the Banach fixed point theorem and argued there that a nonlocal condition can be applied in physics with better effects than classical initial conditions. This work was then followed by a lot of other research, and some basic results on nonlocal differential problems have been obtained [16–18]. Most results, however, were obtained with the assumption that the involved semigroup is compact, and one of the difficulties in these nonlocal problems is how to deal with the compactness of the solution operator at $t = 0$. Many methods and techniques have been developed to avoid this difficulty. We refer the reader to papers [19–21] and the references therein.

From the mathematical point of view, it is natural to combine fractional differential equations with nonlocal conditions. For example, Zhou and Jiao [22] discussed the nonlocal fractional evolution equations based on the Krasnoselskii fixed point theorem with the assumption that the involved semigroup is compact and the nonlocal term is Lipschitz continuous. Li et al. [23] studied the existence of mild solutions to fractional differential equations by using the Hausdorff measure of noncompactness when the semigroup

is equicontinuous and the nonlocal term is compact. Ji and Li [24] also studied nonlocal fractional differential equations in general Banach spaces but without any compactness assumptions to the operator semigroup. Recently, Ji [25] studied the control system governed by a class of abstract nonlocal fractional differential equations. To the best of the authors' knowledge, however, few work has been reported on the existence of solutions for fractional differential inclusions with nonlocal conditions governed by a linear closed operator that generates an equicontinuous semigroup.

In this paper, we assume that X is a real Banach space with norm $\|\cdot\|$. Let $J = [0, b]$ with $b > 0$. We denote by $C(J, X)$ the Banach space of continuous functions from J into X with the norm $\|x\| = \sup_{t \in J} \{\|x(t)\|\}$ for $x \in C(J, X)$. We further denote by $L^1(J, X)$ the space of Bochner-integrable functions from J into X with the norm given by $\|f\|_{L^1} = \int_0^b \|f(t)\| dt$ for $f \in L^1(J, X)$.

Motivated by the aforementioned previous research, we study the existence of mild solutions to the following fractional semilinear *nonlocal* differential inclusion:

$${}^C D_t^q u(t) \in Au(t) + F(t, u(t)), \quad t \in J, \quad (1.1)$$

$$u(0) = g(u), \quad (1.2)$$

where $0 < q \leq 1$, ${}^C D_t^q$ is the Caputo fractional differential operator of order q , $u \in C(J, X)$, $A : D(A) \rightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on X with $D(A)$ representing the domain of the linear operator (usually unbounded) A , F is an upper-Carathéodory multifunction, and the nonlocal term g is a given function from $C(J, X)$ into X .

Using the technique of Hausdorff measure of noncompactness and the multivalued fixed point theorem, we prove some existence results on the fractional semilinear nonlocal differential inclusion (1.1)-(1.2). We assume that the semigroup $\{T(t)\}_{t \geq 0}$ is equicontinuous. Note that the case of compact $\{T(t)\}_{t \geq 0}$ is just a particular case of our assumption. Therefore, the results in the present paper extend to some extent those in [23, 26].

This paper is organized as follows. In Section 2, we present some relevant definitions and facts about fractional derivative and integral, the Hausdorff measure of noncompactness, and the set-valued analysis. In Section 3, we give the existence results of mild solutions for problem (1.1)-(1.2). In Section 4, an example is given to briefly show a potential application of our results.

2 Preliminaries

Let \mathbb{N} , \mathbb{R} , and \mathbb{R}^+ be the sets of positive integers, real numbers, and positive real numbers, respectively. Also, we define the following sets of subsets of X :

$$P(X) = \{B \subseteq X : B \text{ is nonempty and bounded}\},$$

$$P_{cl}(X) = \{B \subseteq X : B \text{ is nonempty and closed}\},$$

$$P_{cp}(X) = \{B \subseteq X : B \text{ is nonempty and compact}\},$$

$$P_{cl,cv}(X) = \{B \subseteq X : B \text{ is nonempty, closed, and convex}\}, \text{ and}$$

$$P_{cp,cv}(X) = \{B \subseteq X : B \text{ is nonempty, compact, and convex}\}.$$

Now let us recall the definitions of fractional derivative and integral.

Definition 1 The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \, ds, \quad t > 0, \quad (2.1)$$

provided that the right-hand side of (2.1) is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2 The Caputo fractional derivative of order $q > 0$ with the lower limit zero for a function f is defined as

$${}^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) \, ds, \quad t > 0, \quad (2.2)$$

where $n \in \mathbb{N}$ is such that $q \in (n-1, n)$.

If f is an abstract function with values in X , then the integrals in the Definitions 1 and 2 are taken in the Bochner sense. In particular, when $0 < q < 1$, we have

$${}^C D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} \, ds, \quad t > 0.$$

Now using the probability density function and its Laplace transform developed in [27], we give the following definition of mild solutions to problem (1.1)-(1.2).

Definition 3 A function $u \in C(J, X)$ is said to be a mild solution of (1.1)-(1.2) if $u(0) = g(u)$ and there exists $f \in L^1(J, X)$ such that $f(t) \in F(t, u(t))$ and

$$u(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds \quad (2.3)$$

for $t \in J$, where

$$\mathcal{T}_q(t) = \int_0^\infty h_q(\theta) T(t^q \theta) \, d\theta, \quad (2.4)$$

$$\mathcal{S}_q(t) = q \int_0^\infty \theta h_q(\theta) T(t^q \theta) \, d\theta. \quad (2.5)$$

Here h_q is the probability density function on $(0, \infty)$ given by

$$h_q(\theta) = \frac{1}{q} \theta^{-1-1/q} \overline{\omega}_q(\theta^{-1/q}), \quad (2.6)$$

where

$$\overline{\omega}_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(\pi nq), \quad \theta \in (0, \infty). \quad (2.7)$$

Note that $h_q(\theta) \geq 0$ for $\theta \in (0, \infty)$ and it satisfies

$$\int_0^\infty h_q(\theta) d\theta = 1, \quad (2.8)$$

$$\int_0^\infty \theta^\nu h_q(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+q\nu)}, \quad \nu \in [0, 1]. \quad (2.9)$$

It should be pointed out that Definition 3 can be regarded as a generalization of many previous definitions of mild solutions for various differential equations [22–26, 28–31], in particular, the definitions used by Zhou and Jiao [22], Li et al. [23], and Ji and Li [24] for the following *nonlocal* fractional semilinear differential equation:

$$\begin{aligned} {}^C D_t^q u(t) &= Au(t) + f(t, u(t)), \quad t \in J, \\ u(0) &= g(u), \end{aligned}$$

and the definition introduced by Wang and Zhou [26] for the following *local* fractional semilinear differential inclusion:

$$\begin{aligned} {}^C D_t^q u(t) &\in Au(t) + F(t, u(t)), \quad t \in J, \\ u(0) &= u_0. \end{aligned}$$

Lemma 1 *The linear operators $\mathcal{T}_q(t)$ and $\mathcal{S}_q(t)$ have the following properties:*

- (1) *For any fixed $t \geq 0$, both $\mathcal{T}_q(t)$ and $\mathcal{S}_q(t)$ are bounded operators, that is, for any $x \in X$, we have [24]*

$$\|\mathcal{T}_q(t)x\| \leq M\|x\|, \quad (2.10)$$

$$\|\mathcal{S}_q(t)x\| \leq \frac{Mq}{\Gamma(1+q)}\|x\|, \quad (2.11)$$

where $M = \sup_{t \in J} \|T(t)\|$.

- (2) *Both $\mathcal{T}_q(t)$ and $\mathcal{S}_q(t)$ are equicontinuous for $t \in J$ if $\{T(t)\}_{t \geq 0}$ is equicontinuous.*

Proof Property (1) follows directly from the definitions of $\mathcal{T}_q(t)$ and $\mathcal{S}_q(t)$, the definition of the constant M , and properties (2.8) and (2.9) of the probability density function. To show property (2), we shall show that both $\{\mathcal{T}_q(t)x : x \in B\}$ and $\{\mathcal{S}_q(t)x : x \in B\}$ are equicontinuous for any bounded subset B of X . To this end, let $0 \leq t < t+h \leq b$, and let $\varepsilon > 0$ be small enough. We then have

$$\begin{aligned} \|\mathcal{T}_q(t+h)x - \mathcal{T}_q(t)x\| &= \left\| \int_0^\infty h_q(\theta) T((t+h)^q \theta) x d\theta - \int_0^\infty h_q(\theta) T(t^q \theta) x d\theta \right\| \\ &\leq \int_0^\infty h_q(\theta) \|T((t+h)^q \theta) x - T(t^q \theta) x\| d\theta \\ &\leq \int_0^\varepsilon h_q(\theta) \|T((t+h)^q \theta) x - T(t^q \theta) x\| d\theta \\ &\quad + \int_\varepsilon^\infty h_q(\theta) \|T((t+h)^q \theta) x - T(t^q \theta) x\| d\theta \end{aligned}$$

$$\leq 2M\|x\| \int_0^\varepsilon h_q(\theta) d\theta + \int_\varepsilon^\infty h_q(\theta) \|T((t+h)^q\theta)x - T(t^q\theta)x\| d\theta.$$

By using (2.8) and the absolute continuity of the integral it is easy to see that the term $2M\|x\| \int_0^\varepsilon h_q(\theta) d\theta$ can be made arbitrarily small by choosing ε sufficiently small. On the other hand, it follows from the Lebesgue dominated convergence theorem and the equicontinuity of $T(t)$ for $t > 0$ that, as $h \rightarrow 0$, we have

$$\int_\varepsilon^\infty h_q(\theta) \|T((t+h)^q\theta)x - T(t^q\theta)x\| d\theta \rightarrow 0.$$

Therefore, $\|\mathcal{T}_q(t+h)x - \mathcal{T}_q(t)x\| \rightarrow 0$ as $h \rightarrow 0$. So $\mathcal{T}_q(t)$ is equicontinuous.

In the same way we can prove that $\|\mathcal{S}_q(t+h)x - \mathcal{S}_q(t)x\| \rightarrow 0$ as $h \rightarrow 0$, uniformly for all $x \in B$, and thus $\mathcal{S}_q(t)$ is also equicontinuous. \square

We recall that for a bounded subset B of the Banach space X , its Hausdorff measure of noncompactness is defined as

$$\beta(B) = \inf\{\varepsilon > 0 : B \text{ has a finite } \varepsilon\text{-net in } X\}. \quad (2.12)$$

Some basic properties of the Hausdorff measure are presented in the following lemmas.

Lemma 2 ([32]) *Let X be a real Banach space. Let $B \subseteq X$ and $C \subseteq X$ be bounded. Then*

- (1) B is relatively compact if and only if $\beta(B) = 0$;
- (2) $\beta(B) = \beta(\overline{B}) = \beta(\text{conv } B)$, where \overline{B} and $\text{conv } B$ represent the closure and the convex hull of B , respectively;
- (3) $\beta(B) \leq \beta(C)$ when $B \subseteq C$;
- (4) $\beta(B + C) \leq \beta(B) + \beta(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;
- (5) $\beta(B \cup C) \leq \max\{\beta(B), \beta(C)\}$;
- (6) $\beta(\lambda(B)) \leq |\lambda|\beta(B)$ for any $\lambda \in \mathbb{R}$;
- (7) If the map $Q : D(Q) \subseteq X \rightarrow Z$ is Lipschitz continuous with constant k , then $\beta_Z(QB) \leq k\beta(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space. and $\beta_Z(\cdot)$ is the Hausdorff measure of noncompactness associated with Z ;
- (8) If $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of bounded and closed nonempty subsets of X and $\lim_{n \rightarrow \infty} \beta(W_n) = 0$, then $\bigcap_{n=1}^\infty W_n$ is nonempty and compact in X .

Lemma 3 ([32]) *If $W \subseteq C(J, X)$ is bounded and equicontinuous, then $\beta(W(t))$ is continuous on J , and*

$$\beta(W) = \sup_{t \in J} \beta(W(t)). \quad (2.13)$$

Lemma 4 ([33]) *If $\{u_n\}_{n=1}^\infty \subset L^1(J, X)$ is uniformly integrable, then $\beta(\{u_n(t)\}_{n=1}^\infty)$ is measurable, and*

$$\beta\left(\left\{\int_0^t u_n(s) ds\right\}_{n=1}^\infty\right) \leq 2 \int_0^t \beta(\{u_n(s)\}_{n=1}^\infty) ds. \quad (2.14)$$

Lemma 5 ([34]) *If $B \subseteq X$ is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty$ in B such that*

$$\beta(B) \leq 2\beta(\{u_n\}_{n=1}^\infty) + \varepsilon. \quad (2.15)$$

In addition, for completeness, we further include some basic definitions and results on multivalued maps. For more details about the multivalued maps, see the books by Deimling [35] and Hu and Papageorgiou [36].

Definition 4 Let X and Y be two topological spaces.

- (1) A multivalued map $F : X \rightarrow P(Y)$ is said to be convex (closed) valued if $F(x)$ is convex (closed) in Y for all $x \in X$. Recall that $P(Y)$ represents the set of all nonempty and bounded subsets of Y .
- (2) F is said to be completely continuous if $F(B)$ is relatively compact for every bounded subset B of X .
- (3) F is said to have a fixed point if there is $x \in X$ such that $x \in F(x)$.
- (4) F is said to be upper semicontinuous (u.s.c.) on X if $F^{-1}(V) = \{x \in X : F(x) \subseteq V\}$ is an open subset of X for every open subset V of Y .
- (5) F is said to be closed if its graph $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of the topological space $X \times Y$, that is, $x_n \rightarrow x$, $y_n \rightarrow y$, and $y_n \in F(x_n)$ imply $y \in F(x)$.

Remark 1 Note that if $D \subset X$ is closed, $F(x)$ is closed for all $x \in D$, and $\overline{F(D)}$ is compact, then F is u.s.c. if and only if F is closed.

Definition 5 A sequence $\{f_n\}_{n=1}^\infty \subset L^1(J, X)$ is said to be semicompact if

- (1) It is integrably bounded, that is, there is $\omega \in L^1(J, \mathbb{R}^+)$ such that

$$\|f_n(t)\| \leq \omega(t) \quad \text{for a.e. } t \in J.$$

- (2) The set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in X for a.e. $t \in J$.

Lemma 6 ([37]) *Every semicompact sequence in $L^1(J, X)$ is weakly compact in the space $L^1(J, X)$.*

Lemma 7 ([26]) *For $\theta \in (0, 1]$ and $0 < a \leq b$, we have*

$$|a^\theta - b^\theta| \leq (b - a)^\theta.$$

3 Main results

Using the Hausdorff measure of noncompactness and the multivalued fixed point theorem, we shall give existence results for the nonlocal initial value problem (1.1)-(1.2). First, we list the hypotheses of our first main result.

- (HA) The C_0 semigroup $\{T(t)\}_{t \geq 0}$ generated by the linear operator A is equicontinuous. We denote

$$M = \sup_{t \in J} \{\|T(t)\|\}.$$

(Hg) The nonlocal term $g : C(J, X) \rightarrow X$ is continuous and compact, and there exists a constant $N > 0$ such that $\|g(u)\| \leq N$ for all $u \in C(J, X)$.

(HF) The multivalued operator $F : J \times X \rightarrow P_{cp,cv}(X)$ satisfies the following hypotheses:

- (1) F is measurable to t for every $x \in X$ and u.s.c. to x for a.e. $t \in J$. For every $u \in C(J, X)$, the set

$$S_F(u) = \{f \in L^1(J, X) : f(t) \in F(t, u(t)), \text{ a.e. } t \in J\}$$

is nonempty.

- (2) There exists a function $m \in L^{1/q_1}(J, \mathbb{R}^+)$ with $q_1 \in (0, q)$ such that, for any $x \in X$,

$$\|F(t, x)\| = \sup\{\|y\| : y \in F(t, x)\} \leq m(t) \quad (3.1)$$

for a.e. $t \in J$.

- (3) There exists a constant $L > 0$ such that, for any bounded subset D of X , we have

$$\beta(F(t, D)) \leq L\beta(D) \quad (3.2)$$

for a.e. $t \in J$.

(H η) There exists a function $\eta \in C(J, \mathbb{R}^+)$ such that, for each constant $\lambda \in (-1, 0)$, we have

$$\eta(t) \geq MN + \frac{qM}{\Gamma(1+q)} \frac{t^{(1+\lambda)(1-q_1)}}{(1+\lambda)^{(1-q_1)}} \|m\|_{L^{1/q_1}[0,t]}, \quad t \in J, \quad (3.3)$$

where M, N, q_1 , and m are from (HA), (Hg), and (HF)(2).

In the proof of the existence results, we also need the following lemmas.

Lemma 8 ([38]) *Let X be a Banach space, and F be a multivalued mapping satisfying (HF)(1). Let $\mathcal{F} : L^1(J, X) \rightarrow C(J, X)$ be a linear continuous mapping. Then the operator*

$$\mathcal{F} \circ S_F : C(J, X) \rightarrow P_{cl,cv}(C(J, X)),$$

where $(\mathcal{F} \circ S_F)(u) = \mathcal{F}(S_F(u))$, is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 9 ([39] (Fixed point theorem)) *If W is a bounded, closed, convex, and compact nonempty subset of X and the map $F : W \rightarrow 2^W$ is upper semicontinuous with $F(x)$ being a closed and convex nonempty subset of W for each $x \in W$, then F has at least one fixed point in W .*

Lemma 10 ([40]) *Suppose that $b \geq 0$, $\sigma > 0$, and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < b$, and suppose that $c(t)$ is nonnegative and locally integrable on $0 \leq t < b$ with*

$$c(t) \leq a(t) + b \int_0^t (t-s)^{\sigma-1} c(s) \, ds, \quad t \in [0, b).$$

Then

$$c(t) \leq a(t) + \mu \int_0^t E'_\sigma(\mu(t-s))a(s) \, ds, \quad t \in [0, b),$$

where

$$\mu = (b\Gamma(\sigma))^{1/\sigma}, \quad E_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^{n\sigma}}{\Gamma(n\sigma + 1)}, \quad E'_\sigma(z) = \frac{d}{dz}E_\sigma(z).$$

Now we are ready to prove the following existence result for the nonlocal fractional differential inclusion (1.1)-(1.2).

Theorem 1 *If hypotheses (HA), (Hg), (HF)(1)(2)(3), and (H η) are satisfied, then the fractional differential inclusion (1.1)-(1.2) has at least one mild solution on J .*

Proof In view of (HF)(1), for each $u \in C(J, X)$, the set $S_F(u)$ is nonempty. So we can define the operator $R: C(J, X) \rightarrow 2^{C(J, X)}$ by

$$R(u) = \left\{ v \in C(J, X) : v(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds, f \in S_F(u) \right\}.$$

Then, obviously, the fixed point of the operator R is a mild solution of (1.1)-(1.2). So we shall show that R satisfies all the conditions of Lemma 9. For convenience, we subdivide the proof into four steps.

Step 1. We show that the values of R are convex and closed subsets in $C(J, X)$.

We first prove that $R(u)$ is convex for arbitrary $u \in C(J, X)$. Indeed, if v_1 and v_2 belong to $R(u)$, then there exist f_1 and f_2 belonging to $S_F(u)$ such that, for every $t \in J$ and $i = 1, 2$,

$$v_i(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f_i(s) \, ds.$$

Let $\lambda \in [0, 1]$. Then, for every $t \in J$ and $i = 1, 2$, we have

$$\lambda v_1(t) + (1-\lambda)v_2(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)(\lambda f_1(s) + (1-\lambda)f_2(s)) \, ds.$$

It is easy to see that $S_F(u)$ is convex since F has convex values. So $\lambda f_1 + (1-\lambda)f_2 \in S_F(u)$, and thus $\lambda v_1 + (1-\lambda)v_2 \in R(u)$. Therefore, $R(u)$ is convex.

We now prove that $R(u)$ is closed for every $u \in C(J, X)$. To this end, suppose that $\{v_n\}_{n=1}^\infty$ is a sequence in $R(u)$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$. Then we need to show that $v \in R(u)$. According to the definition of R , there exists a sequence $\{f_n\}_{n=1}^\infty \subset S_F(u)$ such that, for every $t \in J$, we have

$$v_n(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f_n(s) \, ds. \quad (3.4)$$

For any $n \geq 1$ and almost all $s \in (0, t]$, using assumption (HF)(2), we obtain

$$\|f_n(s)\| \leq m(s).$$

This shows that $\{f_n : n \geq 1\}$ is integrably bounded. Moreover, $\{f_n(t) : n \geq 1\} \subset F(t, u(t))$ implies that $\{f_n(t) : n \geq 1\}$ is relatively compact in X for a.e. $t \in J$. Therefore, the set $\{f_n : n \geq 1\}$ is semicompact. By Lemma 6 it is weakly compact in $L^1(J, X)$. We can assume that f_n converges weakly to a function $f \in L^1(J, X)$. Then by Mazur's lemma there is a sequence $\{g_n\}_{n=1}^\infty \subseteq \overline{\text{conv}}\{f_n : n \geq 1\}$ such that g_n converges strongly to f . Since the values of F are convex and compact, we have that the set $S_F(u)$ is convex and compact. So $\{g_n\}_{n=1}^\infty \subseteq S_F(u)$, and $f \in S_F(u)$.

Moreover, for all $t \in J, s \in (0, t]$ and $n \geq 1$, we have

$$\|(t-s)^{q-1} \mathcal{S}_q(t-s)f_n(s)\| \leq (t-s)^{q-1} \frac{Mq}{\Gamma(1+q)} m(s).$$

Let $\lambda = (q-1)/(1-q_1) \in (-1, 0)$. Using Hölder's inequality, we get

$$\begin{aligned} \int_0^t (t-s)^{q-1} m(s) \, ds &\leq \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} \, ds \right)^{1-q_1} \left(\int_0^t (m(s))^{\frac{1}{q_1}} \, ds \right)^{q_1} \\ &\leq \frac{b^{(1+\lambda)(1-q_1)}}{(1+\lambda)^{(1-q_1)}} \|m\|_{L^{1/q_1}[0,b]} < +\infty. \end{aligned}$$

Therefore, according to the Lebesgue dominated convergence theorem, taking $n \rightarrow \infty$ on both sides of (3.4), we get

$$v(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds, \quad (3.5)$$

which means that $v \in R(u)$.

Step 2. We construct a bounded, convex, closed, and compact nonempty subset $W \subseteq C(J, X)$ such that R maps W into itself.

Let $W_0 = \{u \in C(J, X) : \|u(t)\| \leq \eta(t), t \in J\}$. It is obvious that $W_0 \subseteq C(J, X)$ is bounded, closed, and convex. We claim that $R(W_0) \subseteq W_0$. To prove this, let $u \in W_0$ and $v \in R(u)$. Then there exists $f \in S_F(u)$ such that

$$v(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds.$$

According to (2.10) and assumptions (HF)(2) and (H η), for every $t \in J$, we have

$$\begin{aligned} \|v(t)\| &\leq \|\mathcal{T}_q(t)g(u)\| + \left\| \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds \right\| \\ &\leq MN + \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} m(s) \, ds \\ &\leq MN + \frac{qM}{\Gamma(1+q)} \frac{t^{(1+\lambda)(1-q_1)}}{(1+\lambda)^{(1-q_1)}} \|m\|_{L^{1/q_1}[0,t]} \\ &\leq \eta(t), \end{aligned}$$

which implies that $R(W_0) \subseteq W_0$.

Moreover, we claim that $R(W_0)$ is equicontinuous on J . For $0 \leq t < t+h \leq b$ and any $v \in R(W_0)$, there exist $u \in W_0$ and $f \in S_F(u)$ such that

$$\begin{aligned} \|v(t+h) - v(t)\| &\leq \|\mathcal{T}_q(t+h)g(u) - \mathcal{T}_q(t)g(u)\| + \left\| \int_0^{t+h} (t+h-s)^{q-1} \mathcal{S}_q(t+h-s)f(s) \, ds \right. \\ &\quad \left. - \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds \right\| \\ &\leq G_1 + G_2 + G_3 + G_4, \end{aligned}$$

where

$$\begin{aligned} G_1 &= \|\mathcal{T}_q(t+h)g(u) - \mathcal{T}_q(t)g(u)\|, \\ G_2 &= \left\| \int_0^t [(t+h-s)^{q-1} - (t-s)^{q-1}] \mathcal{S}_q(t+h-s)f(s) \, ds \right\|, \\ G_3 &= \left\| \int_0^t (t-s)^{q-1} [\mathcal{S}_q(t+h-s) - \mathcal{S}_q(t-s)]f(s) \, ds \right\|, \\ G_4 &= \left\| \int_t^{t+h} (t+h-s)^{q-1} \mathcal{S}_q(t+h-s)f(s) \, ds \right\|. \end{aligned}$$

We now show that $G_i \rightarrow 0$ as $h \rightarrow 0$ for $i = 1, 2, 3, 4$. First, by assumption (HA) and Lemma 1(2) we know that $\{\mathcal{T}_q(t) : t \in J\}$ is equicontinuous. Combining this with the compactness of g , we have that $G_1 \rightarrow 0$ as $h \rightarrow 0$, uniformly for all $u \in W_0$.

Next, for G_3 , from the equicontinuity of $\{\mathcal{S}_q(t) : t \in J\}$, we can conclude that

$$\int_0^t \|(t-s)^{q-1} [\mathcal{S}_q(t+h-s) - \mathcal{S}_q(t-s)]f(s)\| \, ds \rightarrow 0$$

as $h \rightarrow 0$, uniformly for all $f \in S_F(u)$ and $u \in W_0$. That is, $G_3 \rightarrow 0$ as $h \rightarrow 0$.

Then for G_2 , we note that $\lambda = (q-1)/(1-q_1) \in (-1, 0)$. So for $s < t \leq t+h$, we have $(t-s)^\lambda \geq (t+h-s)^\lambda$. Applying Lemma 7 and noting that $1-q_1 \in (0, 1)$, we get

$$\left| [(t+h-s)^\lambda]^{1-q_1} - [(t-s)^\lambda]^{1-q_1} \right| \leq [(t-s)^\lambda - (t+h-s)^\lambda]^{1-q_1}.$$

Accordingly, we have

$$\begin{aligned} |(t+h-s)^{q-1} - (t-s)^{q-1}| &= \left| [(t+h-s)^\lambda]^{1-q_1} - [(t-s)^\lambda]^{1-q_1} \right| \\ &\leq [(t-s)^\lambda - (t+h-s)^\lambda]^{1-q_1}. \end{aligned}$$

Therefore, by (3.1), Lemma 1(1), and Lemma 7, we get

$$\begin{aligned} G_2 &\leq \int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}| \|\mathcal{S}_q(t+h-s)f(s)\| \, ds \\ &\leq \frac{Mq}{\Gamma(1+q)} \int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}| m(s) \, ds \\ &\leq \frac{Mq}{\Gamma(1+q)} \left(\int_0^t |(t+h-s)^{q-1} - (t-s)^{q-1}|^{\frac{1}{1-q_1}} \, ds \right)^{1-q_1} \|m\|_{L^{1/q_1}[0,b]} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{Mq}{\Gamma(1+q)} \left(\int_0^t [(t-s)^\lambda - (t+h-s)^\lambda] ds \right)^{1-q_1} \|m\|_{L^{1/q_1}[0,b]} \\
&\leq \frac{Mq}{(1+\lambda)^{(1-q_1)}\Gamma(1+q)} [h^{1+\lambda} + t^{1+\lambda} - (t+h)^{1+\lambda}]^{1-q_1} \|m\|_{L^{1/q_1}[0,b]} \\
&\leq \frac{Mq}{(1+\lambda)^{(1-q_1)}\Gamma(1+q)} h^{(1+\lambda)(1-q_1)} \|m\|_{L^{1/q_1}[0,b]}.
\end{aligned}$$

Therefore, $G_2 \rightarrow 0$ as $h \rightarrow 0$.

Finally, for G_4 , we have

$$\begin{aligned}
G_4 &\leq \int_t^{t+h} |(t+h-s)^{q-1}| \|S_q(t+h-s)f(s)\| ds \\
&\leq \frac{Mq}{\Gamma(1+q)} \int_t^{t+h} (t+h-s)^{q-1} m(s) ds \\
&\leq \frac{Mq}{\Gamma(1+q)} \left(\int_t^{t+h} [(t+h-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \|m\|_{L^{1/q_1}[t,t+h]} \\
&\leq \frac{Mq}{\Gamma(1+q)(1+\lambda)^{(1-q_1)}} h^{(1+\lambda)(1-q_1)} \|m\|_{L^{1/q_1}[t,t+h]}.
\end{aligned}$$

So we also have that $G_4 \rightarrow 0$ as $h \rightarrow 0$. Therefore, we get $\|v(t+h) - v(t)\| \rightarrow 0$ as $h \rightarrow 0$, implying that $R(W_0)$ is equicontinuous on J .

Now, let us define a sequence $\{W_n\}_{n=1}^\infty$ recursively by $W_n = \overline{\text{conv}}R(W_{n-1})$ for all $n \geq 1$. From the above discussions we know that $W_n \subseteq W_{n-1}$ for all $n \geq 1$. Thus, $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of closed, bounded, convex, and equicontinuous subsets of $C(J, X)$. Set $W = \bigcap_{n=1}^\infty W_n$. Then W is a closed, bounded, convex, and equicontinuous subset of $C(J, X)$ and $R(W) \subseteq W$. We claim that W is nonempty and compact in $C(J, X)$. Indeed, by Lemma 2 and (2.15), for arbitrary $\varepsilon > 0$, there exist sequences $\{u_k\}_{k=1}^\infty \subset W_n$ and $\{f_k\}_{k=1}^\infty \subset S_F(u_k)$ such that

$$\begin{aligned}
\beta(W_{n+1}(t)) &= \beta(\overline{\text{conv}}(RW_n)(t)) = \beta((RW_n)(t)) \\
&\leq 2\beta(\{Ru_k(t)\}_{k=1}^\infty) + \varepsilon \\
&\leq 2\beta\left(\mathcal{T}_q(t)g(\{u_k\}_{k=1}^\infty) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)\{f_k(s)\}_{k=1}^\infty ds\right) + \varepsilon \\
&\leq 2\beta(\mathcal{T}_q(t)g(\{u_k\}_{k=1}^\infty)) + 2\beta\left(\int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)\{f_k(s)\}_{k=1}^\infty ds\right) + \varepsilon.
\end{aligned}$$

Since g is compact, by Lemma 2, (2.14), and (3.2) we have

$$\begin{aligned}
\beta(W_{n+1}(t)) &\leq 4 \int_0^t (t-s)^{q-1} \beta(\mathcal{S}_q(t-s)\{f_k(s)\}_{k=1}^\infty) ds + \varepsilon \\
&\leq \frac{4qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \beta(F(s, W_n(s))) ds + \varepsilon \\
&\leq \frac{4qML}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \beta(W_n(s)) ds + \varepsilon.
\end{aligned} \tag{3.6}$$

Since (3.6) is true for arbitrary $\varepsilon > 0$, we must have

$$\beta(W_{n+1}(t)) \leq \frac{4qML}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \beta(W_n(s)) \, ds. \quad (3.7)$$

Because $\{W_n\}_{n=1}^\infty$ is decreasing with respect to n , we define

$$\mu(t) = \lim_{n \rightarrow \infty} \beta(W_n(t)), \quad t \in J.$$

Taking $n \rightarrow \infty$ on both sides of (3.7), we have

$$\mu(t) \leq \frac{4qML}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \mu(s) \, ds.$$

By Lemma 10 we have that $\mu(t) = 0, t \in J$. On the other hand, $\{W_n\}_{n=1}^\infty$ is bounded and equicontinuous. Hence, by Lemma 3 we get $\beta(W_n) = \sup_{t \in J} \beta(W_n(t))$, which implies that $\lim_{n \rightarrow \infty} \beta(W_n) = 0$. Then it follows from Lemma 2(8) that $W = \bigcap_{n=1}^\infty W_n$ is nonempty and compact in $C(J, X)$.

Step 3. We show that the graph of R is closed.

Let $\{u_n\}_{n=1}^\infty \subset W$ with $u_n \rightarrow u$ as $n \rightarrow \infty$, $v_n \in R(u_n)$, and $v_n \rightarrow v$ as $n \rightarrow \infty$. We shall show that $v \in R(u)$. Since $v_n \in R(u_n)$, there exists $f_n \in S_F(u_n)$ such that

$$v_n(t) = \mathcal{T}_q(t)g(u_n) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f_n(s) \, ds.$$

In the sequel, we must show that there exists $f \in S_F(u)$ such that

$$v(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds.$$

Consider the linear operator $\mathcal{F}: L^1(J, X) \rightarrow C(J, X)$ defined as

$$\mathcal{F}(f)(t) = \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds.$$

Clearly, \mathcal{F} is linear and continuous. Hence, it follows from Lemma 8 that $\mathcal{F} \circ S_F$ is a closed graph operator. Moreover, we have

$$v_n(\cdot) - \mathcal{T}_q(\cdot)g(u_n) \in \mathcal{F}(S_F(u_n)).$$

Since $v_n \rightarrow v$, $u_n \rightarrow u$, and g is continuous, we obtain

$$v(\cdot) - \mathcal{T}_q(\cdot)g(u) \in \mathcal{F}(S_F(u)),$$

that is,

$$v(t) - \mathcal{T}_q(t)g(u) = \int_0^t (t-s)^{q-1} \mathcal{S}(t-s)f(s) \, ds$$

for some $f \in S_F(u)$, and thus the graph of R is closed.

Step 4. We show that R is u.s.c. on W .

As a consequence of the previous proof, we have that W is closed and $R(u)$ is closed for all $u \in W$. The set $\overline{R(W)} \subseteq W$ is compact. Moreover, R is closed. According to Remark 1, we can come to the conclusion that R is u.s.c.

Finally, due to the fixed point of Lemma 9, R has at least one point $u \in R(u)$, and u is a mild solution to the fractional semilinear differential inclusion (1.1) with the nonlocal condition (1.2). \square

Remark 2 If A generates an analytic semigroup or a differential semigroup $\{T(t)\}_{t \geq 0}$, then $\{T(t)\}_{t \geq 0}$ is equicontinuous [41]. In applications of partial differential equations, such as parabolic equations and strongly damped wave equations, the corresponding solution semigroups are analytic. Therefore, the results in this paper have wide applicability.

In Theorem 1, hypothesis (HF)(2) may be sometimes difficult to be verified, and also the mapping g needs to be uniformly bounded. Indeed, if g is compact, then g is bounded on a bounded subset. Next, we further give an existence result for the fractional differential inclusion (1.1)-(1.2) under a relatively weak condition of F when g is not uniformly bounded. For this purpose, we introduce the following two hypotheses:

(Hg') The nonlocal term $g : C(J, X) \rightarrow X$ is continuous and compact.

(HF)(2') There exist a function $\alpha \in L^{1/q_1}(J, \mathbb{R}^+)$ for some given $q_1 \in (0, q)$ and an increasing function $\Omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|F(t, x)\| \leq \alpha(t)\Omega(\|x\|)$$

for a.e. $t \in J$ and all $x \in X$.

Theorem 2 Assume that hypotheses (HA), (Hg'), and (HF)(1)(2')(3) are satisfied. Suppose that

$$\limsup_{k \rightarrow \infty} \left\{ \frac{M}{k} \left(\gamma(k) + \frac{M_1 q}{(1 + \lambda)^{(1-q_1)} \Gamma(1+q)} \Omega(k) b^{(1+\lambda)(1-q_1)} \right) \right\} < 1, \quad (3.8)$$

where $\gamma(k) = \sup\{\|g(u)\| : \|u\| \leq k\}$, $M_1 = \|\alpha\|_{L^{1/q_1}[0, b]}$, and $\lambda = (q-1)/(1-q_1) \in (-1, 0)$. Then the fractional differential inclusion (1.1)-(1.2) has at least one mild solution on J .

Proof By (3.8) there exists a constant $k > 0$ such that

$$M \left(\gamma(k) + \frac{M_1 q}{(1 + \lambda)^{(1-q_1)} \Gamma(1+q)} \Omega(k) b^{(1+\lambda)(1-q_1)} \right) < k.$$

Let $W_0 = \{u \in C(J, X) : \|u\| \leq k\}$ and $W_1 = \overline{\text{conv}} R(W_0)$. Then for any $v \in W_1$, there exist $u \in W_0$ and $f \in S_F(u)$ such that

$$v(t) = \mathcal{T}_q(t)g(u) + \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds.$$

From (2.10) and assumption (Hg') we conclude that, for every $t \in J$, we have

$$\begin{aligned}
\|v(t)\| &\leq \|\mathcal{T}_q(t)g(u)\| + \left\| \int_0^t (t-s)^{q-1} \mathcal{S}_q(t-s)f(s) \, ds \right\| \\
&\leq M \left(\gamma(k) + \frac{q}{\Gamma(1+q)} \Omega(k) \int_0^t (t-s)^{q-1} \alpha(s) \, ds \right) \\
&\leq M \left(\gamma(k) + \frac{q}{\Gamma(1+q)} \Omega(k) \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} \, ds \right)^{(1-q_1)} \|\alpha\|_{L^{1/q_1}[0,b]} \right) \\
&\leq M \left(\gamma(k) + \frac{M_1 q}{(1+\lambda)^{(1-q_1)} \Gamma(1+q)} \Omega(k) b^{(1+\lambda)(1-q_1)} \right) \\
&< k.
\end{aligned}$$

It follows that $\|v\| = \sup_{t \in J} \{\|v(t)\|\} < k$. Then $W_1 \subseteq W_0$. Set $W_n = \overline{\text{conv}}R(W_{n-1})$ for all $n \geq 1$. Then we can complete the proof similarly to that of Theorem 1. \square

4 An example

We consider the following fractional partial differential inclusion:

$$\begin{cases} \partial_t^q x(t, y) \in \partial_y^2 x(t, y) + u(t, x(t, y)), \\ x(t, 0) = x(t, \pi) = 0, \\ x(0, y) = \int_0^1 h(s) \sin(1 + |x(s, y)|) \, ds, \end{cases} \quad (4.1)$$

where $X = L^2([0, \pi]; \mathbb{R})$, ∂_t^q is the Caputo fractional partial derivative of order q with $0 < q < 1$, $h \in L^1([0, 1]; \mathbb{R})$, and $u : [0, 1] \times X \rightarrow P(X)$.

We define the operator A by $Az = z''$ with $D(A) = \{z \in X : z \text{ and } z' \text{ are absolutely continuous, } z'' \in X, z(0) = z(\pi) = 0\}$. Then

$$Az = - \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in D(A),$$

where $e_n(z) = (1/\pi)^{1/2} \sin nz$, $0 \leq z \leq \pi$, $n = 1, 2, \dots$. Clearly, A generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in X , which is compact and analytic, and it is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, e_n \rangle e_n, \quad z \in X.$$

This implies that A satisfies assumption (HA).

Then system (4.1) can be reformulated as

$$\begin{cases} {}^C D_t^q x(t) \in Ax(t) + F(t, x(t)), \\ x(0) = g(x), \end{cases} \quad (4.2)$$

where $x(t)(y) = x(t, y)$, $t \in [0, 1]$, $y \in [0, \pi]$, and $F(t, x(t))(y) = u(t, x(t, y))$. The function $g : C([0, 1], X) \rightarrow X$ is given by

$$g(x)(y) = \int_0^1 h(s) \sin(1 + |x(s, y)|) \, ds.$$

If we impose suitable conditions on the above-defined functions for verifying the assumptions in Theorem 1 or Theorem 2, then we can conclude that (4.1) admits at least one mild solution on $[0, 1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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