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Stability of the Jensen equation in C^*-algebras: a fixed point approach

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Abstract
Using fixed point method, we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and also of derivations on C^*-algebras and Lie C^*-algebras for the Jensen equation.
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1 Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1]. In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In [3], Rassias proved a generalization of the Hyers' theorem for additive mappings.

The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. Furthermore, in 1994, a generalization of the Rassias' theorem was obtained by Gǎvruta [4] by replacing the bound $\epsilon(||x||^p + ||y||^p)$ by a general control function $\phi(x, y)$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [5-22].

Theorem 1.1. *Let (X, d) be a complete generalized metric space and J: $X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(*a*) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n_0 \geq n_0$;

(*b*) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(*c*) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(*d*) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In [20], Park proved the Hyers-Ulam stability of the following functional equation:

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y) \tag{1.1}$$

in fuzzy Banach spaces. In this article, using the fixed point alternative approach, we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and also of derivations on C^*-algebras and Lie C^*-algebras for the Jensen Equation (1.1).

2 Stability of homomorphisms in C^*-algebras

Throughout this section, assume that A is a C^*-algebra with the norm $||.||_A$ and that B is a C^*-algebra with the norm $||.||_B$.

For a given mapping f: $A \to B$, we define

$$C_\mu f(x,y) := 2\mu f\left(\frac{x+y}{2}\right)-f(\mu x)-f(\mu y)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y \in A$. Note that a $\mathbb{C}$-linear mapping H: $A \to B$ is called a homomorphism in C^*-algebras, if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x \in A$. Throughout this section, we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras for the functional equation $C_\mu f(x, y) = 0$.

Theorem 2.1. *Let f: $A \to B$ be a mapping with $f(0) = 0$ for which there exists a function ϕ: $A^2 \to [0, \infty)$ such that*

$$\left\|C_\mu f(x,y)\right\|_B \le \varphi(x,y), \tag{2.2}$$

$$\left\|f(xy)-f(x)f(y)\right\|_B \le \varphi(x,y), \tag{2.3}$$

$$\left\|f(x^*)-f(x)^*\right\|_B \le \varphi(x,x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$and all $x, y \in A$. If there exists an $L < \frac{1}{2}$such that

$$\varphi(x,y) \le \frac{L\varphi(2x,2y)}{2} \tag{2.5}$$

for all $x, y \in A$, then there exists a unique C^*-algebra homomorphism H: $A \to B$ such that

$$\left\|f(x)-H(x)\right\|_B \le \frac{\varphi(x,0)}{1-L}. \tag{2.6}$$

Proof. It follows from (2.5) that

$$\lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=\lim_{n\to\infty} L^n\varphi(x,y)=0.$$

Consider the set $X := \{g$: $A \to B; g(0) = 0\}$ and the generalized metric d in X defined by

$$d(f,g)=\inf\{C\in\mathbb{R}^+ : \left\|g(x)-h(x)\right\|_B \le C\varphi(x,0), \forall x \in A\}$$

It is easy to show that (X, d) is complete. Now, we consider a linear mapping $J : A \to A$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right)$$

for all $x \in A$. By [[7], Theorem 3.1], $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in X$. Letting $\mu = 1$ and $y = 0$ in (2.2), we have

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \le \varphi(x, 0) \tag{2.7}$$

for all $x \in A$. It follows from (2.7) that $d(f, Jf) \le 1$. By Theorem 1.1, there exists a mapping H: $A \to B$ satisfying the following:

(1) H is a fixed point of J, that is,

$$H\left(\frac{x}{2}\right) = \frac{1}{2}H(x) \tag{2.8}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set $\Omega = \{g \in X : d(f, g) < \infty\}$. This implies that H is a unique mapping satisfying (2.8) such that there exists $C \in (0, \infty)$ satisfying $||f(x) - H(x)||_B \le C\phi(x,0)$ for all $x \in A$.

(2) $d(J^nf, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x) \tag{2.9}$$

for all $x \in A$.

(3) $d(f,H) \le \dfrac{d(f,Jf)}{1-L}$, which implies the inequality $d(f,H) \le \dfrac{1}{1-L}$. This implies that the inequality (2.6) holds. It follows from (2.2) and (2.9) that

$$\begin{aligned}\left\|2H\left(\frac{x+y}{2}\right) - H(x) - H(y)\right\|_B &= \lim_{n\to\infty}\left\|2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^nf\left(\frac{x}{2^n}\right) - 2^nf\left(\frac{y}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} L^n\varphi(x,y) = 0\end{aligned}$$

for all $x, y \in A$. So $2H\left(\frac{x+y}{2}\right) = H(x) + H(y)$ for all $x, y \in X$. Therefore, the mapping H: $A \to B$ is Jensen additive.

Letting $y = x$ in (2.2), we get $\mu f(x) = f(\mu x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$ So, we get

$$\|\mu H(x) - H(\mu x)\|_B = \lim_{n\to\infty}\left\|2^n\mu f\left(\frac{x}{2^n}\right) - 2^nf\left(\frac{\mu x}{2^n}\right)\right\|_B = 0.$$

So, $\mu H(x) = H(\mu x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$ Thus one can show that the mapping H: $A \to B$ is $\mathbb{C}$-linear. It follows from (2.3) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} 4^n\left\|f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right)f\left(\frac{y}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty}(2L)^n\varphi(x,y) = 0\end{aligned}$$

for all $x \in A$. Furthermore, By (2.4), we have

$$\begin{aligned}\|H(x^*) - H(x)^*\|_B &= \lim_{n\to\infty} 2^n\left\|f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^*\right\|_B \\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} L^n\varphi(x,y) = 0\end{aligned}$$

for all $x \in A$. Thus H: $A \to B$ is a C^*-algebra homomorphism satisfying (2.6), as desired.

Corollary 2.1. *Let* $0 < r < 1$ *and* θ *be nonnegative real numbers and* f: $A \to B$ *be a mapping with* $f(0) = 0$ *such that*

$$\begin{aligned} &\left\| 2\mu f\left(\frac{x+y}{2}\right) - f(\mu x) - f(\mu y) \right\|_B \le \theta\left(\|x\|_A^r + \|y\|_A^r\right), \\ &\|f(xy) - f(x)f(y)\|_B \le \theta\left(\|x\|_A^r + \|y\|_A^r\right) \\ &\|f(x^*) - f(x)^*\|_B \le 2\theta \|x\|_A^r \end{aligned} \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$*and all* $x, y \in A$. *Then the limit* $H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$*exists for all* $x \in A$ *and* H: $A \to B$ *is a unique* C^**-algebra homomorphism such that*

$$\|f(x) - H(x)\|_B \le \frac{2\theta \|x\|_A^r}{2 - 2^r} \tag{2.11}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1, if we take $\varphi(x, y) = \theta\left(\|x\|_A^r + \|y\|_A^r\right)$ for all $x, y \in A$. In fact, if we choose $L = 2^{r-1}$, then we get the desired result.

Theorem 2.2. *Let* f: $A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* ϕ: $A^2 \to [0, \infty)$ *satisfying (2.2), (2.3), and (2.4). If there exists an* $L < 1$ *such that* $\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$*for all* $x, y \in A$, *then there exists a unique* C^**-algebra homomorphism* H: $A \to B$ *such that*

$$\|f(x) - H(x)\|_B \le \frac{L\varphi(x, 0)}{1 - L}. \tag{2.12}$$

for all $x \in A$.

Proof. We consider the linear mapping J: $A \to A$ such that $Jg(x) = \frac{1}{2}g(2x)$ for all $x \in A$. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \le \frac{\varphi(2x, 0)}{2} \le L\varphi(x, 0)$$

for all $x \in X$. Hence $d(f, Jf) \le L$. By Theorem 1.1, there exists a mapping H: $A \to B$ satisfying the following:

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.13}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set $\Omega = \{g \in X: d(f, g) < \infty\}$. This implies that H is a unique mapping satisfying (2.13) such that there exists $C \in (0, \infty)$ satisfying $||f(x) - H(x)||_B \le C\phi(x,0)$ for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x)$ for all $x \in A$.

(3) $d(f, H) \le \frac{d(f, Jf)}{1 - L}$, which implies the inequality $d(f, H) \le \frac{1}{1 - L}$. which implies that the inequality (2.12). The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.2. *Let $r > 1$ and θ be nonnegative real numbers and f: $A \to B$ be a mapping satisfying $f(0) = 0$ and (2.10). Then the limit $H(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x \in A$ and H: $A \to B$ is a unique C^*-algebra homomorphism such that*

$$\left\|f(x) - H(x)\right\|_B \leq \frac{2\theta\,\|x\|_A^r}{2^r - 2} \tag{2.14}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 if we take $\varphi(x, y) = \theta(\|x\|_A^r + \|y\|_A^r)$ for all $x, y \in A$. In fact, if we choose $L = 2^{1-r}$, then we get the desired result.

3 Stability of derivations on C^*-algebras

Throughout this section, assume that A is a C^*-algebra with the norm $||.|_A$. Note that a $\mathbb{C}$-linear mapping δ: $A \to A$ is called a derivation on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

Throughout this section, using the fixed point alternative approach, We prove the Hyers-Ulam stability of derivations on C^*-algebras for the functional equation (1.1).

Theorem 3.1. *Let f: $A \to A$ be a mapping with $f(0) = 0$ for which there exists a function ϕ: $A^2 \to [0, \infty)$ such that*

$$\left\|2\mu f\left(\frac{x+y}{2}\right) - f(\mu x) - f(\mu y)\right\|_A \leq \varphi(x, y) \tag{3.15}$$

$$\left\|f(xy) - f(x)y - xf(y)\right\|_A \leq \varphi(x, y) \tag{3.16}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < \frac{1}{2}$ such that $\varphi(x, y) \leq \frac{L\varphi(2x, 2y)}{2}$ for all $x, y \in A$, then there exists a unique derivation δ: $A \to A$ such that

$$\left\|f(x) - \delta(x)\right\|_A \leq \frac{\varphi(x, 0)}{1 - L}. \tag{3.17}$$

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping δ: $A \to A$ satisfying (3.17). The mapping δ: $A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. It follows from (3.2) that

$$\begin{aligned}\left\|\delta(xy) - \delta(x)y - x\delta(y)\right\|_B &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n}f\left(\frac{y}{2^n}\right)\right\|_B \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq \lim_{n\to\infty} (2L)^n \varphi(x, y) = 0\end{aligned}$$

for all $x, y \in A$. So

$$\delta(xy) - \delta(x)y - x\delta(y)$$

for all $x, y \in A$. Thus δ: $A \to A$ is a derivation satisfying (3.17).

Corollary 3.1. *Let $0 < r < 1$ and θ be nonnegative real numbers and f: $A \to A$ be a mapping with $f(0) = 0$ such that*

$$\left\|2\mu f\left(\frac{x+y}{2}\right)-f(\mu x)-f(\mu y)\right\|_A \le \theta(\|x\|_A^r+\|y\|_A^r), \tag{3.18}$$

$$\|f(xy)-f(x)y-xf(y)\|_A \le \theta(\|x\|_A^r+\|y\|_A^r) \tag{3.19}$$

for all $\mu \in \mathbb{T}^1$*and all* $x, y \in A$. *Then the limit* $H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in A$ *and* $\delta: A \to A$ *is a unique derivation such that*

$$\|f(x)-\delta(x)\| \le \frac{2\theta\,\|x\|_A^r}{2-2^r} \tag{3.20}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 if we take $\varphi(x,y)=\theta(\|x\|_A^r+\|y\|_A^r)$ for all $x, y \in A$. In fact, if we choose $L = 2^{r-1}$, then we get the desired result.

Theorem 3.2. *Let* $f: A \to A$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\phi: A^2 \to [0, \infty)$ *satisfying (3.15) and* (3.2). *If there exists an* $L < 1$ *such that* $\varphi(x,y) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$*for all* $x, y \in A$, *then there exists a unique derivation* $\delta: A \to A$ *such that*

$$\|f(x)-\delta(x)\|_A \le \frac{L\varphi(x,0)}{1-L}. \tag{3.21}$$

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.1.

Corollary 3.2. *Let* $r > 1$ *and* θ *be nonnegative real numbers and* $f: A \to A$ *be a mapping satisfying* $f(0) = 0$, (3.4) *and* (3.5). *Then the limit* $H(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ *exists for all* $x \in A$ *and* $\delta: A \to A$ *is a unique derivation such that*

$$\|f(x)-\delta(x)\|_A \le \frac{2\theta\,\|x\|_A^r}{2^r-2} \tag{3.22}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.2 if we take $\varphi(x,y)=\theta(\|x\|_A^r+\|y\|_A^r)$ for all $x, y \in A$. In fact, if we choose $L = 2^{1-r}$, then we get the desired result.

4 Stability of homomorphisms in Lie C^*-algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x,y] := \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a Lie C^*-algebra (see, [17-19]).

Definition 4.1. *Let* A *and* B *be Lie* C^**-algebras, A* $\mathbb{C}$*-linear mapping* $H: A \to B$ *is called a Lie* C^**-algebra homomorphism if* $H([x,y]) = [H(x),H(y)] = \frac{H(x)H(y)-H(y)H(x)}{2}$*for all* $x, y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with the norm $||.||_A$ and B is a Lie C^*-algebra with the norm $||.||_B$.

We prove the Hyers-Ulam stability of homomorphisms in Lie C^*-algebras for the functional Equation (1.1).

Theorem 4.1. *Let* $f: A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\phi: A^2 \to [0, \infty)$ *satisfying (2.2) such that*

$$\left\|f([x,y]) - [f(x),f(y)]\right\|_B \le \varphi(x,y) \tag{4.23}$$

for all $x, y \in A$. *If there exists an* $L < \frac{1}{2}$ *such that* $\varphi(x,y) \le \frac{L}{2}\varphi(2x,2y)$ *for all* $x, y \in A$, *then there exists a unique Lie C*-algebra homomorphism* H: $A \to B$ *satisfying (2.6).*

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping H: $A \to B$ satisfying (2.6). The mapping H: $A \to B$ is given by $H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in A$. It follows from (4.23) that

$$\begin{aligned}\left\|H([x,y]) - [H(x),H(y)]\right\|_B &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{[x,y]}{4^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right]\right\|_B \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in A$. So $H([x, y]) = [H(x), H(y)]$ for all $x, y \in A$. Thus H: $A \to B$ is a Lie C^*-algebra homomorphism satisfying (2.6), as desired.

Corollary 4.1. *Let* $0 < r < 1$ *and* θ *be nonnegative real numbers, and let* f: $A \to B$ *be a mapping satisfying* $f(0) = 0$ *such that*

$$\left\|C_\mu f(x,y)\right\|_B \le \theta(\|x\|_A^r + \|y\|_A^r), \tag{4.24}$$

$$\left\|f([x,y]) - [f(x),f(y)]\right\|_B \le \theta(\|x\|_A^r + \|y\|_A^r) \tag{4.25}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in A$. *Then there exists a unique Lie C*-algebra homomorphism* H: $A \to B$ *satisfying (2.11).*

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x,y) = \theta(\|x\|_A^r + \|y\|_A^r)$ for all $x, y \in A$. Then $L = 2^{r-1}$ and we get the desired result.

Theorem 4.2. *Let* f: $A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* ϕ: $A^2 \to [0, \infty)$ *satisfying (2.2) and (4.23). If there exists an* $L < 1$ *such that* $\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$ *for all* $x, y \in A$, *then there exists a unique Lie C*-algebra homomorphism* H: $A \to B$ *satisfying (2.12).*

Corollary 4.2. *Let* $r > 1$ *and* θ *be nonnegative real numbers, and let* f: $A \to B$ *be a mapping satisfying* $f(0) = 0$, (4.2) *and* (4.3). *Then there exists a unique Lie C*-algebra homomorphism* H: $A \to B$ *satisfying (2.14).*

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x,y) = \theta(\|x\|_A^r + \|y\|_A^r)$ for all $x, y \in A$. Then $L = 2^{1-r}$ and we get the desired result.

5 Stability of Lie derivations on C*-algebras

Definition 5.1. *Lat A be a Lie C*-algebras, A* $\mathbb{C}$*-linear mapping* δ: $A \to A$ *is called a Lie derivation if* $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ *for all* $x, y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with the norm $||.||_A$. In this section, we prove the Hyers-Ulam stability of derivations on Lie C^*-algebras for the functional Equation (1.1).

Theorem 5.1. *Let* f: $A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* ϕ: $A^2 \to [0, \infty)$ *satisfying (3.15) such that*

$$\left\|f([x,y]) - [f(x),y] - [x,f(y)]\right\|_B \le \varphi(x,y) \tag{5.26}$$

for all $x, y \in A$. If there exists an $L < \frac{1}{2}$ such that $\varphi(x,y) \le \frac{L}{2}\varphi(2x,2y)$ for all $x, y \in A$, then there exists a unique Lie derivation δ: $A \to A$ satisfying (3.17).

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping δ: $A \to A$ satisfying (3.17). The mapping δ: $A \to A$ is given by $\delta(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in A$. It follows from (5.26) that

$$\begin{aligned}\left\|\delta([x,y]) - [\delta(x),y] - [x,\delta(y)]\right\|_A &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{[x,y]}{4^n}\right) - \left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}\right] - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\right\|_A \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} (2L)^n \varphi(x,y) = 0\end{aligned}$$

for all $x, y \in A$. So $\delta([x, y]) = [\delta(x),y] + [x, \delta(y)]$ for all $x, y \in A$. Thus δ: $A \to A$ is a Lie derivation satisfying (3.17), as desired.

Corollary 5.1. *Let* $0 < r < 1$ *and* θ *be nonnegative real numbers, and let* f: $A \to B$ *be a mapping satisfying* $f(0) = 0$ *and* (3.4) *such that*

$$\left\|f([x,y]) - [f(x),y] - [x,f(y)]\right\|_B \le \theta\left(\|x\|_A^r + \|y\|_A^r\right) \tag{5.27}$$

for all $x, y \in A$. Then there exists a unique Lie derivation δ: $A \to A$ satisfying (3.20).

Proof. The proof follows from Theorem 5.1 by taking $\varphi(x,y) = \theta\left(\|x\|_A^r + \|y\|_A^r\right)$ for all $x, y \in A$. Then $L = 2^{r-1}$ and we get the desired result.

Theorem 5.2. *Let f: $A \to B$ be a mapping with $f(0) = 0$ for which there exists a function ϕ: $A^2 \to [0, \infty)$ satisfying (3.15) and (5.26). If there exists an $L < 1$ such that $\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$ for all $x, y \in A$, then there exists a unique Lie derivation δ: $A \to A$ satisfying (3.21).*

Corollary 5.2. *Let $r > 1$ and θ be nonnegative real numbers, and let f: $A \to B$ be a mapping satisfying $f(0) = 0$, (3.4) and (5.27). Then there exists a unique Lie derivation δ: $A \to A$ satisfying (3.22).*

Proof. The proof follows from Theorem 5.2 by taking $\varphi(x,y) = \theta\left(\|x\|_A^r + \|y\|_A^r\right)$ for all $x, y \in A$. Then $L = 2^{1-r}$ and we get the desired result.

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Authors' contributions
All authors conceived of the study participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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