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Research Article

A System of Generalized Mixed Equilibrium Problems and Fixed Point Problems for Pseudocontractive Mappings in Hilbert Spaces

Poom Kumam[1] and Chaichana Jaiboon[2]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, KMUTT, Bangkok 10140, Thailand*
[2] *Department of Mathematics, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, RMUTR, Bangkok 10100, Thailand*

Correspondence should be addressed to Chaichana Jaiboon, chaichana.j@rmutr.ac.th

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We introduce and analyze a new iterative algorithm for finding a common element of the set of fixed points of strict pseudocontractions, the set of common solutions of a system of generalized mixed equilibrium problems, and the set of common solutions of the variational inequalities with inverse-strongly monotone mappings in Hilbert spaces. Furthermore, we prove new strong convergence theorems for a new iterative algorithm under some mild conditions. Finally, we also apply our results for solving convex feasibility problems in Hilbert spaces. The results obtained in this paper improve and extend the corresponding results announced by Qin and Kang (2010) and the previously known results in this area.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let E be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. Let $S : E \rightarrow E$ be a mapping. In the sequel, we will use $F(S)$ to denote the set of *fixed points* of S, that is, $F(S) = \{x \in E : Sx = x\}$.

Definition 1.1. Let $S : E \rightarrow E$ be a mapping. Then S is called

(1) *contraction* if there exists a constant $\alpha \in [0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in E, \tag{1.1}$$

(2) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

Remark 1.2. It is well known that if $E \subset H$ is nonempty, bounded, closed, and convex and S is a nonexpansive mapping on E then $F(S)$ is nonempty; see, for example, [1].

(3) *strongly pseudocontractive* with the coefficient $\tau \in (0, 1)$ if

$$\langle Sx - Sy, x - y \rangle \geq \tau \|x - y\|^2, \quad \forall x, y \in E, \tag{1.3}$$

(4) *strictly pseudocontractive* with the coefficient $k \in [0, 1)$ if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in E; \tag{1.4}$$

for such a case, S is also said to be a *k-strict pseudocontraction,* and if $k = 0$, then S is a nonexpansive mapping,

(5) *pseudocontractive* if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in E. \tag{1.5}$$

The class of strict pseudocontractions falls into the one between classes of nonexpansive mappings and pseudocontractions. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of fixed points for strict pseudocontractions.

In 1967, Browder and Petryshyn [2] introduced a convex combination method to study strict pseudocontractions in Hilbert spaces. On the other hand, Marino and Xu [3] and Zhou [4] introduced and researched some iterative scheme for finding a fixed point of a strict pseudocontraction mapping. More precisely, take $k \in (0, 1)$ and define a mapping S_k by

$$S_k x = kx + (1 - k)Sx, \quad \forall x \in E, \tag{1.6}$$

where S is a strict pseudocontraction. Under appropriate restrictions on k, it is proved the mapping S_k is nonexpansive. Therefore, the techniques of studying nonexpansive mappings can be applied to study more general strict pseudocontractions.

The *domain* of the function $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ is the set

$$\operatorname{dom} \varphi = \{x \in E : \varphi(x) < +\infty\}. \tag{1.7}$$

Let $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ be a proper extended real-valued function and let Φ be a bifunction of $E \times E$ into $\mathcal{R}$ such that $E \cap \operatorname{dom} \varphi \neq \emptyset$, where $\mathcal{R}$ is the set of real numbers.

There exists the *generalized mixed equilibrium problem* for finding $x \in E$ such that

$$\Phi(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $\mathrm{GMEP}(\Phi, \varphi, \Psi)$, that is,

$$\mathrm{GMEP}(\Phi, \varphi, \Psi) = \{x \in E : \Phi(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in E\}. \tag{1.9}$$

We see that x is a solution of problem (1.8) implies that $x \in \operatorname{dom} \varphi$.

Special Examples

(1) If $\Psi = 0$, problem (1.8) is reduced into the *mixed equilibrium problem* for finding $x \in E$ such that

$$\Phi(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.10}$$

Problem (1.10) was studied by Ceng and Yao [5]. The set of solutions of (1.10) is denoted by $\mathrm{MEP}(\Phi, \varphi)$.

(2) If $\varphi = 0$, problem (1.8) is reduced into the *generalized equilibrium problem* for finding $x \in E$ such that

$$\Phi(x, y) + \langle \Psi x, y - x \rangle \geq 0, \quad \forall y \in E. \tag{1.11}$$

Problem (1.11) was studied by Takahashi and Toyoda [6]. The set of solutions of (1.11) is denoted by $\mathrm{GEP}(\Phi, \Psi)$.

(3) If $\Psi = 0$ and $\varphi = 0$, problem (1.8) is reduced into the *equilibrium problem* for finding $x \in E$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in E. \tag{1.12}$$

Problem (1.12) was studied by Blum and Oettli [7]. The set of solutions of (1.12) is denoted by $\mathrm{EP}(\Phi)$.

(4) If $\Phi = 0$, problem (1.8) is reduced into the *mixed variational inequality of Browder type* for finding $x \in E$ such that

$$\langle \Psi x, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.13}$$

Problem (1.13) was studied by Browder [8]. The set of solutions of (1.13) is denoted by $\mathrm{VI}(E, \Psi, \varphi)$.

(5) If $\Phi = 0$ and $\varphi = 0$, problem (1.8) is reduced into the *variational inequality problem* for finding $x \in E$ such that

$$\langle \Psi x, y - x \rangle \geq 0, \quad \forall y \in E. \tag{1.14}$$

Problem (1.14) was studied by Hartman and Stampacchia [9]. The set of solutions of (1.14) is denoted by $\mathrm{VI}(E,\Psi)$. The variational inequality has been extensively studied in the literature. See, for example, [7, 10, 11] and the references therein.

(6) If $\Phi = 0$ and $\Psi = 0$, problem (1.8) is reduced into the *minimize problem* for finding $x \in E$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.15}$$

The set of solutions of (1.15) is denoted by $\mathrm{Argmin}(\varphi)$.

The generalized mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.8). In 1997, Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to initial data when $\mathrm{EP}(\Phi)$ is nonempty and proved a strong convergence theorem. Many authors have proposed some useful methods for solving the $\mathrm{GMEP}(\Phi,\varphi,\Psi)$, $\mathrm{MEP}(\Phi,\varphi)$, and $\mathrm{EP}(\Phi)$; see, for instance, [5, 12–23].

Definition 1.3. Let $B : E \to H$ be a nonlinear mapping. Then B is called

(1) *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in E, \tag{1.16}$$

(2) *β-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in E, \tag{1.17}$$

(3) *η-Lipschitz continuous* if there exists a positive real number η such that

$$\|Bx - By\| \leq \eta\|x - y\|, \quad \forall x, y \in E, \tag{1.18}$$

(4) *β-inverse-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2, \quad \forall x, y \in E. \tag{1.19}$$

Remark 1.4. It is obvious that any β-inverse-strongly monotone mappings B are monotone and $1/\beta$-Lipschitz continuous.

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for a β-inverse-strongly monotone mapping, Takahashi and Toyoda [6] introduced the following iterative scheme:

$$\begin{gathered} x_0 \in E \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_E(x_n - \lambda_n B x_n), \quad \forall n \geq 0, \end{gathered} \tag{1.20}$$

where P_E is the metric projection of H onto E, B is a β-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\beta)$. They showed that if $F(S) \cap \mathrm{VI}(E,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.20) converges weakly to some $q \in F(S) \cap \mathrm{VI}(E,B)$.

On the other hand, Y. Yao and J.-C Yao [24] introduced the following iterative process defined recursively by

$$\begin{gathered} x_1 = x \in E \text{ chosen arbitrary}, \\ y_n = P_E(x_n - \lambda_n B x_n), \\ x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n S P_E(y_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{gathered} \tag{1.21}$$

where B is a β-inverse-strongly monotone mapping, $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in the interval $[0,1]$, and $\{\lambda_n\}$ is a sequence in $(0,2\beta)$. They showed that if $F(S) \cap \mathrm{VI}(E,B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.21) converges strongly to some $q \in F(S) \cap \mathrm{VI}(E,B)$.

Let A be a strongly positive linear bounded operator on H if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.22}$$

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in E} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.23}$$

where A is a linear bounded operator, E is the fixed point set of a nonexpansive mapping S on H, and b is a given point in H. Moreover, it is shown in [25] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n \tag{1.24}$$

converges strongly to $q = P_{F(S)}(I - A + \gamma f)(q)$. Recently, Plubtieng and Punpaeng [26] proposed the following iterative algorithm:

$$\begin{gathered} \Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S u_n. \end{gathered} \tag{1.25}$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution q of the variational inequality:

$$\langle (A - \gamma f) q, x - q \rangle \geq 0, \quad \forall x \in F(S) \cap \mathrm{EP}(\Phi), \tag{1.26}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F(S)\cap \mathrm{EP}(\phi)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.27}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Ceng et al. [27] introduced iterative scheme for finding a common element of the set of solutions of equilibrium problems and the of fixed points of a k-strict pseudocontraction mapping defined in the setting of real Hilbert space H: $x_0 \in H$ and let

$$\begin{gathered}\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Su_n,\end{gathered} \tag{1.28}$$

where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (k,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Further, they proved that $\{x_n\}$ and $\{u_n\}$ converge weakly to $q \in F(S)\cap\mathrm{EP}(\Phi)$, where $q = P_{F(S)\cap\mathrm{EP}(\Phi)}x_0$.

On the other hand, for finding a common element of the set of fixed points of a k-strict pseudocontraction mapping and the set of solutions of an equilibrium problems in a real Hilbert space, Liu [28] introduced the following iterative scheme:

$$\begin{gathered}\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ y_n = \beta_n u_n + (1-\beta_n)Su_n, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)y_n, \quad \forall n \geq 1,\end{gathered} \tag{1.29}$$

where S is a k-strict pseudocontraction mapping and $\{\epsilon_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They proved that under certain appropriate conditions over $\{\epsilon_n\}$, $\{\beta_n\}$, and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to some $q \in F(S)\cap\mathrm{EP}(\Phi)$, which solves some variational inequality problems (1.26).

In 2008, Ceng and Yao [5] introduced an iterative scheme for finding a common fixed point of a finite family of nonexpansive mappings and the set of solutions of a problem (1.8) in Hilbert spaces and obtained the strong convergence theorem which used the following condition.

(G) $K : E \to \mathcal{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology. We note that the condition (G) for the function $K : E \to \mathcal{R}$ is a very strong condition. We also note that the condition (G) does not cover the case $K(x) = \|x\|^2/2$ and $\eta(x, y) = x - y$ for each $(x, y) \in E \times E$. Very recently, Wangkeeree and Wangkeeree [29] introduced a general iterative method for finding a common element of the set of solutions of the mixed equilibrium problems, the set of fixed point of a k-strict pseudocontraction mapping, and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in Hilbert spaces. They obtained a strong convergence theorem except the condition (G) for the sequences generated by these processes.

In 2009, Qin and Kang [30] introduced an explicit viscosity approximation method for finding a common element of the set of fixed points of strict pseudocontraction and the set of solutions of variational inequalities with inverse-strongly monotone mappings in Hilbert spaces. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{aligned} x_1 &\in E, \\ z_n &= P_E(x_n - \mu_n C x_n), \\ y_n &= P_E(x_n - \lambda_n B x_n), \\ x_{n+1} &= \epsilon_n f(x_n) + \beta_n x_n + \gamma_n \left[\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n\right], \quad \forall n \geq 1. \end{aligned} \tag{1.30}$$

Then, they proved that under certain appropriate conditions imposed on $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, and $\{\alpha_n^{(3)}\}$, the sequence $\{x_n\}$ generated by (1.30) converges strongly to $q \in F(S) \cap \mathrm{VI}(E, B) \cap \mathrm{VI}(E, C)$, where $q = P_{F(S)\cap \mathrm{VI}(E,B)\cap \mathrm{VI}(E,C)} f(q)$.

In the present paper, motivated and inspired by Qin and Kang [30], Peng and Yao [21], Plubtieng and Punpaeng [26], and Liu [28], we introduce a new general iterative scheme for finding a common element of the set of fixed points of strict pseudocontractions, the set of common solutions of the system of generalized mixed equilibrium problems, and the set of common solutions of the variational inequalities for inverse-strongly monotone mappings in Hilbert spaces. We obtain a strong convergence theorem for the sequences generated by these processes under some parameter controlling conditions. The results in this paper extend and improve the corresponding recent results of Qin and Kang [30], Peng and Yao [21], Plubtieng and Punpaeng [26], and Liu [28] and many others.

2. Preliminaries

Let H be a real Hilbert space and let E be a nonempty closed convex subset of H. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2, \quad \forall \lambda \in [0, 1],\ \forall x, y \in H. \tag{2.1}$$

For any $x \in H$, there exists a *unique nearest point* in E, denoted by $P_E x$, such that

$$\|x - P_E x\| \leq \|x - y\|, \quad \forall y \in E. \tag{2.2}$$

The mapping P_E is called the *metric projection* of H onto E.

It is well known that P_E is a firmly nonexpansive mapping of H onto E, that is,

$$\langle x - y, P_E x - P_E y\rangle \geq \|P_E x - P_E y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_E x$ is characterized by the following properties: $P_E x \in E$ and

$$\begin{gathered}\langle x - P_E x, y - P_E x\rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_E x\|^2 + \|y - P_E x\|^2\end{gathered} \tag{2.4}$$

for all $x \in H$, $y \in E$.

Lemma 2.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in E$, then,*

$$z = P_E x \Longleftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in E. \tag{2.5}$$

Lemma 2.2. *Let H be a Hilbert space, let E be a nonempty closed convex subset of H, and let B be a mapping of E into H. Let $u \in E$. Then for $\lambda > 0$,*

$$u \in VI(E, B) \Longleftrightarrow u = P_E(u - \lambda B u), \tag{2.6}$$

where P_E is the metric projection of H onto E.

A set-valued mapping $T : H \to 2^H$ is called a *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is called *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of E into H, η-Lipschitz continuous mappings and let $N_E v$ be the *normal cone* to E when $v \in E$, that is,

$$N_E v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in E\}, \tag{2.7}$$

and define a mapping T on E by

$$Tv = \begin{cases} Bv + N_E v, & v \in E, \\ \emptyset, & v \notin E. \end{cases} \tag{2.8}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(E, B)$; see [31].

Lemma 2.3. *Let H be a Hilbert space, let E be a nonempty closed convex subset of H, and let $\Psi : E \to H$ be ρ-inverse-strongly monotone. It $0 < r \leq 2\rho$, then $I - \rho\Psi$ is a nonexpansive mapping in H.*

Proof. For all $x, y \in E$ and $0 < r \leq 2\rho$, we have

$$\begin{aligned}\|(I - r\Psi)x - (I - r\Psi)y\|^2 &= \|(x - y) - r(\Psi x - \Psi y)\|^2 \\ &= \|x - y\|^2 - 2r\langle x - y, \Psi x - \Psi y\rangle + r^2\|\Psi x - \Psi y\|^2 \\ &\leq \|x - y\|^2 - 2r\rho\|\Psi x - \Psi y\| + r^2\|\Psi x - \Psi y\|^2 \\ &= \|x - y\|^2 + r(r - 2\rho)\|\Psi x - \Psi y\|^2 \\ &\leq \|x - y\|^2.\end{aligned} \tag{2.9}$$

So, $I - \rho\Psi$ is a nonexpansive mapping of E into H. □

Lemma 2.4 (see [32]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \tag{2.10}$$

Lemma 2.5 (see [25]). *Let E be a nonempty closed convex subset of H, let f be a contraction of H into itself with $\alpha \in [0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}/\alpha$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H. \tag{2.11}$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\alpha$.

Lemma 2.6 (see [25]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \vartheta \leq \|A\|^{-1}$. Then $\|I - \vartheta A\| \leq 1 - \vartheta\overline{\gamma}$.*

Lemma 2.7 (see [4]). *Let E be a nonempty closed convex subset of a real Hilbert space H and let $S : E \to E$ be a k-strict pseudocontraction mapping with a fixed point. Then $F(S)$ is closed and convex. Define $S_k : E \to E$ by $S_k = kx + (1 - k)Sx$ for each $x \in E$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.*

Lemma 2.8 (see [33]). *Let E be a closed convex subset of a Hilbert space H and let $S : E \to E$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \quad x_n - Sx_n \longrightarrow 0 \quad \text{implies } x = Sx. \tag{2.12}$$

Lemma 2.9 (see [34]). *Let E be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on E. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is* *nonempty. Let δ_n be a sequence of positive number with $\sum_{n=1}^{\infty} \delta_n = 1$. Then a mapping S on E defined by*

$$Sx = \sum_{n=1}^{\infty} \delta_n T_n x \tag{2.13}$$

for $x \in E$ is well defined and nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction Φ, the function φ, and the set E:

(A1) $\Phi(x, x) = 0$ for all $x \in E$;

(A2) Φ is monotone, that is, $\Phi(x, y) + \Phi(y, x) \leq 0$ for all $x, y \in E$;

(A3) for each $x, y, z \in E$, $\lim_{t \to 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;

(A4) for each $x \in E$, $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous;

(A5) for each $y \in E$, $x \mapsto \Phi(x, y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq E$ and $y_x \in E$ such that for any $z \in E \setminus D_x$,

$$\Phi(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.14}$$

(B2) E is a bounded set.

By similar argument as in the proof of Lemma 2.10 in [35], we have the following lemma appearing.

Lemma 2.10. *Let E be a nonempty closed convex subset of H. Let $\Phi : E \times E \to \mathcal{R}$ be a bifunction satisfies (A1)–(A5) and let $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{\Phi} : H \to E$ as follows:*

$$T_r^{\Phi}(x) = \left\{ z \in E : \Phi(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in E \right\} \tag{2.15}$$

for all $z \in H$. Then, the following holds:

(i) *for each $x \in H$, $T_r^{\Phi}(x) \neq \emptyset$;*

(ii) *T_r^{Φ} is single-valued;*

(iii) *T_r^{Φ} is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^{\Phi} x - T_r^{\Phi} y \right\|^2 \leq \left\langle T_r^{\Phi} x - T_r^{\Phi} y, x - y \right\rangle; \tag{2.16}$$

(iv) *$F(T_r^{\Phi}) = MEP(\Phi, \varphi)$;*

(v) *$MEP(\Phi, \varphi)$ is closed and convex.*

Remark 2.11. We remark that Lemma 2.10 is not a consequence of Lemma 3.1 in [5], because the condition of the sequential continuity from the weak topology to the strong topology for the derivative K' of the function $K : E \to \mathcal{R}$ does not cover the case $K(x) = \|x\|^2/2$.

Lemma 2.12 (see [36]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \ge 1$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|l_n - x_n\| = 0$.*

Lemma 2.13 (see [37]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \varrho_n)a_n + \sigma_n, \quad n \ge 1, \tag{2.17}$$

where $\{\varrho_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathcal{R}$ such that

(1) $\sum_{n=1}^{\infty} \varrho_n = \infty$,

(2) $\limsup_{n\to\infty}(\sigma_n/\varrho_n) \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.18}$$

3. Main Results

In this section, we will use the new approximation iterative method to prove a strong convergence theorem for finding a common element of the set of fixed points of strict pseudocontractions, the set of common solutions of the system of generalized mixed equilibrium problems, and the set of a common solutions of the variational inequalities for inverse-strongly monotone mappings in a real Hilbert space.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let Φ_1 and Φ_2 be two bifunctions from $E \times E$ to $\mathcal{R}$ satisfying (A1)–(A5) and let $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $C : E \to H$ be a ξ-inverse-strongly monotone mapping, let $\Psi_1 : E \to H$ be a ρ-inverse-strongly monotone mapping, let $\Psi_2 : E \to H$ be an ω-inverse-strongly monotone mapping, and let $B : E \to H$ be a β-inverse-strongly monotone mapping. Let $f : E \to E$ be an α-contraction with coefficient α $(0 \le \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $S : E \to E$ be a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx + (1-k)Sx$, for all $x \in E$. Suppose that*

$$\Theta := F(S) \cap VI(E,B) \cap VI(E,C) \cap GMEP(\Phi_1, \varphi, \Psi_1) \cap GMEP(\Phi_2, \varphi, \Psi_2) \ne \emptyset. \tag{3.1}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$
\begin{aligned}
&x_1 \in E, \qquad u_n \in E, \qquad v_n \in E,\\
&\Phi_1(u_n, u) + \varphi(x) - \varphi(u_n) + \langle \Psi_1 x_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E,\\
&\Phi_2(v_n, v) + \varphi(x) - \varphi(v_n) + \langle \Psi_2 x_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in E,\\
&k_n = \alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} P_E(x_n - \lambda_n B x_n) + \alpha_n^{(3)} P_E(x_n - \mu_n C x_n) + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n,\\
&x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \epsilon_n A\right)k_n, \quad \forall n \geq 1,
\end{aligned}
\tag{3.2}
$$

where $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r \in (0, 2\rho)$, $s \in (0, 2\omega)$, and $\{\lambda_n\}$ and $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy the following restrictions:

(C1) $\sum_{i=1}^{5} \alpha_n^{(i)} = 1$,

(C2) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty} |\mu_{n+1} - \mu_n| = 0$,

(C5) $d \leq \lambda_n \leq 2\beta$ *and* $e \leq \mu_n \leq 2\xi$, *where* d, e *are two positive constants,*

(C6) $\lim_{n\to\infty} \alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, *where* $i = 1,2,3,4,5$.

Then, $\{x_n\}$ converges strongly to a point $q \in \Theta$ which is the unique solution of the variational inequality:

$$
\left\langle (A - \gamma f)q, x - q \right\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.3}
$$

Equivalently, one has $q = P_\Theta(I - A + \gamma f)(q)$.

Proof. Since $\epsilon_n \to 0$, as $n \to \infty$, we may assume, without loss of generality, that $\epsilon_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. By Lemma 2.6, we know that if $0 \leq \vartheta \leq \|A\|^{-1}$, then $\|I - \vartheta A\| \leq 1 - \vartheta\overline{\gamma}$. We will assume that $\|I - A\| \leq 1 - \overline{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$
\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}. \tag{3.4}
$$

Observe that

$$
\begin{aligned}
\left\langle \left((1-\beta_n)I - \epsilon_n A\right)x, x \right\rangle &= 1 - \beta_n - \epsilon_n \langle Ax, x\rangle\\
&\geq 1 - \beta_n - \epsilon_n \|A\|\\
&\geq 0,
\end{aligned}
\tag{3.5}
$$

and so this shows that $(1-\beta_n)I-\epsilon_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\epsilon_n A\| &= \sup\{|\langle((1-\beta_n)I-\epsilon_n A)x,x\rangle| : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\epsilon_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\le 1-\beta_n-\epsilon_n\overline{\gamma}.\end{aligned}\tag{3.6}$$

Since f is a contraction of H into itself with $\alpha\in[0,1)$, then, we have

$$\begin{aligned}\|P_\Theta(I-A+\gamma f)(x)-P_\Theta(I-A+\gamma f)(y)\| &\le \|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|\\ &\le \|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|\\ &\le (1-\overline{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\ &= (1-(\overline{\gamma}-\gamma\alpha))\|x-y\|, \quad \forall x,y\in H.\end{aligned}\tag{3.7}$$

Since $0<1-(\overline{\gamma}-\gamma\alpha)<1$, it follows that $P_\Theta(I-A+\gamma f)$ is a contraction of H into itself. Therefore the Banach Contraction Mapping Principle implies that there exists a unique element $q\in H$ such that $q=P_\Theta(I-A+\gamma f)(q)$.

Next, we will divide the proof into five steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, let $p\in\Theta$ and by Lemma 2.10, we obtain

$$p=P_E(p-\lambda_n Bp)=P_E(p-\mu_n Cp)=T_r^{\Phi_1}(I-r\Psi_1)p=T_s^{\Phi_2}(I-s\Psi_2)p.\tag{3.8}$$

Note that $u_n=T_r^{\Phi_1}(I-r\Psi_1)x_n\in\operatorname{dom}\varphi$ and $v_n=T_s^{\Phi_2}(I-s\Psi_2)x_n\in\operatorname{dom}\varphi$; we have

$$\begin{aligned}\|u_n-p\| &= \left\|T_r^{\Phi_1}(I-r\Psi_1)x_n-T_r^{\Phi_1}(I-r\Psi_1)p\right\|\le\|x_n-p\|,\\ \|v_n-p\| &= \left\|T_s^{\Phi_2}(I-s\Psi_2)x_n-T_s^{\Phi_2}(I-s\Psi_2)p\right\|\le\|x_n-p\|.\end{aligned}\tag{3.9}$$

Put $z_n = P_E(x_n - \mu_n C x_n)$ and $y_n = P_E(x_n - \lambda_n B x_n)$. For each $\lambda_n \leq 2\beta$ and $\mu_n \leq 2\xi$ by Lemma 2.3, we get that $I - \lambda_n B$ and $I - \mu_n B$ are nonexpansive. Thus, we have

$$\begin{aligned}
\|z_n - p\| &= \|P_E(x_n - \mu_n C x_n) - P_E(p - \mu_n C p)\| \\
&\leq \|(x_n - \mu_n C x_n) - (p - \mu_n C p)\| \\
&= \|(I - \mu_n C)x_n - (I - \mu_{nC})p\| \\
&\leq \|x_n - p\|, \\
\|y_n - p\| &= \|P_E(x_n - \lambda_n B x_n) - P_E(p - \lambda_n B p)\| \\
&\leq \|(x_n - \lambda_n B x_n) - (p - \lambda_n B p)\| \\
&= \|(I - \lambda_n B)v_n - (I - \lambda_n B)p\| \\
&\leq \|v_n - p\| \leq \|x_n - p\|.
\end{aligned} \tag{3.10}$$

From Lemma 2.7, we have that S_k is nonexpansive with $F(S_k) = F(S)$. It follows that

$$\begin{aligned}
\|k_n - p\| &= \left\|\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n\right\| \\
&\leq \alpha_n^{(1)} \|S_k x_n - p\| + \alpha_n^{(2)} \|y_n - p\| + \alpha_n^{(3)} \|z_n - p\| + \alpha_n^{(3)} \|u_n - p\| + \alpha_n^{(3)} \|v_n - p\| \\
&\leq \alpha_n^{(1)} \|x_n - p\| + \alpha_n^{(2)} \|x_n - p\| + \alpha_n^{(3)} \|x_n - p\| + \alpha_n^{(3)} \|x_n - p\| + \alpha_n^{(3)} \|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned} \tag{3.11}$$

which yields that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\epsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \epsilon_n A)(k_n - p)\| \\
&\leq (1 - \beta_n - \epsilon_n\overline{\gamma})\|k_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\leq (1 - \beta_n - \epsilon_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \epsilon_n\|\gamma f(x_n) - Ap\| \\
&\leq (1 - \epsilon_n\overline{\gamma})\|x_n - p\| + \epsilon_n\gamma\|f(x_n) - f(p)\| + \epsilon_n\|\gamma f(p) - Ap\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\epsilon_n\overline{\gamma})\|x_n-p\| + \epsilon_n\gamma\alpha\|x_n-p\| + \epsilon_n\|\gamma f(p)-Ap\| \\
&= (1-(\overline{\gamma}-\alpha\gamma)\epsilon_n)\|x_n-p\| + (\overline{\gamma}-\alpha\gamma)\epsilon_n\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\alpha\gamma} \\
&\leq \max\left\{\|x_n-p\|, \frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\alpha\gamma}\right\} \\
&\leq \qquad \vdots \\
&\leq \max\left\{\|x_1-p\|, \frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\alpha\gamma}\right\}, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{3.12}
$$

Hence, $\{x_n\}$ is bounded, and so are $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, $\{y_n\}$, $\{k_n\}$, $\{f(x_n)\}$, $\{Cx_n\}$, and $\{Bx_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|k_n-x_n\| = 0$.

Observing that $u_n = T_r^{\Phi_1}(I-r\Psi_1)x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_r^{\Phi_1}(I-r\Psi_1)x_{n+1} \in \operatorname{dom}\varphi$, by the nonexpansiveness of $T_r^{\Phi_1}$, we get

$$
\|u_{n+1}-u_n\| = \left\|T_r^{\Phi_1}(I-r\Psi_1)x_{n+1} - T_r^{\Phi_1}(I-r\Psi_1)x_n\right\| \leq \|x_{n+1}-x_n\|. \tag{3.13}
$$

Similarly, let $v_n = T_s^{\Phi_2}(I-s\Psi_2)x_n \in \operatorname{dom}\varphi$ and $v_{n+1} = T_s^{\Phi_2}(I-s\Psi_2)x_{n+1} \in \operatorname{dom}\varphi$; we have

$$
\|v_{n+1}-v_n\| = \left\|T_s^{\Phi_2}(I-s\Psi_2)x_{n+1} - T_s^{\Phi_2}(I-s\Psi_2)x_n\right\| \leq \|x_{n+1}-x_n\|. \tag{3.14}
$$

From $z_n = P_E(x_n-\mu_nCx_n)$ and $y_n = P_E(x_n-\lambda_nBx_n)$; thus, we compute

$$
\begin{aligned}
\|z_{n+1}-z_n\| &= \|P_E(x_{n+1}-\mu_{n+1}Cx_{n+1}) - P_E(x_n-\mu_nCx_n)\|\| \\
&\leq \|(x_{n+1}-\mu_{n+1}Cx_{n+1}) - (x_n-\mu_nCx_n)\| \\
&= \|(x_{n+1}-\mu_{n+1}Cx_{n+1}) - (x_n-\mu_{n+1}Cx_n) + (\mu_n-\mu_{n+1})Cx_n\| \\
&\leq \|(x_{n+1}-\mu_{n+1}Cx_{n+1}) - (x_n-\mu_{n+1}Cx_n)\| + |\mu_n-\mu_{n+1}|\|Cx_n\| \\
&= \|(I-\mu_{n+1}C)x_{n+1} - (I-\mu_{n+1}C)x_n\| + |\mu_n-\mu_{n+1}|\|Cx_n\| \\
&\leq \|x_{n+1}-x_n\| + |\mu_n-\mu_{n+1}|\|Cx_n\|.
\end{aligned} \tag{3.15}
$$

Similarly, we have

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|P_E(x_{n+1}-\lambda_{n+1}Bx_{n+1}) - P_E(x_n-\lambda_nBx_n)\| \\
&\leq \|x_{n+1}-x_n\| + |\lambda_n-\lambda_{n+1}|\|Bx_n\|.
\end{aligned} \tag{3.16}
$$

Also noticing that

$$
\begin{aligned}
k_n &= \alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n \\
k_{n+1} &= \alpha_{n+1}^{(1)} S_k x_{n+1} + \alpha_{n+1}^{(2)} y_{n+1} + \alpha_{n+1}^{(3)} z_{n+1} + \alpha_{n+1}^{(4)} u_{n+1} + \alpha_{n+1}^{(5)} v_{n+1},
\end{aligned}
\tag{3.17}
$$

we compute

$$
\begin{aligned}
\|k_{n+1} - k_n\| \le{} & \alpha_{n+1}^{(1)} \|S_k x_{n+1} - S_k x_n\| + \left|\alpha_{n+1}^{(1)} - \alpha_n^{(1)}\right| \|S_k x_n\| + \alpha_{n+1}^{(2)} \left\|y_{n+1} - y_n\right\| \\
& + \left|\alpha_{n+1}^{(2)} - \alpha_n^{(2)}\right| \left\|y_n\right\| + \alpha_{n+1}^{(3)} \|z_{n+1} - z_n\| + \left|\alpha_{n+1}^{(3)} - \alpha_n^{(3)}\right| \|z_n\| \\
& + \alpha_{n+1}^{(4)} \|u_{n+1} - u_n\| + \left|\alpha_{n+1}^{(4)} - \alpha_n^{(4)}\right| \|u_n\| + \alpha_{n+1}^{(5)} \|v_{n+1} - v_n\| + \left|\alpha_{n+1}^{(5)} - \alpha_n^{(5)}\right| \|v_n\| \\
\le{} & \alpha_{n+1}^{(1)} \|x_{n+1} - x_n\| + \left|\alpha_{n+1}^{(1)} - \alpha_n^{(1)}\right| \|S_k x_n\| + \alpha_{n+1}^{(2)} \left\|y_{n+1} - y_n\right\| \\
& + \left|\alpha_{n+1}^{(2)} - \alpha_n^{(2)}\right| \left\|y_n\right\| + \alpha_{n+1}^{(3)} \|z_{n+1} - z_n\| + \left|\alpha_{n+1}^{(3)} - \alpha_n^{(3)}\right| \|z_n\| \\
& + \alpha_{n+1}^{(4)} \|u_{n+1} - u_n\| + \left|\alpha_{n+1}^{(4)} - \alpha_n^{(4)}\right| \|u_n\| + \alpha_{n+1}^{(5)} \|v_{n+1} - v_n\| + \left|\alpha_{n+1}^{(5)} - \alpha_n^{(5)}\right| \|v_n\|.
\end{aligned}
\tag{3.18}
$$

Substitution of (3.13), (3.14), (3.15), and (3.16) into (3.18) yields that

$$
\begin{aligned}
\|k_{n+1} - k_n\| \le{} & \alpha_{n+1}^{(1)} \|x_{n+1} - x_n\| + \left|\alpha_{n+1}^{(1)} - \alpha_n^{(1)}\right| \|S_k x_n\| \\
& + \alpha_{n+1}^{(2)} \{\|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}| \|Bx_n\|\} + \left|\alpha_{n+1}^{(2)} - \alpha_n^{(2)}\right| \left\|y_n\right\| \\
& + \alpha_{n+1}^{(3)} \{\|x_{n+1} - x_n\| + \left|\mu_n - \mu_{n+1}\right| \|Cx_n\|\} + \left|\alpha_{n+1}^{(3)} - \alpha_n^{(3)}\right| \|z_n\| \\
& + \alpha_{n+1}^{(4)} \|x_{n+1} - x_n\| + \left|\alpha_{n+1}^{(4)} - \alpha_n^{(4)}\right| \|u_n\| + \alpha_{n+1}^{(5)} \|x_{n+1} - x_n\| + \left|\alpha_{n+1}^{(5)} - \alpha_n^{(5)}\right| \|v_n\| \\
\le{} & \|x_{n+1} - x_n\| + M_1 \Big(\left|\alpha_{n+1}^{(1)} - \alpha_n^{(1)}\right| + \left|\alpha_{n+1}^{(2)} - \alpha_n^{(2)}\right| + \left|\alpha_{n+1}^{(3)} - \alpha_n^{(3)}\right| + \left|\alpha_{n+1}^{(4)} - \alpha_n^{(4)}\right| \\
& \qquad + \left|\alpha_{n+1}^{(5)} - \alpha_n^{(5)}\right| + |\lambda_n - \lambda_{n+1}| + \left|\mu_n - \mu_{n+1}\right| \Big),
\end{aligned}
\tag{3.19}
$$

where M_1 is an appropriate constant such that

$$
\begin{aligned}
M_1 \ge \max\Big\{ & \sup_{n\ge1}\{\|S_k x_n\|\}, \sup_{n\ge1}\{\left\|y_n\right\|\}, \sup_{n\ge1}\{\|z_n\|\}, \sup_{n\ge1}\{\|u_n\|\}, \\
& \sup_{n\ge1}\{\|v_n\|\}, \sup_{n\ge1}\{\|Bx_n\|\}, \sup_{n\ge1}\{\|Cx_n\|\} \Big\}.
\end{aligned}
\tag{3.20}
$$

Putting $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$, for all $n \geq 1$, we have

$$l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\epsilon_n \gamma f(x_n) + \left((1-\beta_n)I - \epsilon_n A\right)k_n}{1-\beta_n}. \tag{3.21}$$

Then, we compute

$$\begin{aligned} l_{n+1} - l_n &= \frac{\epsilon_{n+1}\gamma f(x_{n+1}) + \left((1-\beta_{n+1})I - \epsilon_{n+1}A\right)k_{n+1}}{1-\beta_{n+1}} - \frac{\epsilon_n \gamma f(x_n) + \left((1-\beta_n)I - \epsilon_n A\right)k_n}{1-\beta_n} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\epsilon_n}{1-\beta_n}\gamma f(x_n) + k_{n+1} - k_n + \frac{\epsilon_n}{1-\beta_n}Ak_n - \frac{\epsilon_{n+1}}{1-\beta_{n+1}}Ak_{n+1} \\ &= \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - Ak_{n+1}\right) + \frac{\epsilon_n}{1-\beta_n}\left(Ak_n - \gamma f(x_n)\right) + k_{n+1} - k_n. \end{aligned} \tag{3.22}$$

It follows from (3.19) and (3.22) that

$$\begin{aligned} \|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|Ak_{n+1}\|\right) + \frac{\epsilon_n}{1-\beta_n}\left(\|Ak_n\| + \|\gamma f(x_n)\|\right) \\ &\quad + \|k_{n+1} - k_n\| - \|x_{n+1} - x_n\| \\ &\leq \frac{\epsilon_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|Ak_{n+1}\|\right) + \frac{\epsilon_n}{1-\beta_n}\left(\|Ak_n\| + \|\gamma f(x_n)\|\right) \\ &\quad + M_1\Big(\left|\alpha_{n+1}^{(1)} - \alpha_n^{(1)}\right| + \left|\alpha_{n+1}^{(2)} - \alpha_n^{(2)}\right| + \left|\alpha_{n+1}^{(3)} - \alpha_n^{(3)}\right| + \left|\alpha_{n+1}^{(4)} - \alpha_n^{(4)}\right| \\ &\qquad + \left|\alpha_{n+1}^{(5)} - \alpha_n^{(5)}\right| + |\lambda_n - \lambda_{n+1}| + |\mu_n - \mu_{n+1}|\Big). \end{aligned} \tag{3.23}$$

This together with (C2), (C3), (C4), and (C6) implies that

$$\limsup_{n\to\infty}\left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{3.24}$$

Hence, by Lemma 2.12, we obtain $\|l_n - x_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n - x_n\| = 0. \tag{3.25}$$

Moreover, we also get

$$\begin{aligned}\lim_{n\to\infty}\|u_{n+1}-u_n\| &= \lim_{n\to\infty}\|v_{n+1}-v_n\| = \lim_{n\to\infty}\|z_{n+1}-z_n\| = \lim_{n\to\infty}\|y_{n+1}-y_n\| \\ &= \lim_{n\to\infty}\|k_{n+1}-k_n\| = 0.\end{aligned} \tag{3.26}$$

Observe that

$$x_{n+1}-x_n = \epsilon_n\left(\gamma f(x_n) - Ax_n\right) + \left(1-\beta_n-\epsilon_n\overline{\gamma}\right)(k_n-x_n). \tag{3.27}$$

By conditions (C2), (C3), and (3.25), we have

$$\lim_{n\to\infty}\|k_n-x_n\| = 0. \tag{3.28}$$

Step 3. We claim that the following statements hold:

(s1) $\lim_{n\to\infty}\|x_n-u_n\| = 0$;

(s2) $\lim_{n\to\infty}\|x_n-y_n\| = 0$;

(s3) $\lim_{n\to\infty}\|x_n-z_n\| = 0$;

(s4) $\lim_{n\to\infty}\|x_n-v_n\| = 0$.

For $p\in\Theta$, we compute

$$\begin{aligned}\|z_n-p\|^2 &= \|P_E(x_n-\mu_nCx_n) - P_E(p-\mu_nCp)\|^2 \\ &\le \|(x_n-\mu_nCx_n) - (p-\mu_nCp)\|^2 \\ &= \|(x_n-p) - \mu_n(Cx_n-Cp)\|^2 \\ &\le \|x_n-p\|^2 - 2\mu_n\langle x_n-p, Cx_n-Cp\rangle + \mu_n^2\|Cx_n-Cp\|^2 \\ &\le \|x_n-p\|^2 + \mu_n(\mu_n-2\xi)\|Cx_n-Cp\|^2 \\ &= \|x_n-p\|^2 - \mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2.\end{aligned} \tag{3.29}$$

By the same way, we can get

$$\|y_n-p\|^2 \le \|x_n-p\|^2 - \lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2. \tag{3.30}$$

We note that

$$\begin{aligned}
\|u_n-p\|^2 &= \left\|T_r^{\Phi_1}(I-r\Psi_1)x_n - T_r^{\Phi_1}(I-r\Psi_1)p\right\|^2 \\
&\le \|(I-r\Psi_1)x_n-(I-r\Psi_1)p\|^2 \\
&= \|(x_n-p)-r(\Psi_1x_n-\Psi_1)p\|^2 \\
&= \|x_n-p\|^2-2r\langle x_n-p,\Psi_1x_n-\Psi_1p\rangle + r^2\|\Psi_1x_n-\Psi_1p\|^2 \\
&\le \|x_n-p\|^2-2r\rho\|\Psi_1x_n-\Psi_1p\| + r^2\|\Psi_1x_n-\Psi_1p\|^2 \\
&= \|x_n-p\|^2+r(r-2\rho)\|\Psi_1x_n-\Psi_1p\|^2 \\
&= \|x_n-p\|^2-r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2.
\end{aligned} \tag{3.31}$$

Similarly, we have

$$\|v_n-p\|^2 \le \|x_n-p\|^2-s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2. \tag{3.32}$$

Observe that

$$\begin{aligned}
\|k_n-p\|^2 &\le \alpha_n^{(1)}\|S_kx_n-p\|^2+\alpha_n^{(2)}\|y_n-p\|^2+\alpha_n^{(3)}\|z_n-p\|^2+\alpha_n^{(4)}\|u_n-p\|^2+\alpha_n^{(5)}\|v_n-p\|^2 \\
&\le \alpha_n^{(1)}\|x_n-p\|^2+\alpha_n^{(2)}\|y_n-p\|^2+\alpha_n^{(3)}\|z_n-p\|^2+\alpha_n^{(4)}\|u_n-p\|^2+\alpha_n^{(5)}\|v_n-p\|^2.
\end{aligned} \tag{3.33}$$

Substituting (3.29), (3.30), (3.31), and (3.32) into (3.33), we obtain

$$\begin{aligned}
\|k_n-p\|^2 &\le \alpha_n^{(1)}\|x_n-p\|^2+\alpha_n^{(2)}\left\{\|x_n-p\|^2-\lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2\right\} \\
&\quad + \alpha_n^{(3)}\left\{\|x_n-p\|^2-\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2\right\} \\
&\quad + \alpha_n^{(4)}\left\{\|x_n-p\|^2-r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2\right\} \\
&\quad + \alpha_n^{(5)}\left\{\|x_n-p\|^2-s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2\right\} \\
&= \|x_n-p\|^2-\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2-\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2 \\
&\quad - \alpha_n^{(4)}r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2-\alpha_n^{(5)}s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2.
\end{aligned} \tag{3.34}$$

It follows from (3.2) and (3.34) that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad= \|\epsilon_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\epsilon_n A)k_n-p\|^2 \\
&\quad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2 \\
&\quad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\epsilon_n\overline{\gamma}) \\
&\qquad\times\Big\{\|x_n-p\|^2-\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2-\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2 \\
&\qquad\quad-\alpha_n^{(4)}r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2-\alpha_n^{(5)}s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2\Big\} \\
&\quad= \epsilon_n\|\gamma f(x_n)-Ap\|^2+(1-\epsilon_n\overline{\gamma})\|x_n-p\|^2-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2 \\
&\quad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(2)}\lambda_n(2\beta-\lambda_n)\|Bx_n-Bp\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}r(2\rho-r)\|\Psi_1x_n-\Psi_1p\|^2 \\
&\qquad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}s(2\omega-s)\|\Psi_2x_n-\Psi_2p\|^2 \\
&\quad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2.
\end{aligned}
\tag{3.35}
$$

It follows from (C5) that

$$
\begin{aligned}
&(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\mu_n(2\xi-\mu_n)\|Cx_n-Cp\|^2 \\
&\qquad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad= \epsilon_n\|\gamma f(x_n)-Ap\|^2+(\|x_n-p\|-\|x_{n+1}-p\|)(\|x_n-p\|+\|x_{n+1}-p\|) \\
&\qquad\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|).
\end{aligned}
\tag{3.36}
$$

From (C2), (C6), and (3.25), we have

$$\lim_{n\to\infty}\|Cx_n-Cp\|=0. \tag{3.37}$$

Since $s \in (0, 2\omega)$, we also have

$$\begin{aligned}
&(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}s(2\omega-s)\left\|\Psi_2x_n-\Psi_2p\right\|^2 \\
&\quad\le \epsilon_n\left\|\gamma f(x_n)-Ap\right\|^2+\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2 \\
&\quad\le \epsilon_n\left\|\gamma f(x_n)-Ap\right\|^2+\|x_{n+1}-x_n\|\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right).
\end{aligned}\tag{3.38}$$

From (C2), (C6), and (3.25), we obtain

$$\lim_{n\to\infty}\left\|\Psi_2x_n-\Psi_2p\right\|=0.\tag{3.39}$$

Similarly, from (3.37) and (3.39), we can prove that

$$\lim_{n\to\infty}\left\|Bx_n-Bp\right\|=\lim_{n\to\infty}\left\|\Psi_1x_n-\Psi_1p\right\|=0.\tag{3.40}$$

On the other hand, let $p \in \Theta$ for each $n \geq 1$; we get $p = T_r^{\Phi_1}(I - r\Psi_1)p$. By Lemma 2.10(iii), that is, $T_r^{\Phi_1}$ is firmly nonexpansive, we obtain

$$\begin{aligned}
&\left\|u_n-p\right\|^2 \\
&\quad=\left\|T_r^{\Phi_1}(I-r\Psi_1)x_n-T_r^{\Phi_1}(I-r\Psi_1)p\right\|^2 \\
&\quad\le\left\langle (I-r\Psi_1)x_n-(I-r\Psi_1)p,u_n-p\right\rangle \\
&\quad=\frac{1}{2}\left\{\left\|(I-r\Psi_1)x_n-(I-r\Psi_1)p\right\|^2+\left\|u_n-p\right\|^2-\left\|(I-r\Psi_1)x_n-(I-r\Psi_1)p-(u_n-p)\right\|^2\right\} \\
&\quad\le\frac{1}{2}\left\{\left\|x_n-p\right\|^2+\left\|u_n-p\right\|^2-\left\|(x_n-u_n)-r(\Psi_1x_n-\Psi_1p)\right\|^2\right\} \\
&\quad\le\frac{1}{2}\left\{\left\|x_n-p\right\|^2+\left\|u_n-p\right\|^2-\|x_n-u_n\|^2+2r\|x_n-u_n\|\left\|\Psi_1x_n-\Psi_1p\right\|-r^2\left\|\Psi_1x_n-\Psi_1p\right\|^2\right\}.
\end{aligned}\tag{3.41}$$

So, we obtain

$$\left\|u_n-p\right\|^2\le\left\|x_n-p\right\|^2-\|x_n-u_n\|^2+2r\|x_n-u_n\|\left\|\Psi_1x_n-\Psi_1p\right\|.\tag{3.42}$$

Observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_E(x_n - \lambda_n Bx_n) - P_E(p - \lambda_n Bp)\|^2 \\
&\le \langle (I - \lambda_n B)x_n - (I - \lambda_n B)p, y_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|(I - \lambda_n B)x_n - (I - \lambda_n B)p\|^2 + \|y_n - p\|^2 \\
&\qquad - \|(I - \lambda_n B)x_n - (I - \lambda_n B)p - (y_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|(x_n - y_n) - \lambda_n(Bx_n - Bp)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 - \lambda_n^2 \|Bx_n - Bp\|^2 \\
&\qquad + 2\lambda_n \langle x_n - y_n, Bx_n - Bp\rangle \Big\},
\end{aligned} \tag{3.43}$$

and hence

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \|x_n - y_n\| \|Bx_n - Bp\|. \tag{3.44}$$

By using the same argument in (3.42) and (3.44), we can prove that

$$\begin{aligned}
\|v_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s\|x_n - v_n\| \|\Psi_2 x_n - \Psi_2 p\|, \\
\|z_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\| \|Cx_n - Cp\|.
\end{aligned} \tag{3.45}$$

Substituting (3.42), (3.44), and (3.45) into (3.33), we obtain

$$\begin{aligned}
\|k_n - p\|^2 &\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\|y_n - p\|^2 + \alpha_n^{(3)}\|z_n - p\|^2 + \alpha_n^{(4)}\|u_n - p\|^2 + \alpha_n^{(5)}\|v_n - p\|^2 \\
&\le \alpha_n^{(1)}\|x_n - p\|^2 + \alpha_n^{(2)}\Big\{ \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\|x_n - y_n\| \|Bx_n - Bp\| \Big\} \\
&\quad + \alpha_n^{(3)}\Big\{ \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\mu_n\|x_n - z_n\| \|Cx_n - Cp\| \Big\} \\
&\quad + \alpha_n^{(4)}\Big\{ \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\| \|\Psi_1 x_n - \Psi_1 p\| \Big\} \\
&\quad + \alpha_n^{(5)}\Big\{ \|x_n - p\|^2 - \|x_n - v_n\|^2 + 2s\|x_n - v_n\| \|\Psi_2 x_n - \Psi_2 p\| \Big\} \\
&= \|x_n - p\|^2 - \alpha_n^{(2)}\|x_n - y_n\|^2 + 2\lambda_n\alpha_n^{(2)}\|x_n - y_n\| \|Bx_n - Bp\| \\
&\quad - \alpha_n^{(3)}\|x_n - z_n\|^2 + 2\mu_n\alpha_n^{(3)}\|x_n - z_n\| \|Cx_n - Cp\| \\
&\quad - \alpha_n^{(4)}\|x_n - u_n\|^2 + 2r\alpha_n^{(4)}\|x_n - u_n\| \|\Psi_1 x_n - \Psi_1 p\| \\
&\quad - \alpha_n^{(5)}\|x_n - v_n\|^2 + 2s\alpha_n^{(5)}\|x_n - v_n\| \|\Psi_2 x_n - \Psi_2 p\|.
\end{aligned} \tag{3.46}$$

From Lemma 2.4, (3.2), and (3.46), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\epsilon_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\epsilon_n A)(k_n-p)\right\|^2\\
&\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\epsilon_n\overline{\gamma})\|k_n-p\|^2\\
&\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2\\
&\quad+(1-\beta_n-\epsilon_n\overline{\gamma})\Big\{\|x_n-p\|^2-\alpha_n^{(2)}\|x_n-y_n\|^2+2\lambda_n\alpha_n^{(2)}\|x_n-y_n\|\|Bx_n-Bp\|\\
&\qquad-\alpha_n^{(3)}\|x_n-z_n\|^2+2\mu_n\alpha_n^{(3)}\|x_n-z_n\|\|Cx_n-Cp\|\\
&\qquad-\alpha_n^{(4)}\|x_n-u_n\|^2+2r\alpha_n^{(4)}\|x_n-u_n\|\|\Psi_1x_n-\Psi_1p\|\\
&\qquad-\alpha_n^{(5)}\|x_n-v_n\|^2+2s\alpha_n^{(5)}\|x_n-v_n\|\|\Psi_2x_n-\Psi_2p\|\Big\}\\
&= \epsilon_n\|\gamma f(x_n)-Ap\|^2+(1-\epsilon_n\overline{\gamma})\|x_n-p\|^2\\
&\quad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(2)}\|x_n-y_n\|^2+2(1-\beta_n-\epsilon_n\overline{\gamma})\lambda_n\alpha_n^{(2)}\|x_n-y_n\|\|Bx_n-Bp\|\\
&\quad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\|x_n-z_n\|^2+2(1-\beta_n-\epsilon_n\overline{\gamma})\mu_n\alpha_n^{(3)}\|x_n-z_n\|\|Cx_n-Cp\|\\
&\quad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|^2+2r(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|\|\Psi_1x_n-\Psi_1p\|\\
&\quad-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}\|x_n-v_n\|^2+2s(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}\|x_n-v_n\|\|\Psi_2x_n-\Psi_2p\|\\
&\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(2)}\|x_n-y_n\|^2\\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\lambda_n\alpha_n^{(2)}\|x_n-y_n\|\|Bx_n-Bp\|-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(3)}\|x_n-z_n\|^2\\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\mu_n\alpha_n^{(3)}\|x_n-z_n\|\|Cx_n-Cp\|-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|^2\\
&\quad+2r(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|\|\Psi_1x_n-\Psi_1p\|-(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}\|x_n-v_n\|^2\\
&\quad+2s(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}\|x_n-v_n\|\|\Psi_2x_n-\Psi_2p\|.
\end{aligned}\tag{3.47}$$

It follows that

$$\begin{aligned}
(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|^2 &\le \epsilon_n\|\gamma f(x_n)-Ap\|^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\lambda_n\alpha_n^{(2)}\|x_n-y_n\|\|Bx_n-Bp\|\\
&\quad+2(1-\beta_n-\epsilon_n\overline{\gamma})\mu_n\alpha_n^{(3)}\|x_n-z_n\|\|Cx_n-Cp\|\\
&\quad+2r(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(4)}\|x_n-u_n\|\|\Psi_1x_n-\Psi_1p\|\\
&\quad+2s(1-\beta_n-\epsilon_n\overline{\gamma})\alpha_n^{(5)}\|x_n-v_n\|\|\Psi_2x_n-\Psi_2p\|.
\end{aligned}\tag{3.48}$$

From (C2), (C6), (3.37), (3.39), (3.40), and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we also have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.49}$$

From (3.47) and by using the same argument above, we can prove that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.50}$$

Applying (3.28), (3.49), and (3.50), we obtain

$$\lim_{n\to\infty} \|k_n - u_n\| = \lim_{n\to\infty} \|k_n - y_n\| = \lim_{n\to\infty} \|k_n - z_n\| = \lim_{n\to\infty} \|k_n - v_n\| = 0. \tag{3.51}$$

Step 4. We claim that $\limsup_{n\to\infty} \langle (A - \gamma f)q, q - x_n\rangle \leq 0$, where $q = P_\Theta(I - A + \gamma f)(q)$ is the unique solution of the variational inequality $\langle (A - \gamma f)q, x - q\rangle \geq 0$, for all $x \in \Theta$.

To show the above inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (A - \gamma f)q, q - x_n\rangle = \lim_{i\to\infty} \langle (A - \gamma f)q, q - x_{n_i}\rangle. \tag{3.52}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $z \in E$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$. We claim that $z \in \Theta$.

That is, we will prove that

$$z \in F(S) \cap \mathrm{VI}(E, C) \cap \mathrm{VI}(E, B) \cap \mathrm{GMEP}(\Phi_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\Phi_2, \varphi, \Psi_2). \tag{3.53}$$

Assume also that $\lambda_n \to \lambda \in [d, 2\beta]$ and $\mu_n \to \mu \in [e, 2\xi]$.

Define a mapping $Q : E \to E$ by

$$\begin{aligned} Qx = {} & \alpha^{(1)} S_k x + \alpha^{(2)} P_E(1 - \lambda B)x + \alpha^{(3)} P_E(1 - \mu C)x + \alpha^{(4)} T_r^{\Phi_1}(I - r\Psi_1)x \\ & + \alpha^{(5)} T_s^{\Phi_2}(I - r\Psi_2)x, \quad \forall x \in E, \end{aligned} \tag{3.54}$$

where $\lim_{n\to\infty}\alpha_n^{(i)} = \alpha^{(i)} \in (0,1)$, where $i = 1,2,3,4,5$. Since $\sum_{i=1}^{5} \alpha_n^{(i)} = 1$ and by Lemma 2.9, we have that Q is nonexpansive and

$$\begin{aligned} F(Q) &= F(S_k) \cap F(P_E(1 - \lambda B)) \cap F(P_E(1 - \mu C)) \cap F\left(T_r^{\Phi_1}(I - r\Psi_1)\right) \cap F\left(T_s^{\Phi_2}(I - r\Psi_2)\right) \\ &= F(S) \cap \mathrm{VI}(E, C) \cap \mathrm{VI}(E, B) \cap \mathrm{GMEP}(\Phi_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\Phi_2, \varphi, \Psi_2). \end{aligned} \tag{3.55}$$

Notice that

$$
\begin{aligned}
&\|Qx_n - x_n\| \\
&\le \|Qx_n - k_n\| + \|k_n - x_n\| \\
&= \Big\| \Big[\alpha^{(1)} S_k x_n + \alpha^{(2)} P_E(1-\lambda B)x_n + \alpha^{(3)} P_E(1-\mu C)x_n + \alpha^{(4)} T_r^{\Phi_1}(I - r\Psi_1)x_n \\
&\qquad + \alpha^{(5)} T_s^{\Phi_2}(I - r\Psi_2)x_n\Big] - \Big[\alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} P_E(1-\lambda_n B)x_n + \alpha_n^{(3)} P_E(1-\mu_n C)x_n \\
&\qquad\qquad + \alpha_n^{(4)} T_r^{\Phi_1}(I - r\Psi_1)x_n + \alpha_n^{(5)} T_s^{\Phi_2}(I - r\Psi_2)x_n\Big] \Big\| + \|k_n - x_n\| \\
&\le \left|\alpha^{(1)} - \alpha_n^{(1)}\right| \|S_k x_n\| + \alpha^{(2)} \|P_E(I-\lambda B)x_n - P_E(I-\lambda_n B)x_n\| + \left|\alpha^{(2)} - \alpha_n^{(2)}\right| \|P_E(I-\lambda_n B)x_n\| \\
&\quad + \alpha^{(3)} \|P_E(I-\mu C)x_n - P_E(I-\mu_n C)x_n\| + \left|\alpha^{(3)} - \alpha_n^{(3)}\right| \|P_E(I-\mu_n C)x_n\| \\
&\quad + \left|\alpha^{(4)} - \alpha_n^{(4)}\right| \left\|T_r^{\Phi_1}(I - r\Psi_1)x_n\right\| + \left|\alpha^{(5)} - \alpha_n^{(5)}\right| \left\|T_s^{\Phi_2}(I - r\Psi_2)x_n\right\| + \|k_n - x_n\| \\
&\le \left|\alpha^{(1)} - \alpha_n^{(1)}\right| \|S_k x_n\| + \alpha^{(2)} |\lambda_n - \lambda| \|Bx_n\| + \left|\alpha^{(2)} - \alpha_n^{(2)}\right| \|P_E(I-\lambda_n B)x_n\| \\
&\quad + \alpha^{(3)} |\mu_n - \mu| \|Cx_n\| + \left|\alpha^{(3)} - \alpha_n^{(3)}\right| \|P_E(I-\mu_n C)x_n\| \\
&\quad + \left|\alpha^{(4)} - \alpha_n^{(4)}\right| \left\|T_r^{\Phi_1}(I - r\Psi_1)x_n\right\| + \left|\alpha^{(5)} - \alpha_n^{(5)}\right| \left\|T_s^{\Phi_2}(I - r\Psi_2)x_n\right\| + \|k_n - x_n\| \\
&\le K_1 \left(\sum_{i=1}^{5} \left|\alpha^{(i)} - \alpha_n^{(i)}\right| + |\lambda_n - \lambda| + |\mu_n - \mu| \right) + \|k_n - x_n\|,
\end{aligned}
\tag{3.56}
$$

where K_1 is an appropriate constant such that

$$
\begin{aligned}
K_1 \ge \max\Bigg\{ &\sup_{n\ge1}\left\{\left\|T_r^{\Phi_1}(I - r\Psi_1)x_n\right\|\right\}, \sup_{n\ge1}\left\{\left\|T_s^{\Phi_2}(I - r\Psi_2)x_n\right\|\right\}, \sup_{n\ge1}\{\|P_E(I-\lambda_n B)x_n\|\}, \\
&\sup_{n\ge1}\{\|P_E(I-\mu_n C)x_n\|\}, \sup_{n\ge1}\{\|Bx_n\|\}, \sup_{n\ge1}\{\|Cx_n\|\}, \sup_{n\ge1}\{\|S_k x_n\|\} \Bigg\}.
\end{aligned}
\tag{3.57}
$$

From (C4), (C6), and (3.28), we obtain

$$
\lim_{n\to\infty} \|x_n - Qx_n\| = 0. \tag{3.58}
$$

Since $P_\Theta(I - A + \gamma f)(q)$ is a contraction with the coefficient $\alpha \in [0,1)$, we have that there exists a unique fixed point. We use q to denote the unique fixed point to the mapping $P_\Theta(I - A + \gamma f)(q)$. That is, $q = P_\Theta(I - A + \gamma f)(q)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$

which converges weakly to z. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup z$. It follows from (3.58), that

$$\lim_{n\to\infty} \|x_{n_i} - Qx_{n_i}\| = 0. \tag{3.59}$$

It follows from Lemma 2.8 that $z \in F(Q)$. By (3.55), we have $z \in \Theta$.

Hence from (3.52) and (2.4), we arrive at

$$\begin{aligned} \limsup_{n\to\infty} \langle (A-\gamma f)q, q-x_n\rangle &= \limsup_{n\to\infty} \langle (A-\gamma f)q, q-x_{n_i}\rangle \\ &= \langle (A-\gamma f)q, q-z\rangle \le 0. \end{aligned} \tag{3.60}$$

On the other hand, we have

$$\begin{aligned} \langle (A-\gamma f)q, q-x_{n+1}\rangle &= \langle (A-\gamma f)q, x_n-x_{n+1}\rangle + \langle (A-\gamma f)q, q-x_n\rangle \\ &\le \|(A-\gamma f)q\| \|x_n-x_{n+1}\| + \langle (A-\gamma f)q, q-x_n\rangle. \end{aligned} \tag{3.61}$$

From (3.25) and (3.60), we obtain that

$$\limsup_{n\to\infty} \langle (A-\gamma f)q, q-x_{n+1}\rangle \le 0. \tag{3.62}$$

Step 5. We claim that $\lim_{n\to\infty}\|x_n - q\| = 0$.

Indeed, by (3.2) and using Lemmas 2.6 and 2.14, we observe that

$$\begin{aligned} \|x_{n+1}-q\|^2 &= \|\epsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \epsilon_n A)k_n - q\|^2 \\ &= \|((1-\beta_n)I-\epsilon_n A)(k_n-q) + \beta_n(x_n-q) + \epsilon_n(\gamma f(x_n) - Aq)\|^2 \\ &\le \left\|(1-\beta_n)\frac{((1-\beta_n)I-\epsilon_n A)}{(1-\beta_n)}(k_n-q) + \beta_n(x_n-q)\right\|^2 \\ &\quad + 2\epsilon_n\langle \gamma f(x_n) - Aq, x_{n+1}-q\rangle \\ &\le (1-\beta_n)\left\|\frac{((1-\beta_n)I-\epsilon_n A)}{1-\beta_n}(k_n-q)\right\|^2 + \beta_n\|x_n-q\|^2 \end{aligned}$$

$$
\begin{aligned}
&+2\epsilon_n\gamma\langle f(x_n)-f(q),x_{n+1}-q\rangle+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\\
&\le(1-\beta_n)\left\|\frac{((1-\beta_n)I-\epsilon_nA)}{1-\beta_n}(k_n-q)\right\|^2+\beta_n\|x_n-q\|^2\\
&\quad+2\epsilon_n\gamma\alpha\|x_n-q\|\|x_{n+1}-q\|+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\\
&\le\frac{\|(1-\beta_n)I-\epsilon_nA\|^2}{1-\beta_n}\|k_n-q\|^2+\beta_n\|x_n-q\|^2\\
&\quad+\epsilon_n\gamma\alpha\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\\
&\le\frac{\|(1-\beta_n)I-\epsilon_nA\|^2}{1-\beta_n}\|x_n-q\|^2+\beta_n\|x_n-q\|^2\\
&\quad+\epsilon_n\gamma\alpha\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\\
&\le\left(\frac{((1-\beta_n)-\overline{\gamma}\epsilon_n)^2}{1-\beta_n}+\beta_n+\epsilon_n\gamma\alpha\right)\|x_n-q\|^2\\
&\quad+\epsilon_n\gamma\alpha\|x_{n+1}-q\|^2+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\\
&\le\left(1-(2\overline{\gamma}-\alpha\gamma)\epsilon_n+\frac{\overline{\gamma}^2\epsilon_n^2}{1-\beta_n}\right)\|x_n-q\|^2\\
&\quad+\epsilon_n\gamma\alpha\|x_{n+1}-q\|^2+2\epsilon_n\langle\gamma f(q)-Aq,x_{n+1}-q\rangle,
\end{aligned}
\tag{3.63}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-q\|^2\le&\left(1-\frac{2(\overline{\gamma}-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n}\right)\|x_n-q\|^2\\
&+\frac{\epsilon_n}{1-\alpha\gamma\epsilon_n}\left\{\frac{\overline{\gamma}^2\epsilon_n}{1-\beta_n}\|x_n-q\|^2+2\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\right\}.
\end{aligned}
\tag{3.64}
$$

Taking

$$
\begin{gathered}
\sigma_n=\frac{\epsilon_n}{1-\alpha\gamma\epsilon_n}\left\{\frac{\overline{\gamma}^2\epsilon_n}{1-\beta_n}\|x_n-q\|^2+2\langle\gamma f(q)-Aq,x_{n+1}-q\rangle\right\}\\
\varrho_n=\frac{2(\overline{\gamma}-\alpha\gamma)\epsilon_n}{1-\alpha\gamma\epsilon_n},
\end{gathered}
\tag{3.65}
$$

then, we can rewrite (3.64) as

$$
\|x_{n+1}-q\|^2\le(1-\varrho_n)\|x_n-q\|^2+\sigma_n,
\tag{3.66}
$$

and we can see that $\sum_{n=1}^{\infty} \varrho_n = \infty$ and $\limsup_{n\to\infty}(\sigma_n/\varrho_n) \leq 0$. Applying Lemma 2.13 to (3.66), we conclude that $\{x_n\}$ converges strongly to q in norm. This completes the proof. □

If the mapping S is nonexpansive, then $S_k = S_0 = S$. We can obtain the following result from Theorem 3.1 immediately.

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let Φ_1 and Φ_2 be two bifunctions from $E \times E$ to $\mathcal{R}$ satisfying (A1)–(A5) and let $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $C : E \to H$ be a ξ-inverse-strongly monotone mapping, let $\Psi_1 : E \to H$ be a ρ-inverse-strongly monotone mapping, let $\Psi_2 : E \to H$ be an ω-inverse-strongly monotone mapping and let $B : E \to H$ be a β-inverse-strongly monotone mapping. Let $f : E \to E$ be an α-contraction with coefficient $\alpha\,(0 \leq \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $S : E \to E$ be a nonexpansive mapping with a fixed point. Suppose that*

$$\Theta := F(S) \cap VI(E,B) \cap VI(E,C) \cap GMEP(\Phi_1, \varphi, \Psi_1) \cap GMEP(\Phi_2, \varphi, \Psi_2) \neq \emptyset. \tag{3.67}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm (3.2), where $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r \in (0, 2\rho)$, $s \in (0, 2\omega)$, and $\{\lambda_n\}$ and $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy (C1)–(C6) in Theorem 3.1. Then, $\{x_n\}$ converges strongly to a point $q \in \Theta$ which is the unique solution of the variational inequality:

$$\langle (A - \gamma f)q, x - q\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.68}$$

Equivalently, one has $q = P_\Theta(I - A + \gamma f)(q)$.

If $\varphi = 0$, $\Psi_1 = \Psi_2 = 0$, $A = I$, $\gamma \equiv 1$, and $\gamma_n = 1 - \epsilon_n - \beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let Φ_1 and Φ_2 be two bifunctions from $E \times E$ to $\mathcal{R}$ satisfying (A1)–(A4). Let $C : E \to H$ be a ξ-inverse-strongly monotone mapping and let $B : E \to H$ be a β-inverse-strongly monotone mapping. Let $f : E \to E$ be an α-contraction with coefficient $\alpha\,(0 \leq \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $S : E \to E$ be a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx + (1-k)Sx$, for all $x \in E$. Suppose that*

$$\Theta := F(S) \cap VI(E,C) \cap VI(E,B) \cap EP(\Phi_1) \cap EP(\Phi_2) \neq \emptyset. \tag{3.69}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$
\begin{gathered}
x_1 \in E, \qquad u_n \in E, \qquad v_n \in E, \\
\Phi_1(u_n, u) + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in E, \\
\Phi_2(v_n, v) + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in E, \\
z_n = P_E(x_n - \mu_n C x_n), \\
y_n = P_E(x_n - \lambda_n B x_n), \\
k_n = \alpha_n^{(1)} S_k x_n + \alpha_n^{(2)} y_n + \alpha_n^{(3)} z_n + \alpha_n^{(4)} u_n + \alpha_n^{(5)} v_n, \\
x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n k_n, \quad \forall n \geq 1,
\end{gathered}
\tag{3.70}
$$

where $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $r \in (0,\infty)$, $s \in (0,\infty)$, and $\{\lambda_n\}$ and $\{\mu_n\}$ are positive sequences. Assume that the control sequences satisfy the condition (C1)–(C6) in Theorem 3.1 and $\epsilon_n + \beta_n + \gamma_n = 1$. Then, $\{x_n\}$ converges strongly to a point $q \in \Theta$, where $q = P_\Theta f(q)$.

If $B = 0$, $C = 0$, and $\Phi_1(u_n, u) = \Phi_1(v_n, v) = 0$ in Corollary 3.3, then $P_E = I$ and we get $u_n = y_n = x_n$ and $v_n = z_n = x_n$; hence we can obtain the following result immediately.

Corollary 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $S : E \to E$ be a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx + (1 - k)Sx$, for all $x \in E$. Suppose that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$
\begin{gathered}
x_1 \in E, \\
k_n = \alpha_n S_k x_n + (1 - \alpha_n) x_n, \\
x_{n+1} = \epsilon_n f(x_n) + \beta_n x_n + \gamma_n k_n, \quad \forall n \geq 1,
\end{gathered}
\tag{3.71}
$$

where $\{\epsilon_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n\}$ are sequences in $(0,1)$. Assume that the control sequences satisfy the conditions (C2) and (C3), $\lim_{n\to\infty}\alpha_n = \alpha \in (0,1)$ in Theorem 3.1, and $\epsilon_n + \beta_n + \gamma_n = 1$. Then, $\{x_n\}$ converges strongly to a point $q \in F(S)$, where $q = P_{F(S)} f(q)$.

Finally, we consider the following *Convex Feasibility Problem* (CFP):

$$
\text{finding an } x \in \bigcap_{i=1}^{M} C_i, \tag{3.72}
$$

where $M \geq 1$ is an integer and each C_i is assumed to be the of solutions of equilibrium problem with the bifunction $\Phi_i, i = 1,2,3,\ldots,M$ and the solution set of the variational inequality problem. There is a considerable investigation on CEP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [38, 39], computer tomography [40], and radiation therapy treatment planning [41].

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Theorem 3.5. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let be a Φ_i bifunction from $E \times E$ to $\mathcal{R}$ satisfying (A1)–(A5) and let $\varphi : E \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $C_i : E \to H$ be an ξ_i-inverse-strongly monotone mapping for each $i \in \{1, 2, 3, \dots, N\}$. Let $f : E \to E$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $S : E \to E$ be a k-strict pseudocontraction with a fixed point. Define a mapping $S_k : E \to E$ by $S_k x = kx + (1-k)Sx$, for all $x \in E$. Suppose that*

$$\mathcal{F} := F(S) \cap \left(\bigcap_{i=1}^{N} VI(E, C_i) \right) \cap \left(\bigcap_{j=1}^{M} MEP(\Phi_j, \varphi, \Psi_j) \right) \neq \emptyset. \tag{3.73}$$

Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{aligned}
&x_1 \in E, \qquad u_M \in E,\\
&\Phi_1(u_{n,N+1}, u_1) + \varphi(x) - \varphi(u_{n,N+1}) + \langle \Psi_1 x_n, u - u_{n,N+1} \rangle + \frac{1}{r_1} \langle u_1 - u_{n,N+1}, u_{n,N+1} - x_n \rangle \geq 0, \quad \forall u_1 \in E,\\
&\Phi_2(u_{n,N+2}, u_2) + \varphi(x) - \varphi(u_{n,N+2}) + \langle \Psi_2 x_n, u - u_{n,N+2} \rangle + \frac{1}{r_2} \langle u_2 - u_{n,N+2}, u_{n,N+2} - x_n \rangle \geq 0, \quad \forall u_2 \in E,\\
&\vdots\\
&\Phi_N(u_{n,M}, u_M) + \varphi(x) - \varphi(u_{n,M}) + \langle \Psi_M x_n, u - u_{n,M} \rangle + \frac{1}{r_N} \langle u_M - u_{n,M}, u_{n,M} - x_n \rangle \geq 0, \quad \forall u_M \in E,\\
&z_{n,1} = P_E(x_n - \mu_{n,1} C_1 x_n),\\
&z_{n,2} = P_E(x_n - \mu_{n,2} C_2 x_n),\\
&\vdots\\
&z_{n,N} = P_E(x_n - \mu_{n,N} C_N x_n),\\
&k_n = \alpha_{n,0} S_k x_n + \sum_{i=1}^{N} \alpha_{n,i} z_{n,i} + \sum_{j=N+1}^{M} \alpha_{n,j} u_{n,j},\\
&x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \epsilon_n A) k_n, \quad \forall n \geq 1,
\end{aligned} \tag{3.74}$$

where $\alpha_{n,0}, \alpha_{n,1}, \alpha_{n,2}, \alpha_{n,3}, \alpha_{n,N}, \dots, \alpha_{n,N+1}, \dots, \alpha_{n,M} \in (0,1)$ such that $\sum_{i=0}^{M} \alpha_{n,i} = 1$, $\{\mu_{n,i}\}$ are positive sequences, and $\{\epsilon_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that the control sequences *satisfy the following restrictions:*

(C1) $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\lim_{n\to\infty} |\mu_{n+1,i} - \mu_{n,i}| = 0$, *for each* $1 \leq i \leq N$,

(C4) $e_i \leq \mu_{n,i} \leq 2\xi_i$, *where* e_i *is some positive constant for each* $1 \leq i \leq N$,

(C5) $\lim_{n\to\infty} \alpha_{n,i} = \alpha_i \in (0,1)$, *for each* $1 \leq i \leq M$.

Then, $\{x_n\}$ *converges strongly to a point* $q \in \mathcal{F}$ *which is the unique solution of the variational inequality:*

$$\langle (A - \gamma f)q, x - q\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{3.75}$$

Equivalently, one has $q = P_{\mathcal{F}}(I - A + \gamma f)(q)$.

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