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On Chebyshev polynomials and their applications

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Abstract
The main purpose of this paper is, using some properties of the Chebyshev polynomials, to study the power sum problems for the $\sin x$ and $\cos x$ functions and to obtain some interesting computational formulas.

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Keywords: Chebyshev polynomials; trigonometric power sums; computational formulas

1 Introduction

As is well known, the Chebyshev polynomials of the first kind $\{T_n(x)\}$ and the Chebyshev polynomials of the second kind $\{U_n(x)\}$ are defined by $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$, $U_1(x) = 2x$ and $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ for all integers $n \geq 0$. If we write $\alpha(x) = x + \sqrt{x^2 - 1}$ and $\beta(x) = x - \sqrt{x^2 - 1}$, then the explicit expressions of $T_n(x)$ and $U_n(x)$ are given by

$$T_n(x) = \frac{1}{2}\bigl(\alpha^n(x) + \beta^n(x)\bigr) = \frac{n}{2}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}, \quad |x| < 1 \tag{1}$$

and

$$U_n(x) = \frac{1}{\alpha - \beta}\bigl(\alpha^{n+1}(x) - \beta^{n+1}(x)\bigr) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k}, \quad |x| < 1. \tag{2}$$

If we take $x = \cos\theta$, then

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{3}$$

These polynomials are very important in different areas of mathematics, and many scholars have studied various properties of them and obtained a series of interesting and important results. Some related papers can be found in [1–13]. Recently, Li Xiaoxue [6] established some identities involving the power sums of $T_n(x)$ and $U_n(x)$, by virtue of which she showed some divisible properties involving the Chebyshev polynomials of the first kind and the second kind, as follows:

$$U_0(x)U_2(x)U_4(x)\cdots U_{2n}(x)\cdot\sum_{m=1}^{h} T_{2m}^{2n+1}(x)\equiv 0 \bmod \bigl(U_{2h}(x)-1\bigr).$$

In a private communication with professor Wenpeng Zhang, he suggested us to give some explicit formulas for the following trigonometric power sums:

$$\sum_{a=0}^{q-1}\cos^{2n}\left(\frac{\pi a}{q}\right)\quad\text{and}\quad\sum_{a=0}^{q-1}\sin^{2n}\left(\frac{\pi a}{q}\right).$$

As far as we know, it seems that nobody has studied these problems yet. In this paper, we shall use the properties of the Chebyshev polynomials of the first kind to obtain some closed formulas for the above trigonometric power sums. These results are stated in the following theorems.

Theorem 1 *Let q and n be positive integers with $q\geq 2$. Then we have*

$$\sum_{a=0}^{q-1}\cos^{2n}\left(\frac{\pi a}{q}\right)=\frac{2q}{4^n}\cdot\sum_{k=0}^{[\frac{n}{q}]}\binom{2n}{n-qk}-\frac{q}{4^n}\cdot\binom{2n}{n},$$

where $\binom{m}{n}=\frac{m!}{n!(m-n)!}$, and $[x]$ denotes the greatest integer $\leq x$.

Theorem 2 *Let q and n be positive integers with $q\geq 2$. Then we have*

$$\sum_{a=0}^{q-1}\sin^{2n}\left(\frac{\pi a}{q}\right)=\frac{2q}{4^n}\sum_{k=0}^{[\frac{n}{q}]}(-1)^{kq}\binom{2n}{n-qk}-\frac{q}{4^n}\cdot\binom{2n}{n}.$$

Theorem 3 *Let q be a positive integer with $2\mid q$. Then, for any non-negative integer n, we have the identity*

$$\sum_{a=0}^{q-1}\sin^{2n+1}\left(\frac{\pi a}{q}\right)=\frac{1}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}\cot\left(\frac{\pi(2k+1)}{2q}\right).$$

It is clear that from Theorem 1 and Theorem 2 we may immediately deduce the following corollary.

Corollary *For any positive integers q and n with $q>n$, we have the identity*

$$\sum_{a=0}^{q-1}\cos^{2n}\left(\frac{\pi a}{q}\right)=\sum_{a=0}^{q-1}\sin^{2n}\left(\frac{\pi a}{q}\right)=\frac{q}{4^n}\cdot\binom{2n}{n}.$$

Remark 1 In order to make our result described in Theorem 3 look simple and nice, we only consider the case $2\mid q$. For the case $2\nmid q$, we can also give a corresponding result, but the formula in this case is much more complicated. In fact, by substituting the cosine function in Theorem 3 for the sine function, we can obtain that, for positive integer q

with $2 \mid q$,

$$\sum_{a=0}^{q-1} \cos^{2n+1}\left(\frac{\pi a}{q}\right) = 1 + \sum_{a=1}^{\frac{q}{2}-1} \cos^{2n+1}\left(\frac{\pi a}{q}\right) + \sum_{a=1}^{\frac{q}{2}-1} \cos^{2n+1}\left(\frac{\pi (q-a)}{q}\right).$$

Remark 2 As was pointed out by the reviewer, the corresponding equivalent versions of Theorem 1 and Theorem 2 also appear in [10], and our methods are completely different from the ones described in [10].

2 Several simple lemmas

To complete the proofs of our theorems, we need some new properties of Chebyshev polynomials, which we summarize as the following lemmas.

Lemma 1 *For any non-negative integer k, we have the identities*

$$T_{2k}(\sin x) = (-1)^k \cos(2kx)$$

and

$$T_{2k+1}(\sin x) = (-1)^k \sin\bigl((2k+1)x\bigr).$$

Proof Taking $x = \sin(\theta)$ in (1) and noting that $x^2 - 1 = -\cos^2(\theta) = i^2 \cos^2(\theta)$, from Euler's formula, we have

$$\begin{aligned}
T_{2k}\bigl(\sin(\theta)\bigr) &= \frac{1}{2}\Bigl[\Bigl(\sin(\theta) + \sqrt{\sin^2(\theta) - 1}\Bigr)^{2k} + \Bigl(\sin(\theta) - \sqrt{\sin^2(\theta) - 1}\Bigr)^{2k}\Bigr] \\
&= \frac{1}{2}\Bigl[\bigl(\sin(\theta) + i\cos(\theta)\bigr)^{2k} + \bigl(\sin(\theta) - i\cos(\theta)\bigr)^{2k}\Bigr] \\
&= \frac{i^{2k}}{2}\Bigl[\bigl(\cos(\theta) - i\sin(\theta)\bigr)^{2k} + \bigl(\cos(\theta) + i\sin(\theta)\bigr)^{2k}\Bigr] \\
&= \frac{(-1)^k}{2}\bigl[\cos(2k\theta) - i\sin(2k\theta) + \cos(2k\theta) + i\sin(2k\theta)\bigr] \\
&= (-1)^k \cdot \cos(2k\theta).
\end{aligned}$$

This proves the first formula of Lemma 1.

Similarly, we also have the identity

$$\begin{aligned}
&T_{2k+1}\bigl(\sin(\theta)\bigr) \\
&\quad= \frac{1}{2}\Bigl[\Bigl(\sin(\theta) + \sqrt{\sin^2(\theta) - 1}\Bigr)^{2k+1} + \Bigl(\sin(\theta) - \sqrt{\sin^2(\theta) - 1}\Bigr)^{2k+1}\Bigr] \\
&\quad= \frac{1}{2}\Bigl[\bigl(\sin(\theta) + i\cos(\theta)\bigr)^{2k+1} + \bigl(\sin(\theta) - i\cos(\theta)\bigr)^{2k+1}\Bigr] \\
&\quad= \frac{1}{2}\Bigl[i^{2k+1}\bigl(\cos(\theta) - i\sin(\theta)\bigr)^{2k+1} + (-i)^{2k+1}\bigl(\cos(\theta) + i\sin(\theta)\bigr)^{2k+1}\Bigr] \\
&\quad= \frac{(-1)^k}{2}\Bigl[i\cos\bigl((2k+1)\theta\bigr) + \sin\bigl((2k+1)\theta\bigr) - i\cos\bigl((2k+1)\theta\bigr) + \sin\bigl((2k+1)\theta\bigr)\Bigr] \\
&\quad= (-1)^k \cdot \sin\bigl((2k+1)\theta\bigr).
\end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2 *For any non-negative integer n, we have the identities*

$$x^{2n} = \frac{\binom{2n}{n}}{4^n} T_0(x) + \frac{2}{4^n} \cdot \sum_{k=1}^{n} \binom{2n}{n-k} \cdot T_{2k}(x)$$

and

$$x^{2n+1} = \frac{1}{4^n} \cdot \sum_{k=0}^{n} \binom{2n+1}{n-k} \cdot T_{2k+1}(x).$$

Proof This is Lemma 4 in Ma Rong and Zhang Wenpeng [7]. □

Lemma 3 *For any non-negative integer k and positive integer q with $q \nmid (2k+1)$, we have the identities*

$$\sum_{a=0}^{q-1} \cos\left(\frac{\pi(2k+1)a}{q}\right) = 1 \quad \textit{and} \quad \sum_{a=0}^{q-1} \sin\left(\frac{\pi(2k+1)a}{q}\right) = \cot\left(\frac{\pi(2k+1)}{2q}\right).$$

Proof Let $e(x) = e^{2\pi i x}$, note that $e(\frac{(2k+1)}{2}) = -1$, then, applying the summation formula for geometric series, we have

$$\begin{aligned} \sum_{a=0}^{q-1} e\left(\frac{(2k+1)a}{2q}\right) &= \frac{-2}{e(\frac{(2k+1)}{2q}) - 1} = \frac{-2e(\frac{-(2k+1)}{4q})}{e(\frac{(2k+1)}{4q}) - e(\frac{-(2k+1)}{4q})} \\ &= \frac{-2\cos(\frac{\pi(2k+1)}{2q}) + 2i\sin(\frac{\pi(2k+1)}{2q})}{2i\sin(\frac{\pi(2k+1)}{2q})} = 1 + i\cot\left(\frac{\pi(2k+1)}{2q}\right). \end{aligned} \tag{4}$$

Applying Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ and comparing the real and imaginary parts in (4), we may immediately deduce Lemma 3. □

3 Proofs of the theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1.

Proof of Theorem 1 Replace x by $\cos(\frac{\pi a}{q})$ in the first formula of Lemma 2 and make the summation for a with $0 \le a \le q-1$, from (3) we have

$$\begin{aligned} \sum_{a=0}^{q-1} \cos^{2n}\left(\frac{\pi a}{q}\right) &= \frac{\binom{2n}{n}}{4^n} \sum_{a=0}^{q-1} 1 + \frac{2}{4^n} \cdot \sum_{k=1}^{n} \binom{2n}{n-k} \cdot \sum_{a=0}^{q-1} T_{2k}\left(\cos\left(\frac{\pi a}{q}\right)\right) \\ &= \frac{\binom{2n}{n}}{4^n} \cdot q + \frac{2}{4^n} \cdot \sum_{k=1}^{n} \binom{2n}{n-k} \cdot \sum_{a=0}^{q-1} \cos\left(\frac{2\pi k a}{q}\right). \end{aligned} \tag{5}$$

Note the trigonometric identity

$$\sum_{a=0}^{q-1} \left(\cos\left(\frac{2\pi n a}{q}\right) + i\sin\left(\frac{2\pi n a}{q}\right)\right) = \sum_{a=0}^{q-1} e\left(\frac{na}{q}\right) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{if } q \nmid n. \end{cases} \tag{6}$$

Combining (5) and (6) we have

$$\begin{aligned}\sum_{a=0}^{q-1}\cos^{2n}\left(\frac{\pi a}{q}\right) &= \frac{\binom{2n}{n}}{4^n}\cdot q+\frac{2}{4^n}\cdot\sum_{\substack{k=1\\ q\mid k}}^{n}\binom{2n}{n-k}\cdot q\\ &= \frac{2q}{4^n}\cdot\sum_{k=0}^{[\frac{n}{q}]}\binom{2n}{n-qk}-q\cdot\frac{\binom{2n}{n}}{4^n}.\end{aligned}$$

This proves Theorem 1. □

Now we prove Theorem 2.

Proof of Theorem 2 Replace x by $\sin(\frac{\pi a}{q})$ in the first formula of Lemma 2 and summarize on $0\le a\le q-1$, from (6) and Lemma 1 we have

$$\begin{aligned}\sum_{a=0}^{q-1}\sin^{2n}\left(\frac{\pi a}{q}\right) &= \frac{\binom{2n}{n}}{4^n}\sum_{a=0}^{q-1}1+\frac{2}{4^n}\cdot\sum_{k=1}^{n}\binom{2n}{n-k}\cdot\sum_{a=0}^{q-1}T_{2k}\left(\sin\left(\frac{\pi a}{q}\right)\right)\\ &= \frac{\binom{2n}{n}}{4^n}\cdot q+\frac{2}{4^n}\cdot\sum_{k=1}^{n}\binom{2n}{n-k}\cdot\sum_{a=0}^{q-1}(-1)^k\cos\left(\frac{2\pi ka}{q}\right)\\ &= \frac{\binom{2n}{n}}{4^n}\cdot q+\frac{2}{4^n}\cdot\sum_{\substack{k=1\\ q\mid k}}^{n}(-1)^k\binom{2n}{n-k}\cdot\sum_{a=0}^{q-1}1\\ &= \frac{2q}{4^n}\sum_{k=0}^{[\frac{n}{q}]}(-1)^{kq}\binom{2n}{n-qk}-\frac{\binom{2n}{n}}{4^n}\cdot q.\end{aligned}$$

This proves Theorem 2. □

We now use the similar methods of proving Theorem 1 and Theorem 2 to complete the proof of Theorem 3.

Proof of Theorem 3 Note that if $2\mid q$, then $q\nmid(2k+1)$ for any non-negative integer k. So, by substituting $\sin(\frac{\pi a}{q})$ for x in the second formula of Lemma 2 and making the summation for a with $0\le a\le q-1$, with the help of Lemma 1 and Lemma 3, we deduce Theorem 3 immediately. □

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Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors read and approved the final manuscript.

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