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Growth of meromorphic solutions of certain types of q-difference differential equations

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Abstract

In this paper, relying on Nevanlinna theory of the value distribution of meromorphic functions, we mainly study meromorphic solutions of certain types of q-difference differential equations, obtain estimates of the growth order of their meromorphic solutions, and give a number of examples to show what our results are the best possible in certain senses. Improvements and extensions of some results in the literature are presented.

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1 Introduction and main results

In this paper, we assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory [1]. In addition, we use notations $\lambda(\frac{1}{f})$ and $\rho(f)$ to denote the exponent of convergence of the pole-sequence and the order of growth of meromorphic function $f(z)$, respectively. We denote by $S(r,f)$ any quantify satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. We define the logarithmic measure of E to be

$$\operatorname{lm}(E) = \int_{E\cap(1,\infty)} \frac{dr}{r}.$$

A set $E \subset (1,\infty)$ is said to have finite logarithmic measure if $\operatorname{lm}(E) < \infty$. Further, we recall the definitions of the truncated exponent of convergence of the pole-sequence and the lower order in complex plane:

$$\overline{\lambda}\left(\frac{1}{f}\right) = \limsup_{r\to\infty} \frac{\log^+ \overline{N}(r,f)}{\log r}, \qquad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

There has been a lot of work on the growth order of meromorphic solution to certain types of complex differential equations and complex difference equations (or complex functional equations); see [2–10]. Malmquist [8] investigated the existence of transcendental meromorphic solutions of a complex differential equation and obtained the following result.

Theorem A ([8]) *Let*

$$\frac{df(z)}{dz} = R\bigl(z,f(z)\bigr) = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{i=0}^{p} a_i(z)f^i(z)}{\sum_{j=0}^{q} b_j(z)f^j(z)}, \tag{1.1}$$

where $P(z,f(z))$ and $Q(z,f(z))$ are relatively prime polynomials in $f(z)$, the coefficients $a_i(z)$ $(i = 0,\dots,p)$ and $b_j(z)$ $(j = 0,\dots,q)$ are rational functions. If equation (1.1) *admits a transcendental meromorphic solution, then $q = 0$ and $p \le 2$.*

Recently, Gundersen et al. [7] considered meromorphic solutions of a functional equation of the form

$$f(qz) = R\bigl(z,f(z)\bigr) = \frac{\sum_{i=0}^{k} a_i(z)f^i(z)}{\sum_{j=0}^{l} b_j(z)f^j(z)}, \tag{1.2}$$

where the coefficients $a_i(z)$ $(i = 0,\dots,k)$ and $b_j(z)$ $(j = 0,\dots,l)$ are of growth $S(r,f)$, and q $(|q| > 1)$ is a constant. In fact, they obtained the following theorem.

Theorem B ([7]) *Suppose that $f(z)$ is a transcendental meromophic of equation* (1.2) *with $|q| > 1$. Assuming that $d := \deg_f R(z,f(z)) = \max\{k,l\} \ge 1$, $a_k(z) \not\equiv 0$, $b_l(z) \not\equiv 0$, and that $R(z,f(z))$ is irreducible in $f(z)$. Then*

$$\rho(f) = \frac{\log d}{\log q}.$$

In this paper, we continue to investigate the growth order of meromorphic solutions to certain types of complex q-difference differential equations and generalize Theorems A and B. Now, we state our results as follows.

Theorem 1.1 *Suppose that $f(z)$ is a solution of the equation*

$$\bigl(f'(qz)\bigr)^n = R\bigl(z,f(z)\bigr) = \frac{\sum_{i=0}^{k} a_i(z)f^i(z)}{\sum_{j=0}^{l} b_j(z)f^j(z)} \tag{1.3}$$

with meromorphic coefficients $a_i(z)$ $(i = 0,\dots,k)$ and $b_j(z)$ $(j = 0,\dots,l)$ of growth $S(r,f)$ and a constant $q \in \mathbb{C}\backslash\{0\}$, assuming that $d := \deg_f R(z,f(z)) = \max\{k,l\} \ge 1$, $a_k(z) \not\equiv 0$, $b_l(z) \not\equiv 0$, and that $R(z,f(z))$ is irreducible in $f(z)$. Then one of the following cases holds.

(i) *For $|q| > 1$, if $f(z)$ is a transcendental meromorphic solution of equation* (1.3) *and $d > 2n$, then*

$$\frac{\log d - \log 2n}{\log |q|} \le \mu(f) \le \rho(f).$$

If $f(z)$ is a transcendental entire solution of equation (1.3) *and $d > n$, then*

$$\frac{\log d - \log n}{\log |q|} \le \mu(f) \le \rho(f).$$

(ii) *For* $|q| < 1$, *if* $f(z)$ *is a transcendental meromorphic solution of equation* (1.3), *then* $d \leq 2n$, *and*

$$\rho(f) \leq \frac{\log 2n - \log d}{-\log |q|}.$$

If $f(z)$ *is a transcendental entire solution of equation* (1.3), *then* $d \leq n$, *and*

$$\rho(f) \leq \frac{\log n - \log d}{-\log |q|}.$$

(iii) *For* $|q| = 1$, *if* $f(z)$ *is a transcendental meromorphic solution of equation* (1.3), *then* $d \leq 2n$. *Furthermore, if* $n < d \leq 2n$, *then* $\overline{\lambda}(\frac{1}{f}) = \lambda(\frac{1}{f}) = \rho(f)$. *If* $f(z)$ *is a transcendental entire solution of equation* (1.3), *then* $d \leq n$.

Example 1.1 The function $f(z) = \frac{e^z+1}{e^z-1}$ is a solution to the q-difference differential equation

$$f'(3z) = -\frac{(f^2(z) - 1)^3}{2(3f^2(z) + 1)^2},$$

where $d = 6$, $n = 1$, $q = 3$, so that $d > 2n$, $|q| > 1$. Then $\frac{\log d - \log 2n}{\log |q|} = 1 = \mu(f) = \rho(f)$.

Example 1.2 The function $f(z) = \cos z$ is a solution to the q-difference differential equation

$$\left(f'(2z)\right)^2 = -4f^4(z) + 4f^2(z),$$

where $d = 4$, $n = 2$, $q = 2$, so that $d > n$, $|q| > 1$. Then $\frac{\log d - \log n}{\log |q|} = 1 = \mu(f) = \rho(f)$.

Example 1.3 The function $f(z) = \frac{e^z}{z+1}$ is a solution to the q-difference differential equation

$$\left(f'(z/3)\right)^3 = \frac{27(z + 1)z^3}{(z + 3)^6} f(z),$$

where $d = 1$, $n = 3$, $q = \frac{1}{3}$, so that $d < 2n$, $|q| < 1$. Then $1 = \rho(f) < \frac{\log 2n - \log d}{-\log |q|} = 1 + \frac{\log 2}{\log 3}$.

Example 1.4 The function $f(z) = e^{z^2} + 1$ is a solution to the q-difference differential equation

$$\left(f'(z/2)\right)^2 = z^4 f(z) - z^4,$$

where $d = 1$, $n = 4$, $q = \frac{1}{2}$, so that $d < n$, $|q| < 1$. Then $\rho(f) = 2 = \frac{\log n - \log d}{-\log |q|}$.

Example 1.5 The function $f(z) = \frac{e^z+1}{z}$ is a solution to the q-difference differential equation

$$f'(-z) = -\frac{f(z) + 1}{z^2 f(z) - z},$$

where $d = 1$, $n = 1$, $q = -1$, so that $n = d < 2n$, $|q| = 1$.

Example 1.6 The function $f(z) = e^z + 1$ is a solution to the q-difference differential equation

$$f'(z) = f(z) - 1,$$

where $d = 1$, $n = 1$, $q = 1$, so that $d = n$, $|q| = 1$.

Theorem 1.2 *Suppose that $f(z)$ is a solution of the equation*

$$\bigl(f'(qz)\bigr)^n = R\bigl(z,f\bigl(p(z)\bigr)\bigr) = \frac{\sum_{i=0}^{k} a_i(z)f^i(p(z))}{\sum_{j=0}^{l} b_j(z)f^j(p(z))} \tag{1.4}$$

with meromorphic coefficients $a_i(z)$ $(i = 0,\dots,k)$ and $b_j(z)$ $(j = 0,\dots,l)$ of growth $S(r,f)$, a constant $q \in \mathbb{C}\backslash\{0\}$, and $p(z) = c_m z^m + c_{m-1}z^{m-1} + \cdots + c_0$, where c_m $(\neq 0)$, $c_{m-1},\dots,c_0$ are complex constants, and m (≥ 2) is an integer. Assume that $d := \deg_f R(z,f(z)) = \max\{k,l\} \geq 1$, $a_k(z) \not\equiv 0$, $b_l(z) \not\equiv 0$, and that $R(z,f(z))$ is irreducible in $f(z)$. Then, if $f(z)$ is a transcendental meromorphic solution of equation (1.4) *and $d \leq 2n$, then*

$$T\bigl(r,f(z)\bigr) = O\bigl((\log r)^{\alpha}\bigr),$$

where

$$\alpha = \frac{\log 2n - \log d}{\log m}.$$

If $f(z)$ is a transcendental entire solution of equation (1.4) *and $d \leq n$, then*

$$T\bigl(r,f(z)\bigr) = O\bigl((\log r)^{\alpha}\bigr),$$

where

$$\alpha = \frac{\log n - \log d}{\log m}.$$

Theorem 1.3 *Suppose that $f(z)$ is a solution of equation*

$$\sum_{s=1}^{n} \alpha_s(z) f^{(\lambda_s)}(q_s z) = R\bigl(z,f(z)\bigr) = \frac{\sum_{i=0}^{k} a_i(z)f^i(z)}{\sum_{j=0}^{l} b_j(z)f^j(z)} \tag{1.5}$$

with meromorphic coefficients $\alpha_s(z)$ $(s = 1,\dots,n)$, $a_i(z)$ $(i = 0,\dots,k)$, and $b_j(z)$ $(j = 0,\dots,l)$ of growth $S(r,f)$, distinct constants q_s with $|q_s| \geq 1$, and finite nonnegative integers λ_s. Suppose that $a_k(z) \not\equiv 0$, $b_l(z) \not\equiv 0$, and $R(z,f(z))$ is irreducible in $f(z)$. Denote

$$d := \deg_f R\bigl(z,f(z)\bigr) = \max\{k,l\} \geq 1; \qquad \lambda = \sum_{s=1}^{n} \lambda_s; \qquad |q| = \max_{1\leq s\leq n}\bigl\{|q_s|\bigr\} > 1.$$

Then if $f(z)$ is a transcendental meromorphic solution of equation (1.5) *and $d > \lambda + n$, then*

$$\frac{\log d - \log(\lambda + n)}{\log |q|} \leq \mu(f) \leq \rho(f).$$

If $f(z)$ is a transcendental entire solution of equation (1.5) *and $d > n$, then*

$$\frac{\log d - \log n}{\log|q|} \leq \mu(f) \leq \rho(f).$$

Example 1.7 The function $f(z) = \frac{1}{e^z+1}$ is a solution to the q-difference differential equation

$$f''(2z) = \frac{-f^2(z)(f(z)-1)^2(2f(z)-1)}{(2f^2(z)-2f(z)+1)^3},$$

where $d = 6$, $n = 1$, $\lambda = 2$, so that $d > \lambda + n = 3$, $|q| = 2 > 1$. Then $1 = \frac{\log d - \log(\lambda+n)}{\log|q|} = \mu(f) = \rho(f)$.

Example 1.8 The function $f(z) = e^z + 1$ is a solution to the q-difference differential equation

$$f'(z) + f''(3z) = f^3(z) - 3f^2(z) + 4f(z) - 2,$$

where $d = 3$, $n = 2$, so that $d > n$, $|q| = \max\{|q_1|, |q_2|\} = 3 > 1$. Then $1 - \frac{\log 2}{\log 3} = \frac{\log d - \log n}{\log|q|} < \mu(f) = \rho(f) = 1$.

Theorem 1.4 *Suppose that $f(z)$ is a solution of the equation*

$$\sum_{s=1}^{n} \alpha_s(z) f^{(\lambda_s)}(q_s z) = R(z, f(p(z))) = \frac{\sum_{i=0}^{k} a_i(z) f^i(p(z))}{\sum_{j=0}^{l} b_j(z) f^j(p(z))} \tag{1.6}$$

with meromorphic coefficients $\alpha_s(z)$ ($s = 1, \ldots, n$), $a_i(z)$ ($i = 0, \ldots, k$), and $b_j(z)$ ($j = 0, \ldots, l$) of growth $S(r,f)$, distinct nonzero constants q_s, finite nonnegative integers λ_s, and $p(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0$, where c_m ($\neq 0$), $c_{m-1}, \ldots, c_0$ are complex constants, and m (≥ 2) is an integer. Suppose that $a_k(z) \not\equiv 0$, $b_l(z) \not\equiv 0$, and $R(z,f(z))$ is irreducible in $f(z)$. Denote

$$d := \deg_f R(z, f(z)) = \max\{k, l\} \geq 1; \qquad \lambda = \sum_{s=1}^{n} \lambda_s; \qquad |q| = \max_{1 \leq s \leq n} \{|q_s|\} > 0.$$

Then if $f(z)$ is a transcendental meromorphic solution of equation (1.6) *and $d \leq \lambda + n$, then*

$$T(r, f(z)) = O((\log r)^\alpha),$$

where

$$\alpha = \frac{\log(\lambda + n) - \log d}{\log m}.$$

If $f(z)$ is a transcendental entire solution of equation (1.6) *and $d \leq n$, then*

$$T(r, f(z)) = O((\log r)^\alpha),$$

where

$$\alpha = \frac{\log n - \log d}{\log m}.$$

Beardon [3] studied entire solutions of the generalized function equation

$$f(qz) = qf(z)f'(z), \quad f(0) = 0, \tag{1.7}$$

where q is a nonzero complex number. First, we give some notations. The formal series $\mathcal{O}$ and $\mathcal{I}$ are defined by $\mathcal{O} := 0 + 0z + 0z^2 + \cdots$ and $\mathcal{I} := 0 + 1z + 0z^2 + 0z^3 + \cdots$. We also introduce the sets $\mathcal{K}_p = \{z : z^p = p + 2\}$ ($p = 1, 2, \ldots$) and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots$. Clearly, $\mathcal{K}_p$ contains exactly p points, which are equally spaced around the circle $|z| = r_p$, where $r_p = (p+2)^{\frac{1}{p}} > 1$ and $r_p \in \mathcal{K}_p$. Also, since $x^{-1}\log(x+2)$ is decreasing when $x > 1$. we see that $r_1 = 3 > r_2 = 2 > \cdots > 1$ and $r_p \to 1$ as $p \to \infty$. Based on these notations, Beardon obtained the following two main theorems.

Theorem C ([3]) *Any transcendental solution of* (1.7) *is of the form*

$$f(z) = z + z\bigl(bz^p + \cdots\bigr),$$

where p is a positive integer, $b \neq 0$, and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (1.7) *are $\mathcal{O}$ and $\mathcal{I}$.*

Theorem D ([3]) *For each positive integer p, there is a unique real entire function*

$$F_p = z\bigl(1 + z^p + b_2 z^{2p} + b_3 z^{3p} + \cdots\bigr)$$

that is a solution of (1.7) *for each q in $\mathcal{K}_p$. Further, if $q \in \mathcal{K}_p$, then the only transcendental solutions of* (1.7) *are the linear conjugates of F_p.*

More recently, Zhang [10] investigated the growth of solutions of (1.7) and obtained the following theorem.

Theorem E ([10]) *Suppose that $f(z)$ is a transcendental solution of* (1.7) *for $k \in \mathcal{K}$. Then the order of growth $\rho(f) \leq \frac{\log 2}{\log |q|}$.*

In this paper, we generalize equation (1.7) and investigate the growth of solution of certain types of q-difference differential equations and obtain the following results.

Theorem 1.5 *Let q be a complex constant satisfying $|q| > 1$. Suppose that $f(z)$ is a solution to the equation*

$$\sum_{s=1}^{n} \alpha_s(z) f^{(\lambda_s)}(z) = \frac{A(qz, f(qz))}{B(z, f(z))}, \tag{1.8}$$

where $A(z,y)$ and $B(z,y)$ are rational functions with meromorphic coefficients of growth $S(r,f)$ such that $A(z,y)$ and $B(z,y)$ are irreducible in y. Denote

$$\lambda = \sum_{s=1}^{n} \lambda_s; \qquad 1 \leq a := \deg_f A \leq \deg_f B =: b.$$

Then,

(i) *if $f(z)$ is a transcendental meromorphic solution of equation* (1.8), *then*

$$\rho(f) \leq \frac{\log(b+\lambda+n)-\log a}{\log|q|}.$$

Furthermore, if $b > a+\lambda+n$, then

$$\frac{\log(b-\lambda-n)-\log a}{\log|q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(b+\lambda+n)-\log a}{\log|q|}.$$

(ii) *If $f(z)$ is a transcendental entire solution of equation* (1.8), *then*

$$\rho(f) \leq \frac{\log(b+n)-\log a}{\log|q|}.$$

Furthermore, if $b > a+n$, then

$$\frac{\log(b-n)-\log a}{\log|q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(b+n)-\log a}{\log|q|}.$$

Example 1.9 The function $f(z) = \tan z$ is a solution to the q-difference differential equation

$$f''(z) = \frac{f^2(2z)}{\frac{2f(z)}{f^6(z)-f^4(z)-f^2(z)+1}},$$

where $a = 2$, $b = 6$, $n = 1$, $q = 2$, $\lambda = 2$, so that $b > a+\lambda+n = 5$. Then $\frac{\log 3}{\log 2} - 1 = \frac{\log(b-\lambda-n)-\log a}{\log|q|} < \mu(f) = \rho(f) = 1 < \frac{\log(b+\lambda+n)-\log a}{\log|q|} = 2\frac{\log 3}{\log 2} - 1$.

Example 1.10 The function $f(z) = ze^z$ is a solution to the q-difference differential equation

$$f'(z) + f''(z) = \frac{f^2(3z)+f(3z)}{\frac{9f^5(z)}{(2z+3)z^3} + \frac{3f^2(z)}{(2z+3)z}},$$

where $a = 2$, $b = 5$, $n = 2$, $q = 3$, so that $b > a+n = 4$. Then $1 - \frac{\log 2}{\log 3} = \frac{\log(b-n)-\log a}{\log|q|} < \mu(f) = \rho(f) = 1 < \frac{\log(b+n)-\log a}{\log|q|} = \frac{\log 7-\log 2}{\log 3}$.

Theorem 1.6 *Let q be a complex constant satisfying $|q| > 1$. Suppose that $f(z)$ is a solution to the equation*

$$\bigl(f'(z)\bigr)^n = \frac{A(qz,f(qz))}{B(z,f(z))}, \tag{1.9}$$

where $A(z,y)$ and $B(z,y)$ are rational functions with meromorphic coefficients of growth $S(r,f)$ such that $A(z,y)$ and $B(z,y)$ are irreducible in y. Denote $1 \leq a := \deg_f A \leq \deg_f B =: b$. Then,

(i) *if $f(z)$ is a transcendental meromorphic solution of equation* (1.9), *then*

$$\rho(f) \leq \frac{\log(b+2n)-\log a}{\log|q|}.$$

Furthermore, if $b > a + 2n$, *then*

$$\frac{\log(b-2n)-\log a}{\log|q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(b+2n)-\log a}{\log|q|}.$$

(ii) *If* $f(z)$ *is a transcendental entire solution of equation* (1.9), *then*

$$\rho(f) \leq \frac{\log(b+n)-\log a}{\log|q|}.$$

Furthermore, if $b > a + n$, *then*

$$\frac{\log(b-n)-\log a}{\log|q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(b+n)-\log a}{\log|q|}.$$

Example 1.11 The function $f(z) = \tan z$ is a solution to the q-difference differential equation

$$\left(f'(z)\right)^2 = \frac{f(2z)+1}{\frac{f^2(z)-2f(z)-1}{f^6(z)+f^4(z)-f^2(z)-1}},$$

where $a = 1$, $b = 6$, $n = 2$, $q = 2$, so that $b > a + 2n = 5$. Then $1 = \frac{\log(b-2n)-\log a}{\log|q|} = \mu(f) = \rho(f) < \frac{\log(b+2n)-\log a}{\log|q|} = 1 + \frac{\log 5}{\log 2}$.

Example 1.12 The function $f(z) = ze^z$ is a solution to the q-difference differential equation

$$\left(f'(z)\right)^2 = \frac{f^2(4z)+f(4z)}{\frac{16f^6(z)}{(z+1)^2z^4} + \frac{4f^2(z)}{z(z+1)^2}},$$

where $a = 2$, $b = 6$, $n = 2$, $q = 4$, so that $b > a + n = 4$. Then $\frac{1}{2} = \frac{\log(b-n)-\log a}{\log|q|} < \mu(f) = \rho(f) = \frac{\log(b+n)-\log a}{\log|q|} = 1$.

2 Some lemmas

Lemma 2.1 (See [4], Lemma 4) *Let* $f(z)$ *be a transcendental meromorphic function, and* $p(z) = a_kz^k + a_{k-1}z^{k-1} + \cdots + a_1z + a_0$ $(a_k \neq 0)$ *be a nonconstant polynomial of degree* k. *Given* $0 < \delta < |a_k|$, *let* $\lambda = |a_k| + \delta$ *and* $\mu = |a_k| - \delta$, *then, for any given* $\varepsilon > 0$,

$$(1-\varepsilon)T\left(\mu r^k, f(z)\right) \leq T\left(r, f\left(p(z)\right)\right) \leq (1+\varepsilon)T\left(\lambda r^k, f(z)\right)$$

for sufficiently large r.

Lemma 2.2 (See [7], Lemma 3.1) *Let* $\Phi : (1,\infty) \to (0,\infty)$ *be an increasing function, and let* $f(z)$ *be a nonconstant meromorphic function. If for some real constant* $\alpha \in (0,1)$, *there exist real constants* $K_1 > 0$ *and* $K_2 \geq 1$ *such that*

$$T\left(r, f(z)\right) \leq K_1\Phi(r) + K_2T\left(\alpha r, f(z)\right) + S(\alpha r, f),$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log\alpha} + \limsup_{r\to\infty}\frac{\log\Phi(r)}{\log r}.$$

Lemma 2.3 (See [9], Lemma 2.2) *Let $\Phi : (r_0, \infty) \to (1, \infty)$, where $r_0 \geq 1$, be an increasing function. If for some real constant $\alpha > 1$, there exists a real number $K > 1$ such that $\Phi(\alpha r) > K\Phi(r)$, then*

$$\liminf_{r\to\infty} \frac{\log \Phi(r)}{\log r} \geq \frac{\log K}{\log \alpha}.$$

Lemma 2.4 (See [5], Lemma 3) *Let $\Psi(r)$ be a function of r ($r \geq r_0$), positive and bounded in every finite interval. Suppose that $\Psi(\mu r^m) \leq A\Psi(r) + B(r \geq r_0)$, where μ (> 0), m (> 1), A (≥ 1), and B are constants. Then $\Psi(r) = O((\log r)^\alpha)$ with $\alpha = \frac{\log A}{\log m}$, unless $A = 1$ and $B > 0$; and if $A = 1$ and $B > 0$, then, for any $\varepsilon > 0$, $\Psi(r) = O((\log r)^\varepsilon)$.*

The following lemma is proved by Bergweiler et al. [2], p. 2.

Lemma 2.5

$$T\big(r,f(qz)\big) = T\big(|q|r,f(z)\big) + O(1), \qquad \overline{N}\big(r,f(qz)\big) = \overline{N}\big(|q|r,f(z)\big) + O(1)$$

for any meromorphic function $f(z)$ and any nonzero constant q.

3 Proof of Theorems 1.1-1.2

Proof of Theorem 1.1 If $|q| > 1$ and $f(z)$ is a transcendental meromorphic solution of (1.3), then by applying the Valiron-Mohon'ko identity (see [11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.3) that

$$\begin{aligned} T\big(r,R\big(z,f(z)\big)\big) &= T\left(r, \frac{\sum_{i=0}^{k} a_i(z)f^i(z)}{\sum_{j=0}^{l} b_j(z)f^j(z)}\right) \\ &= dT\big(r,f(z)\big) + S(r,f) \\ &= T\big(r, \big(f'(qz)\big)^n\big) \\ &\leq n\big[T\big(r,f(qz)\big) + \overline{N}\big(r,f(qz)\big) + S\big(r,f(qz)\big)\big] \\ &= n\big[T\big(|q|r,f(z)\big) + \overline{N}\big(|q|r,f(z)\big) + S\big(|q|r,f(z)\big)\big] \\ &\leq 2nT\big(|q|r,f(z)\big) + S\big(|q|r,f(z)\big), \end{aligned}$$

that is,

$$dT\big(r,f(z)\big) + S(r,f) \leq 2nT\big(|q|r,f(z)\big) + S\big(|q|r,f(z)\big). \tag{3.1}$$

By (3.1), for any small $\varepsilon > 0$,

$$d(1-\varepsilon)T\big(r,f(z)\big) \leq 2n(1+\varepsilon)T\big(|q|r,f(z)\big) \tag{3.2}$$

for sufficiently large $r \notin E$, where $\mathrm{lm}(E) < \infty$. By an application of [6], Lemma 5, with $\beta > 1$ and (3.2) we see that

$$d(1-\varepsilon)T\big(r,f(z)\big) \leq 2n(1+\varepsilon)T\big(\beta|q|r,f(z)\big) \tag{3.3}$$

for all $r \geq r_0$. If $d \leq 2n$, then since $\beta|q| > 1$, estimate (3.3) is trivial. So we only have to consider the case where $d > 2n$. Then $\frac{d(1-\varepsilon)}{2n(1+\varepsilon)} > 1$. It follows from Lemmas 2.3 and (3.3) that

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} \geq \frac{\log(d(1-\varepsilon)) - \log(2n(1+\varepsilon))}{\log \beta|q|}.$$

As $\varepsilon \to 0^+$ and $\beta \to 1^+$, we have

$$\rho(f) \geq \mu(f) \geq \frac{\log d - \log 2n}{\log |q|}.$$

If $|q| > 1$ and $f(z)$ is a transcendental entire solution of (1.3), then similarly to (3.1), for any small $\varepsilon > 0$, we have

$$d(1-\varepsilon)T\bigl(r,f(z)\bigr) \leq n(1+\varepsilon)T\bigl(|q|r,f(z)\bigr) \tag{3.4}$$

for sufficiently large $r \notin E$, where $\operatorname{lm}(E) < \infty$. If $d > n$, similarly to the previous argument, we conclude that

$$\rho(f) \geq \mu(f) \geq \frac{\log d - \log n}{\log |q|}.$$

If $|q| < 1$ and $f(z)$ is a transcendental meromorphic solution of (1.3), then applying [6], Lemma 5, and (3.2), we obtain that there exists $\alpha > 1$ such that

$$\alpha|q| < 1 \quad \text{and} \quad d(1-\varepsilon)T\bigl(r,f(z)\bigr) \leq 2n(1+\varepsilon)T\bigl(\alpha|q|r,f(z)\bigr) \tag{3.5}$$

for all $r \geq r_0$. Since $\alpha|q| < 1$, if $d > 2n$, then $\frac{2n(1+\varepsilon)}{d(1-\varepsilon)} < 1$, a contradiction to (3.5). Thus, we have $d \leq 2n$. Then $\frac{2n(1+\varepsilon)}{d(1-\varepsilon)} > 1$, and from Lemma 2.2 we have that

$$\rho(f) \leq \frac{\log(2n(1+\varepsilon)) - \log(d(1-\varepsilon))}{-\log \alpha|q|}.$$

As $\varepsilon \to 0^+$ and $\alpha \to 1^+$, we have

$$\rho(f) \leq \frac{\log 2n - \log d}{-\log |q|}.$$

If $|q| < 1$ and $f(z)$ is a transcendental entire solution of (1.3), then similarly to the previous argument, we have

$$d \leq n \quad \text{and} \quad \rho(f) \leq \frac{\log n - \log d}{-\log |q|}.$$

If $|q| = 1$ and $f(z)$ is a transcendental meromorphic solution of (1.3), then by the proof of (3.1) we conclude that

$$\begin{aligned} dT\bigl(r,f(z)\bigr) + S(r,f) &\leq n\bigl[T\bigl(r,f(z)\bigr) + \overline{N}\bigl(r,f(z)\bigr) + S(r,f)\bigr] \\ &\leq 2nT\bigl(r,f(z)\bigr) + S(r,f). \end{aligned}$$

From this inequality we have $d \leq 2n$. If $n < d \leq 2n$, then

$$\begin{aligned}\frac{d-n}{n}T\bigl(r,f(z)\bigr)+S(r,f) &\leq \overline{N}\bigl(r,f(z)\bigr)+S(r,f)\\ &\leq N\bigl(r,f(z)\bigr)+S(r,f)\\ &\leq T\bigl(r,f(z)\bigr)+S(r,f),\end{aligned}$$

that is, $\overline{\lambda}(\frac{1}{f}) = \lambda(\frac{1}{f}) = \rho(f)$. If $|q| = 1$ and $f(z)$ is a transcendental entire solution of (1.3), then we similarly obtain that $d \leq n$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 If $f(z)$ is a transcendental meromorphic solution of (1.4), then by the Valiron-Mohon'ko identity ([11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.4) that

$$\begin{aligned}T\bigl(r,R\bigl(z,f\bigl(p(z)\bigr)\bigr)\bigr) &= T\left(r,\frac{\sum_{i=0}^{k}a_i(z)f^i(p(z))}{\sum_{j=0}^{l}b_j(z)f^j(p(z))}\right)\\ &= dT\bigl(r,f\bigl(p(z)\bigr)\bigr)+S\bigl(r,f\bigl(p(z)\bigr)\bigr)\\ &= T\bigl(r,\bigl(f'(qz)\bigr)^n\bigr)\\ &\leq n\bigl[T\bigl(r,f(qz)\bigr)+\overline{N}\bigl(r,f(qz)\bigr)+S\bigl(r,f(qz)\bigr)\bigr]\\ &= n\bigl[T\bigl(|q|r,f(z)\bigr)+\overline{N}\bigl(|q|r,f(z)\bigr)+S\bigl(|q|r,f(z)\bigr)\bigr]\\ &\leq 2nT\bigl(|q|r,f(z)\bigr)+S\bigl(|q|r,f(z)\bigr),\end{aligned}$$

that is,

$$dT\bigl(r,f\bigl(p(z)\bigr)\bigr)+S\bigl(r,f\bigl(p(z)\bigr)\bigr) \leq 2nT\bigl(|q|r,f(z)\bigr)+S\bigl(|q|r,f(z)\bigr). \tag{3.6}$$

By Lemma 2.1, for given $0 < \delta < |c_m|$ and $\mu = |c_m| - \delta$ and for any small $\varepsilon > 0$,

$$d(1-\varepsilon)T\bigl(\mu r^m,f(z)\bigr) \leq 2n(1+\varepsilon)T\bigl(|q|r,f(z)\bigr) \tag{3.7}$$

for sufficiently large $r \notin E$, where $\operatorname{lm}(E) < \infty$. An application of [6], Lemma 5, with $\beta > 1$ and (3.7) yields

$$d(1-\varepsilon)T\bigl(\mu r^m,f(z)\bigr) \leq 2n(1+\varepsilon)T\bigl(\beta|q|r,f(z)\bigr)$$

for $r \geq r_0$. Put $R = \beta|q|r$. Then the last inequality can be rewritten as

$$T\left(\frac{\mu R^m}{\beta^m|q|^m},f(z)\right) \leq \frac{2n(1+\varepsilon)}{d(1-\varepsilon)}T\bigl(R,f(z)\bigr). \tag{3.8}$$

If $d \leq 2n$, then $\frac{2n(1+\varepsilon)}{d(1-\varepsilon)} \geq 1$. Since $\frac{\mu}{\beta^m|q|^m} > 0$, $m \geq 2$, by Lemma 2.4 we get that

$$T\bigl(r,f(z)\bigr) = O\bigl((\log r)^\alpha\bigr),$$

where

$$\begin{aligned}\alpha &= \frac{\log(2n(1+\varepsilon)) - \log(d(1-\varepsilon))}{\log m} \\ &= \frac{\log 2n - \log d}{\log m} + \frac{\log(1+\varepsilon) - \log(1-\varepsilon)}{\log m} \\ &\to \frac{\log 2n - \log d}{\log m} \quad (\varepsilon \to 0).\end{aligned}$$

If $f(z)$ is a transcendental entire solution of (1.4) and $d \leq n$, then we similarly have

$$T\bigl(r, f(z)\bigr) = O\bigl((\log r)^{\alpha}\bigr),$$

where

$$\alpha = \frac{\log n - \log d}{\log m}.$$

This completes the proof of Theorem 1.2. □

4 Proof of Theorems 1.3-1.4

Proof of Theorem 1.3 If $f(z)$ is a transcendental meromorphic solution of (1.5), then by applying the Valiron-Mohon'ko identity ([11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.5), $|q_s| > 1$, and $|q| = \max_{1\leq s\leq n}\{|q_s|\} > 1$ that

$$\begin{aligned}T\bigl(r, R\bigl(z, f(z)\bigr)\bigr) &= T\left(r, \frac{\sum_{i=0}^{k} a_i(z) f^i(z)}{\sum_{j=0}^{l} b_j(z) f^j(z)}\right) \\ &= dT\bigl(r, f(z)\bigr) + S(r,f) \\ &= T\left(r, \sum_{s=1}^{n} \alpha_s(z) f^{(\lambda_s)}(q_s z)\right) \\ &\leq \sum_{s=1}^{n} T\bigl(r, f^{(\lambda_s)}(q_s z)\bigr) + S(r,f) \\ &\leq \sum_{s=1}^{n} \bigl[T\bigl(r, f(q_s z)\bigr) + \lambda_s \overline{N}\bigl(r, f(q_s z)\bigr) + S\bigl(r, f(q_s z)\bigr)\bigr] + S(r,f) \\ &= \sum_{s=1}^{n} \bigl[T\bigl(|q_s| r, f(z)\bigr) + \lambda_s \overline{N}\bigl(|q_s| r, f(z)\bigr) + S\bigl(|q_s| r, f(z)\bigr)\bigr] + S(r,f) \\ &\leq \sum_{s=1}^{n} \bigl[(1+\lambda_s) T\bigl(|q_s| r, f(z)\bigr) + S\bigl(|q_s| r, f(z)\bigr)\bigr] + S(r,f) \\ &\leq \sum_{s=1}^{n} (1+\lambda_s) T\bigl(|q| r, f(z)\bigr) + S\bigl(|q| r, f(z)\bigr) + S(r,f) \\ &= (\lambda + n) T\bigl(|q| r, f(z)\bigr) + S\bigl(|q| r, f(z)\bigr) + S(r,f),\end{aligned}$$

that is,

$$dT\big(r,f(z)\big) + S(r,f) \leq (\lambda + n)T\big(|q|r,f(z)\big) + S\big(|q|r,f(z)\big) \tag{4.1}$$

for sufficiently large $r \notin E$, where $\mathrm{lm}(E) < \infty$. By using (4.1) and [6], Lemma 5, with $\beta > 1$, for any given $\varepsilon > 0$, we have

$$d(1-\varepsilon)T\big(r,f(z)\big) \leq (\lambda + n)(1+\varepsilon)T\big(\beta|q|r,f(z)\big) \tag{4.2}$$

for all $r \geq r_0$. Since $\beta|q| > 1$, if $d \leq \lambda + n$, then estimate (4.2) is trivial. So we only have to consider the case where $d > \lambda + n$. Then $\frac{d(1-\varepsilon)}{(\lambda+n)(1+\varepsilon)} > 1$. It follows from Lemma 2.3 and (4.2) that

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} \geq \frac{\log(d(1-\varepsilon)) - \log((\lambda+n)(1+\varepsilon))}{\log \beta|q|}.$$

As $\varepsilon \to 0^+$ and $\beta \to 1^+$, we have

$$\rho(f) \geq \mu(f) \geq \frac{\log d - \log(\lambda + n)}{\log|q|}.$$

Similarly, if $f(z)$ is a transcendental entire solution of (1.5) and $d > n$, we have

$$\rho(f) \geq \mu(f) \geq \frac{\log d - \log n}{\log|q|}.$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4 If $f(z)$ is a transcendental meromorphic solution of (1.6), similarly to (4.1), by applying the Valiron-Mohon'ko identity ([11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.6) and $|q| = \max_{1\leq s\leq n}\{|q_s|\} > 0$ that

$$dT\big(r,f\big(p(z)\big)\big) + S\big(r,f\big(p(z)\big)\big) \leq (\lambda + n)T\big(|q|r,f(z)\big) + S\big(|q|r,f(z)\big). \tag{4.3}$$

By Lemma 2.1, for given $0 < \delta < |c_m|$ and $\mu = |c_m| - \delta$ and for any small $\varepsilon > 0$,

$$d(1-\varepsilon)T\big(\mu r^m,f(z)\big) \leq (\lambda + n)(1+\varepsilon)T\big(|q|r,f(z)\big) \tag{4.4}$$

for sufficiently large $r \notin E$, where $\mathrm{lm}(E) < \infty$. An application of [6], Lemma 5, with $\beta > 1$ and (4.4) yields

$$d(1-\varepsilon)T\big(\mu r^m,f(z)\big) \leq (\lambda + n)(1+\varepsilon)T\big(\beta|q|r,f(z)\big) \tag{4.5}$$

for $r \geq r_0$. Set $R = \beta|q|r$. Then (4.5) can be rewritten as

$$T\left(\frac{\mu R^m}{\beta^m|q|^m},f(z)\right) \leq \frac{(\lambda+n)(1+\varepsilon)}{d(1-\varepsilon)}T\big(R,f(z)\big). \tag{4.6}$$

If $d \leq \lambda + n$, then $\frac{(\lambda+n)(1+\varepsilon)}{d(1-\varepsilon)} \geq 1$. Since $\frac{\mu}{\beta^m |q|^m} > 0$, $m \geq 2$, from Lemma 2.4 we get that

$$T\bigl(r, f(z)\bigr) = O\bigl((\log r)^\alpha\bigr),$$

where

$$\begin{aligned}\alpha &= \frac{\log((\lambda+n)(1+\varepsilon)) - \log(d(1-\varepsilon))}{\log m} \\ &= \frac{\log(\lambda+n) - \log d}{\log m} + \frac{\log(1+\varepsilon) - \log(1-\varepsilon)}{\log m} \\ &\to \frac{\log(\lambda+n) - \log d}{\log m} \quad (\varepsilon \to 0).\end{aligned}$$

If $f(z)$ is a transcendental entire solution of (1.6) and $d \leq n$, we similarly have

$$T\bigl(r, f(z)\bigr) = O\bigl((\log r)^\alpha\bigr),$$

where

$$\alpha = \frac{\log n - \log d}{\log m}.$$

This completes the proof of Theorem 1.4. □

5 Proof of Theorems 1.5-1.6

Proof of Theorem 1.5 If $f(z)$ is a transcendental meromorphic solution of (1.8), by applying the Valiron-Mohon'ko identity ([11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.8) that

$$\begin{aligned}T\bigl(r, A\bigl(qz, f(qz)\bigr)\bigr) &= aT\bigl(r, f(qz)\bigr) + S\bigl(r, f(qz)\bigr) \\ &= aT\bigl(|q|r, f(z)\bigr) + S\bigl(|q|r, f(z)\bigr) \\ &= T\left(r, B\bigl(z, f(z)\bigr) \sum_{s=1}^{n} \alpha_s(z) f^{(\lambda_s)}(z)\right) \\ &\leq T\bigl(r, B\bigl(z, f(z)\bigr)\bigr) + T\left(r, \sum_{s=1}^{n} f^{(\lambda_s)}(z)\right) + S(r, f) \\ &\leq bT\bigl(r, f(z)\bigr) + \sum_{s=1}^{n} \bigl[T\bigl(r, f(z)\bigr) + \lambda_s \overline{N}\bigl(r, f(z)\bigr) + S(r, f)\bigr] + S(r, f) \\ &\leq bT\bigl(r, f(z)\bigr) + \sum_{s=1}^{n} (1 + \lambda_s) T\bigl(r, f(z)\bigr) + S(r, f) \\ &= bT\bigl(r, f(z)\bigr) + (\lambda + n) T\bigl(r, f(z)\bigr) + S(r, f) \\ &= (b + \lambda + n) T\bigl(r, f(z)\bigr) + S(r, f),\end{aligned}$$

that is,

$$aT\bigl(|q|r, f(z)\bigr) + S\bigl(|q|r, f(z)\bigr) \leq (b + \lambda + n) T\bigl(r, f(z)\bigr) + S(r, f),$$

that is,

$$aT\bigl(r,f(z)\bigr) + S(r,f) \le (b+\lambda+n)T\left(\frac{r}{|q|},f(z)\right) + S\left(\frac{r}{|q|},f\right) \tag{5.1}$$

for sufficiently large $r \notin E$, where $\operatorname{lm}(E) < \infty$. By using [6], Lemma 5, and $\frac{1}{|q|} < 1$, from (5.1) we obtain that there exists $\beta > 1$ such that

$$\frac{\beta}{|q|} < 1 \quad \text{and} \quad a(1-\varepsilon)T\bigl(r,f(z)\bigr) \le (b+\lambda+n)(1+\varepsilon)T\left(\frac{\beta r}{|q|},f(z)\right)$$

for all $r \ge r_0$. Since $1 \le a \le b$, then $\frac{(b+\lambda+n)(1+\varepsilon)}{a(1-\varepsilon)} \ge 1$, and from Lemma 2.2 we get that

$$\rho(f) \le \frac{\log((b+\lambda+n)(1+\varepsilon)) - \log(a(1-\varepsilon))}{-\log\frac{\beta}{|q|}}.$$

As $\varepsilon \to 0^+$ and $\beta \to 1^+$, we have

$$\rho(f) \le \frac{\log(b+\lambda+n) - \log a}{\log|q|}.$$

On the other hand, by applying the Valiron-Mohon'ko identity ([11], Theorem 2.2.5), Lemma 2.5, and [1], Theorem 3.1, it follows from (1.8) that

$$\begin{aligned}
T\bigl(r,B\bigl(z,f(z)\bigr)\bigr) &= bT\bigl(r,f(z)\bigr) + S(r,f) \\
&= T\left(r,\frac{A(qz,f(qz))}{\sum_{s=1}^{n}\alpha_s(z)f^{(\lambda_s)}(z)}\right) \\
&\le T\bigl(r,A\bigl(qz,f(qz)\bigr)\bigr) + T\left(r,\sum_{s=1}^{n}\alpha_s(z)f^{(\lambda_s)}(z)\right) + O(1) \\
&\le aT\bigl(r,f(qz)\bigr) + T\left(r,\sum_{s=1}^{n}f^{(\lambda_s)}(z)\right) + S\bigl(r,f(qz)\bigr) + S(r,f) \\
&\le aT\bigl(|q|r,f(z)\bigr) + \sum_{s=1}^{n}\bigl[T\bigl(r,f(z)\bigr) + \lambda_s\overline{N}\bigl(r,f(z)\bigr) + S(r,f)\bigr] + S\bigl(|q|r,f\bigr) \\
&\le aT\bigl(|q|r,f(z)\bigr) + \sum_{s=1}^{n}(1+\lambda_s)T\bigl(r,f(z)\bigr) + S\bigl(|q|r,f\bigr) + S(r,f) \\
&\le aT\bigl(|q|r,f(z)\bigr) + (\lambda+n)T\bigl(r,f(z)\bigr) + S\bigl(|q|r,f\bigr) + S(r,f),
\end{aligned}$$

that is,

$$bT\bigl(r,f(z)\bigr) + S(r,f) \le aT\bigl(|q|r,f(z)\bigr) + (\lambda+n)T\bigl(r,f(z)\bigr) + S\bigl(|q|r,f\bigr).$$

If $b > a+\lambda+n$, then for any given $\varepsilon > 0$, this inequality can be rewritten as

$$(b-\lambda-n)(1-\varepsilon)T\bigl(r,f(z)\bigr) \le a(1+\varepsilon)T\bigl(|q|r,f(z)\bigr) \tag{5.2}$$

for sufficiently large $r \notin E$, where $\mathrm{lm}(E) < \infty$. By using [6], Lemma 5, with $\beta > 1$ from (5.2) we have that

$$(b-\lambda-n)(1-\varepsilon)T\bigl(r,f(z)\bigr) \le a(1+\varepsilon)T\bigl(\beta|q|r,f(z)\bigr) \tag{5.3}$$

for all $r \ge r_0$. Since $\beta|q| > 1$, $\frac{(b-\lambda-n)(1-\varepsilon)}{a(1+\varepsilon)} > 1$, and it follows from Lemma 2.3 and (5.3) that

$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} \ge \frac{\log((b-\lambda-n)(1-\varepsilon)) - \log(a(1+\varepsilon))}{\log \beta|q|}.$$

As $\varepsilon \to 0^+$ and $\beta \to 1^+$, we have

$$\rho(f) \ge \mu(f) \ge \frac{\log(b-\lambda-n) - \log a}{\log|q|}.$$

Similarly, if $f(z)$ is a transcendental entire solution of (1.8) and $b > a+n$, we have

$$\frac{\log(b-n) - \log a}{\log|q|} \le \mu(f) \le \rho(f) \le \frac{\log(b+n) - \log a}{\log|q|}.$$

This completes the proof of Theorem 1.5. □

Proof of Theorem 1.6 Using the same method as in the proof of Theorem 1.5, the conclusion of Theorem 1.6 follows immediately. We omit the proof here. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
M-FC, Y-YJ, and Z-SG completed the main part of this article, M-FC, Z-SG corrected the main theorems. All authors read and approved the final manuscript.

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