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Existence of positive solutions for fractional differential equation with integral boundary conditions on the half-line

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Abstract

This paper considers the existence of positive solutions for fractional-order nonlinear differential equation with integral boundary conditions on the half-infinite interval. By using the fixed point theorem in a cone, sufficient conditions for the existence of at least one or at least two positive solutions of a boundary value problem are established. These theorems also reveal the properties of solutions on the half-line.

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1 Introduction

Boundary value problems are often studies in the areas of applied mathematics and physics. With the development of technology, applications of boundary value problems on the infinite interval attract increasing attention; see [1–4] and the references therein. Recently, fractional differential equations have also aroused great interest; see [5–8]. At the same time, the existence of positive solutions for nonlinear fractional differential equation boundary value problems have been widely studied by many authors; see [9–17] and the references therein.

In [3], the authors, using fixed point theorems in a cone, established the existence of one positive solution and three positive solutions for the following second-order nonlinear boundary value problems with integral boundary conditions on an infinite interval:

$$\begin{cases} \frac{1}{p(t)}(p(t)u'(t))' + f(t,u(t)) = 0, \quad t \in (0,+\infty), \\ a_1 u(0) - b_1 \lim_{t\to 0^+} p(t)u'(t) = \int_0^{+\infty} g_1(u(s))\psi(s)\,\mathrm{d}s, \\ a_2 \lim_{t\to +\infty} u(t) + b_2 \lim_{t\to +\infty} p(t)u'(t) = \int_0^{+\infty} g_2(u(s))\psi(s)\,\mathrm{d}s, \end{cases}$$

where $f \in C((0,+\infty) \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$, f may be singular at $t = 0$, $g_1, g_2 : [0,+\infty) \to [0,+\infty)$ and $\psi : [0,+\infty) \to (0,+\infty)$ are continuous, $\int_0^{+\infty} \psi(s)\,\mathrm{d}s < +\infty$, $p \in C[0,+\infty) \cap C^1(0,+\infty)$ with $p(t) > 0$ on $(0,+\infty)$ and $\int_0^{+\infty} \frac{\mathrm{d}s}{p(s)} < +\infty$, $a_1 + a_2 > 0$, and $b_i > 0$ for $i = 1, 2$. Iterative schemes for approximating the solutions of a nonlinear fractional boundary value problem on the half-line were presented in [15]. The authors, based on the monotone iterative technique, obtained the existence of positive solutions of the following fractional boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, & t \in (0,+\infty), \\ u(0) = 0, & \lim_{t\to+\infty} D_{0+}^{\alpha-1}u(t) = \lambda > 0, \end{cases}$$

where $1 < \alpha < 2$, and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. For an overview of the literature on differential equations boundary value problems, see [6–8] and the references therein.

Motivated by all the works mentioned, we study the following fractional boundary value problem on the half-line:

$$\begin{cases} \frac{1}{p(t)}(p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t))' + f(t,u(t)) = 0, \quad t \in (0,+\infty), \\ a_1 u(0) - b_1 \lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) = \int_0^{+\infty} g_1(u(s))\psi_1(s)\,\mathrm{d}s, \\ a_2 \lim_{t\to+\infty} u(t) + b_2 \lim_{t\to+\infty} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) = \int_0^{+\infty} g_2(u(s))\psi_2(s)\,\mathrm{d}s, \end{cases} \tag{1.1}$$

where ${}^{\mathrm{C}}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $p \in C^1([0,+\infty),(0,+\infty))$, $f : (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function and may be singular at $t = 0$; $a_i > 0$, $b_i > 0$, $g_i \in C([0,+\infty),[0,+\infty))$, and $\psi_i \in L^1([0,+\infty))$ is nonnegative for $i = 1,2$.

We assume that the following conditions are satisfied:

(H0) $\lim_{t\to+\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{p(s)}\,\mathrm{d}s < +\infty$, $\frac{b_2}{a_2} > M$, where

$$M = \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,+\infty)} \int_0^t \frac{(t-s)^{\alpha-1}}{p(s)}\,\mathrm{d}s. \tag{1.2}$$

(H1) There exist functions $h \in C([0,+\infty),[0,+\infty))$ and $\nu \in C((0,+\infty),(0,+\infty))$ such that

$$f(t,u) \le \nu(t)h(u), \quad t \in (0,+\infty); \qquad \int_0^{+\infty} p(s)\nu(s)\,\mathrm{d}s < +\infty.$$

2 Preliminaries

In this section, we present some useful definitions and the related theorems.

Definition 2.1 (See [5, 7]) Let $\alpha > 0$. For a function $u : (0,+\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral operator of order α of u is defined by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)\,\mathrm{d}s,$$

provided that the integral exists.

Definition 2.2 (See [5, 7]) The Caputo derivative of order α for a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$${}^{\mathrm{C}}D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\delta+1-n}}\,\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$ and $n - 1 < \alpha < n$.

If $\alpha = n$, then ${}^{\mathrm{C}}D_{0+}^{\alpha} u(t) = u^{(n)}(t)$.

Lemma 2.1 (See [7]) *Let $\alpha > 0$. Then the differential equation*

$$ {}^{\mathrm{C}}D_{0+}^{\alpha} h(t) = 0 $$

has solutions

$$ h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1, $$

where n is the smallest integer greater than or equal to α.

Lemma 2.2 *If* (H0) *holds and $y \in C((0, +\infty), [0, +\infty))$ with $\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s < +\infty$, then the fractional boundary value problem*

$$ \begin{cases} \frac{1}{p(t)}(p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha} u(t))' + y(t) = 0, \quad t \in (0, +\infty), \\ a_1 u(0) - b_1 \lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha} u(t) = \int_0^{+\infty} g_1(u(s))\psi_1(s)\,\mathrm{d}s, \\ a_2 \lim_{t\to+\infty} u(t) + b_2 \lim_{t\to+\infty} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha} u(t) = \int_0^{+\infty} g_2(u(s))\psi_2(s)\,\mathrm{d}s \end{cases} \tag{2.1} $$

has a unique solution

$$ \begin{aligned} u(t) = {} & \int_0^{+\infty} G(t,s)p(s)y(s)\,\mathrm{d}s + F_1(t) \int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ & + F_2(t) \int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s, \end{aligned} $$

where

$$ G(t,s) = \frac{1}{\rho} \begin{cases} (b_1 + a_1 \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r)(b_2 + a_2 \lim_{\tau\to+\infty} \int_s^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r), & 0 \le t \le s < +\infty, \\ (b_1 + a_1 \int_0^s \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r)(b_2 + a_2 \lim_{\tau\to+\infty} \int_s^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r) & \\ \quad - b_1 a_2 \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r, & 0 \le s < t < +\infty, \end{cases} $$

$$ \rho = a_1 b_2 + a_2 b_1 + a_1 a_2 \lim_{\tau\to+\infty} \int_0^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r, $$

$$ F_1(t) = \frac{1}{\rho}\left(b_2 + a_2 \lim_{\tau\to+\infty} \int_0^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r - a_2 \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r \right), $$

and

$$ F_2(t) = \frac{1}{\rho}\left(b_1 + a_1 \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r \right). $$

Proof It is well known that the fractional differential equation in (2.1) is equivalent to the integral equation

$$ p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha} u(t) = \lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha} u(t) - \int_0^t p(s)y(s)\,\mathrm{d}s. \tag{2.2} $$

Hence,

$$u(t) = \left(I_{0+}^{\alpha}\frac{1}{p(t)}\right)\lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) - I_{0+}^{\alpha}\left(\frac{1}{p(t)}\int_0^t p(s)y(s)\,\mathrm{d}s\right) + c_0$$
$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\,\mathrm{d}s\lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(s)y(s)\,\mathrm{d}r\,\mathrm{d}s + c_0, \tag{2.3}$$

where $c_0 \in \mathbb{R}$ is any constant. It follows from (2.2) and (2.3) that

$$\lim_{t\to+\infty} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) = \lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) - \int_0^{+\infty} p(s)y(s)\,\mathrm{d}s$$

and

$$\lim_{t\to+\infty} u(t) = \lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t)\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\,\mathrm{d}s$$
$$- \lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(s)y(s)\,\mathrm{d}r\,\mathrm{d}s + c_0.$$

By the boundary conditions in (2.1) we have

$$c_0 = \frac{1}{\rho}\Big[a_2b_1\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(r)y(r)\,\mathrm{d}r\,\mathrm{d}s$$
$$+ b_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s + b_1\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s$$
$$+ \left(a_2\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\,\mathrm{d}s + b_2\right)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s\Big]$$

and

$$\lim_{t\to 0^+} p(t)\,{}^{\mathrm{C}}D_{0+}^{\alpha}u(t) = \frac{1}{\rho}\bigg(a_1a_2\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(r)y(r)\,\mathrm{d}r\,\mathrm{d}s$$
$$+ a_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s$$
$$+ a_1\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s - a_2\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s\bigg).$$

Substituting them into (2.3), we get

$$u(t) = \frac{1}{\rho}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\,\mathrm{d}s\bigg(a_1a_2\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(r)y(r)\,\mathrm{d}r\,\mathrm{d}s$$
$$+ a_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s\bigg)$$
$$+ \frac{1}{\rho}\bigg(a_2b_1\lim_{t\to+\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(r)y(r)\,\mathrm{d}r\,\mathrm{d}s + b_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s\bigg)$$
$$- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\int_0^s p(r)y(r)\,\mathrm{d}r\,\mathrm{d}s$$
$$+ F_1(t)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s + F_2(t)\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{1}{\rho}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(s)}\,\mathrm{d}s\right)\left[a_1a_2 \lim_{t\to+\infty}\int_0^t\left(\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)p(s)y(s)\,\mathrm{d}s\right.\\
&\quad\left. + a_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s\right]\\
&\quad + \frac{1}{\rho}\left[a_2b_1 \lim_{t\to+\infty}\int_0^t\left(\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)p(s)y(s)\,\mathrm{d}s + b_1b_2\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s\right]\\
&\quad - \int_0^t\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r p(s)y(s)\,\mathrm{d}s\\
&\quad + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s.
\end{aligned}
$$

By (1.2), $M < \frac{b_2}{a_2}$, and $\int_0^{+\infty} p(s)y(s)\,\mathrm{d}s < +\infty$, we have

$$
p(s)y(s)\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r \le Mp(s)y(s) \in L^1\bigl([0,+\infty)\bigr).
$$

Hence,

$$
\lim_{t\to+\infty}\int_0^t p(s)y(s)\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\,\mathrm{d}s = \int_0^{+\infty} p(s)y(s)\lim_{t\to+\infty}\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\,\mathrm{d}s.
$$

Therefore, the unique solution of the fractional boundary value problem (2.1) is

$$
\begin{aligned}
u(t) &= \int_0^{+\infty} G(t,s)p(s)y(s)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s\\
&\quad + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s.
\end{aligned}
$$

□

For convenience, we denote

$$
\begin{aligned}
&G_M = \frac{(b_1+a_1M)(b_2+a_2M)}{a_1b_2+a_2b_1}, \qquad G_m = \frac{b_1(b_2-a_2M)}{a_1b_2+a_2b_1+a_1a_2M},\\
&F_{1M} = \frac{b_2+a_2M}{a_1b_2+a_2b_1}, \qquad F_{1m} = \frac{b_2-a_2M}{a_1b_2+a_2b_1+a_1a_2M},\\
&F_{2M} = \frac{b_1+a_1M}{a_1b_2+a_2b_1}, \qquad F_{2m} = \frac{b_1}{a_1b_2+a_2b_1+a_1a_2M},\\
&\gamma_0 = \min\left\{\frac{G_m}{G_M}, \frac{F_{1m}}{F_{1M}}, \frac{F_{2m}}{F_{2M}}\right\}, \qquad \overline{G} = \frac{b_1b_2}{\rho},\\
&\overline{F}_1 = \frac{b_2}{\rho}, \qquad \overline{F}_2 = \frac{1}{\rho}\left(b_1 + a_1\lim_{\tau\to+\infty}\int_0^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right).
\end{aligned}
$$

Lemma 2.3 *If* (H0) *holds, then $G(t,s)$, $F_1(t)$, and $F_2(t)$ defined in Lemma* 2.2 *satisfy*

(1) *$G(t,s)$ is a continuous function and $G(t,s) > 0$ for $(t,s) \in [0,+\infty) \times [0,+\infty)$;*
(2) *$F_1(t)$, $F_2(t)$ are continuous functions, and $F_1(t), F_2(t) \ge 0$ for $t \in [0,+\infty)$;*
(3)

$$
\begin{aligned}
&G_m \le G(t,s) \le G_M \quad \textit{for } (t,s) \in [0,+\infty) \times [0,+\infty),\\
&F_{im} \le F_i(t) \le F_{iM} \quad \textit{for } t \in [0,+\infty), i = 1,2;
\end{aligned}
$$

(4) *there exist constants* $0 < l_1 < l_2 < +\infty$ *such that*

$$G(t,s) \geq \gamma_0 G_M \quad \textit{for } (t,s) \in [l_1, l_2] \times [0,+\infty),$$
$$F_i(t) \geq \gamma_0 F_{iM} \quad \textit{for } t \in [l_1, l_2] \textit{ and } i = 1,2;$$

(5) *for any* $s \in [0,+\infty)$, $\lim_{t\to+\infty} G(t,s) = \overline{G} < +\infty$, $\lim_{t\to+\infty} F_1(t) = \overline{F}_1 < +\infty$, $\lim_{t\to+\infty} F_2(t) = \overline{F}_2 < +\infty$.

Proof (1) For $0 \leq t \leq s$, it is easy to see that $G(t,s) > 0$.

For $0 \leq s < t$, by (H0) and (1.2) we have

$$G(t,s) \geq \frac{1}{\rho}\left(b_1 b_2 - b_1 a_2 \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right) \geq \frac{b_1 a_2}{\rho}\left(\frac{b_2}{a_2} - M\right) > 0.$$

Hence, $G(t,s) > 0$ for $(t,s) \in [0,+\infty) \times [0,+\infty)$.

It is easy to see that $G(t,s)$ is a continuous function.

(2) It follows from (1.2) and (H0) that (2) holds.

(3) By (H0), for $t, s \in [0,+\infty)$, we have

$$G(t,s) \geq \frac{1}{\rho}\left(b_1 b_2 - b_1 a_2 \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right) \geq \frac{b_1 a_2}{\rho}\left(\frac{b_2}{a_2} - M\right) = G_m$$

and

$$G(t,s) \leq \frac{1}{\rho}\left(b_1 + a_1 \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)\left(b_2 + a_2 \lim_{\tau\to+\infty} \int_s^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right) \leq G_M.$$

It is easy to see that $F_{im} \leq F_i(t) \leq F_{iM}$ for $t \in [0,+\infty)$, $i = 1,2$.

(4) By (3) there exist constants $0 < l_1 < l_2 < +\infty$ such that

$$G(t,s) \geq G_m = \frac{G_m}{G_M} \cdot G_M \geq \gamma_0 G_M \quad \text{for } (t,s) \in [l_1, l_2] \times [0,+\infty).$$

Similarly, we have

$$F_i(t) \geq \gamma_0 F_{iM} \quad \text{for } t \in [l_1, l_2] \text{ and } i = 1,2.$$

(5) By (H0) and $0 < \alpha < 1$, for any $s \in [0,+\infty)$, we can show that

$$\begin{aligned}
\lim_{t\to+\infty} G(t,s) &= \frac{1}{\rho} \lim_{t\to+\infty}\left(\left(b_1 + a_1 \int_0^s \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)\left(b_2 + a_2 \lim_{\tau\to+\infty} \int_s^\tau \frac{(\tau-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)\right. \\
&\quad \left. - b_1 a_2 \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right) \\
&= \frac{1}{\rho}\left(\left(b_1 + a_1 \int_0^s \lim_{t\to+\infty} \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)\left(b_2 + a_2 \lim_{t\to+\infty} \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)\right. \\
&\quad \left. - b_1 a_2 \lim_{t\to+\infty} \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)
\end{aligned}$$

$$= \frac{1}{\rho}\left(b_1\left(b_2 + a_2 \lim_{t\to+\infty}\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right) - b_1 a_2 \lim_{t\to+\infty}\int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)p(r)}\,\mathrm{d}r\right)$$
$$= \overline{G}.$$

It is obvious that $\lim_{t\to+\infty} F_1(t) = \overline{F}_1 < +\infty$ and $\lim_{t\to+\infty} F_2(t) = \overline{F}_2 < +\infty$. □

Let

$$E = \left\{u \in C[0,+\infty) : \lim_{t\to+\infty} u(t) < +\infty\right\} \tag{2.4}$$

be a Banach space with the norm $\|u\| = \sup_{t\in[0,+\infty)} |u(t)|$, and

$$P = \left\{u \in E : u(t) \geq 0, t \in [0,+\infty), \inf_{t\in[l_1,l_2]} u(t) \geq \gamma_0 \|u\|\right\}$$

be a cone in E.

For $r > 0$, we denote

$$K_r = \left\{u \in P : \|u\| < r\right\}, \qquad \partial K_r = \left\{u \in P : \|u\| = r\right\},$$
$$S_r := \sup\left\{h(u) : 0 \leq u \leq r\right\}, \qquad S_r' := \sup\left\{g_1(u) : 0 \leq u \leq r\right\},$$

and

$$S_r'' := \sup\left\{g_2(u) : 0 \leq u \leq r\right\}.$$

It follows from (H1) that S_r, S_r', and $S_r'' < +\infty$.

We define the operator $T : P \to E$ by

$$Tu(t) = \int_0^{+\infty} G(t,s)p(s)f\big(s,u(s)\big)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s$$
$$+ F_2(t)\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s, \quad t \in [0,+\infty).$$

We can easily get the following Lemma 2.4 from Lemma 2.2.

Lemma 2.4 *If $u \in P$, then the boundary value problem* (1.1) *is equivalent to the integral equation*

$$u(t) = \int_0^{+\infty} G(t,s)p(s)f\big(s,u(s)\big)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s$$
$$+ F_2(t)\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s, \quad t \in [0,+\infty).$$

Lemma 2.5 (See [1, 18]) *Let E be defined by* (2.4), *and $\Omega \subset E$. Then Ω is relatively compact in E if the following conditions hold*:

(a) *Ω is uniformly bounded in E*;

(b) *the functions belonging to M are equicontinuous on any compact interval of* $[0,+\infty)$;

(c) *the functions from* Ω *are equiconvergent, that is, for any given* $\varepsilon > 0$, *there exists* $T(\varepsilon) > 0$ *such that* $|f(t) - f(+\infty)| < \varepsilon$ *for any* $t > T(\varepsilon)$ *and* $f \in \Omega$.

Lemma 2.6 *If* (H0) *and* (H1) *hold, then* $T : P \to P$ *is completely continuous.*

Proof We divide the proof into three steps.

Step 1: We show that $T : P \to P$ is well defined.

For $u \in P$, there exists a constant $r_0 > 0$ such that $\|u\| \le r_0$. By (H1) and Lemma 2.3, for $t, s \in [0, +\infty)$, we have

$$G(t,s)p(s)f\bigl(s,u(s)\bigr) \le G(t,s)p(s)\nu(s)h(u) \le G_M p(s)\nu(s)S_{r_0}.$$

Since $G(t,s)$, $F_1(t)$, $F_2(t)$ are continuous with respect to t, by using the Lebesgue dominated convergence theorem, for $t_0 \in [0, +\infty)$, we have

$$\begin{aligned}\lim_{t\to t_0} Tu(t) &= \lim_{t\to t_0}\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + \lim_{t\to t_0} F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + \lim_{t\to t_0} F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \\ &= \int_0^{+\infty} G(t_0,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + F_1(t_0)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + F_2(t_0)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s.\end{aligned}$$

So, $Tu \in C[0, +\infty)$, and we get

$$\begin{aligned}\lim_{t\to+\infty} Tu(t) &= \int_0^{+\infty} \overline{G}p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + \overline{F}_1\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + \overline{F}_2\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s < +\infty.\end{aligned}$$

It is obvious that $Tu(t) \ge 0$, $t \in [0, +\infty)$, by Lemma 2.3. Moreover,

$$\begin{aligned}\inf_{t\in[l_1,l_2]} Tu(t) &\ge \inf_{t\in[l_1,l_2]}\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + \inf_{t\in[l_1,l_2]} F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + \inf_{t\in[l_1,l_2]} F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s.\end{aligned}$$

By Lemma 2.3 (4) we have

$$\begin{aligned}\inf_{t\in[l_1,l_2]} Tu(t) &\ge \gamma_0 \sup_{t\in[0,+\infty)}\biggl(\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s\biggr) \\ &\ge \gamma_0\|Tu\|.\end{aligned}$$

Hence, $T : P \to P$ is well defined.

Step 2: We can verify that $T : P \to P$ is continuous.

Let $u_n, u \in P$ and $\|u_n - u\| \to 0$ as $n \to +\infty$. Then there exists a constant $r_1 > 0$ such that $\|u_n\|, \|u\| \leq r_1$. We have

$$\begin{aligned}
0 &\leq G_M p(s)\big|f\big(s, u_n(s)\big) - f\big(s, u(s)\big)\big| + F_{1M}\big|g_1\big(u_n(s)\big) - g_1\big(u(s)\big)\big|\psi_1(s) \\
&\quad + F_{2M}\big|g_2\big(u_n(s)\big) - g_2\big(u(s)\big)\big|\psi_2(s) \\
&\leq G_M p(s)v(s)\big(h\big(u_n(s)\big) + h\big(u(s)\big)\big) + F_{1M}\big(g_1\big(u_n(s)\big) + g_1\big(u(s)\big)\big)\psi_1(s) \\
&\quad + F_{2M}\big(g_2\big(u_n(s)\big) + g_2\big(u(s)\big)\big)\psi_2(s) \\
&\leq 2G_M S_{r_1} p(s)v(s) + 2F_{1M}S'_{r_1}\psi_1(s) + 2F_{2M}S''_{r_1}\psi_2(s) \\
&\in L^1\big([0, +\infty)\big)
\end{aligned}$$

and, for $s \in [0, +\infty)$,

$$\begin{aligned}
&f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \to 0 \quad \text{as } n \to +\infty, \\
&g_i\big(u_n(s)\big) - g_i\big(u(s)\big) \to 0 \quad \text{as } n \to +\infty, i = 1, 2.
\end{aligned}$$

Then, by the Lebesgue dominated convergence theorem we have

$$\begin{aligned}
\|Tu_n - Tu\| \leq{}& \int_0^{+\infty} G_M p(s)\big|f\big(s, u_n(s)\big) - f\big(s, u(s)\big)\big|\,\mathrm{d}s \\
&+ F_{1M}\int_0^{+\infty}\big|g_1\big(u_n(s)\big) - g_1\big(u(s)\big)\big|\psi_1(s)\,\mathrm{d}s \\
&+ F_{2M}\int_0^{+\infty}\big|g_2\big(u_n(s)\big) - g_2\big(u(s)\big)\big|\psi_2(s)\,\mathrm{d}s \\
\to{}& 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

Therefore, $T : P \to P$ is a continuous operator.

Step 3: We can show that $T : P \to P$ is relatively compact.

Let Ω be a bounded subset of P. Then there exists a constant $r_2 > 0$ such that $\|u\| \leq r_2$ for each $u \in \Omega$.

By Lemma 2.3 and (H1) we have

$$\begin{aligned}
\|Tu\| ={}& \sup_{t\in[0,+\infty)}\bigg|\int_0^{+\infty} G(t,s)p(s)f\big(s, u(s)\big)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s \\
&+ F_2(t)\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s\bigg| \\
\leq{}& \int_0^{+\infty} G_M p(s)v(s)h\big(u(s)\big)\,\mathrm{d}s + F_{1M}\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s \\
&+ F_{2M}\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s \\
\leq{}& G_M S_{r_2}\int_0^{+\infty} p(s)v(s)\,\mathrm{d}s + F_{1M}S'_{r_2}\int_0^{+\infty}\psi_1(s)\,\mathrm{d}s + F_{2M}S''_{r_2}\int_0^{+\infty}\psi_2(s)\,\mathrm{d}s \\
<{}& +\infty.
\end{aligned}$$

So, $T(\Omega)$ is uniformly bounded.

For any $\overline{T} \in (0, +\infty)$, since $G(t,s)$, $F_1(t)$, and $F_2(t)$ are continuous, we have that G is uniformly continuous on $[0, \overline{T}] \times [0, \overline{T}]$ and F_1 and F_1 are uniformly continuous on $[0, \overline{T}]$. This implies that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, when $t_1, t_2 \in [0, \overline{T}]$, whenever $|t_2 - t_1| < \delta$ and $s \in [0, \overline{T}]$, we have

$$\begin{aligned} &\bigl|G(t_2,s) - G(t_1,s)\bigr| < \varepsilon, \\ &\bigl|F_i(t_2) - F_i(t_1)\bigr| < \varepsilon, \quad i = 1, 2. \end{aligned}$$

Therefore, for $t_1, t_2 \in [0, \overline{T}]$, whenever $|t_2 - t_1| < \delta$ and $u \in \Omega$, we can show that

$$\begin{aligned} &\bigl|Tu(t_1) - Tu(t_2)\bigr| \\ &\quad \leq \int_0^{+\infty} \bigl|G(t_1,s) - G(t_2,s)\bigr| p(s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &\qquad + \bigl|F_1(t_1) - F_1(t_2)\bigr| \int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\qquad + \bigl|F_2(t_1) - F_2(t_2)\bigr| \int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \\ &\quad < \varepsilon \biggl(S_{r_2} \int_0^{+\infty} p(s)v(s)\,\mathrm{d}s + S'_{r_2} \int_0^{+\infty} \psi_1(s)\,\mathrm{d}s + S''_{r_2} \int_0^{+\infty} \psi_2(s)\,\mathrm{d}s \biggr). \end{aligned}$$

Hence, $T(\Omega)$ is locally equicontinuous on $[0, +\infty)$.

Since

$$\begin{aligned} Tu(+\infty) = {}& \int_0^{+\infty} \overline{G} p(s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s + \overline{F}_1 \int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ & + \overline{F}_2 \int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s. \end{aligned}$$

By Lemma 2.3 we conclude that

$$\begin{aligned} &\bigl|Tu(t) - Tu(+\infty)\bigr| \\ &\quad \leq \int_0^{+\infty} \bigl|G(t,s) - \overline{G}\bigr| p(s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s + \bigl|F_1(t) - \overline{F}_1\bigr| \int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\qquad + \bigl|F_2(t) - \overline{F}_2\bigr| \int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \\ &\quad \to 0 \quad \text{as } t \to +\infty. \end{aligned}$$

Hence, $T(\Omega)$ is equiconvergent at infinity.

By Lemma 2.5 we obtain that $T : P \to P$ is completely continuous. □

Lemma 2.7 (See [19]) *Let E be a Banach space, $P \subseteq E$ be a cone, and Ω_1, Ω_2 be two bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, or*

(ii) *$\|Tx\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, and $\|Tx\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$,*

holds. Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 The existence of positive solutions

For convenience, we give the following notation:

$$h^{\varphi} = \limsup_{u\to\varphi} \frac{h(u)}{u}, \qquad f_{\varphi} = \liminf_{u\to\varphi} \inf_{t\in[l_1,l_2]} \frac{f(t,u)}{u},$$
$$g_i^{\varphi} = \limsup_{u\to\varphi} \frac{g_i(u)}{u}, \qquad g_{i,\varphi} = \liminf_{u\to\varphi} \frac{g_i(u)}{u},$$

where $\varphi = 0$ or $+\infty$, and $i = 1, 2$. We denote

$$A = \max\left\{G_M \int_0^{+\infty} p(s)\nu(s)\,\mathrm{d}s, F_{1M}\int_0^{+\infty} \psi_1(s)\,\mathrm{d}s, F_{2M}\int_0^{+\infty} \psi_2(s)\,\mathrm{d}s\right\},$$
$$B = \gamma_0 \cdot \min\left\{G_m \int_{l_1}^{l_2} p(s)\,\mathrm{d}s, F_{1m}\int_{l_1}^{l_2} \psi_1(s)\,\mathrm{d}s, F_{2m}\int_{l_1}^{l_2} \psi_2(s)\,\mathrm{d}s\right\}.$$

Theorem 3.1 *Suppose that* (H0) *and* (H1) *hold. If*

$$A\big(h^0 + g_1^0 + g_2^0\big) < 1 < B(f_{+\infty} + g_{1,+\infty} + g_{2,+\infty}),$$

then the boundary value problem (1.1) *has at least one positive solution.*

Proof Since $A(h^0 + g_1^0 + g_2^0) < 1$, then there exists a constant $r_1 > 0$ such that, for $u \le r_1$, we have

$$h(u) \le \left(h^0 + \frac{\varepsilon_1}{3}\right)u, \qquad g_i(u) \le \left(g_i^0 + \frac{\varepsilon_1}{3}\right)u, \quad i = 1, 2, \tag{3.1}$$

where ε_1 satisfies $A(h^0 + g_1^0 + g_2^0 + \varepsilon_1) \le 1$.

Therefore, for any $t \in [0, +\infty)$, $u \in \partial K_{r_1}$, we can get

$$\begin{aligned}
\big|Tu(t)\big| &= \left|\int_0^{+\infty} G(t,s)p(s)f\big(s,u(s)\big)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\big(u(s)\big)\psi_1(s)\,\mathrm{d}s\right.\\
&\quad \left. + F_2(t)\int_0^{+\infty} g_2\big(u(s)\big)\psi_2(s)\,\mathrm{d}s\right|\\
&\le \int_0^{+\infty} G_M p(s)\nu(s)\left(h_0 + \frac{\varepsilon_1}{3}\right)u(s)\,\mathrm{d}s + F_{1M}\int_0^{+\infty}\left(g_1^0 + \frac{\varepsilon_1}{3}\right)u(s)\psi_1(s)\,\mathrm{d}s\\
&\quad + F_{2M}\int_0^{+\infty}\left(g_2^0 + \frac{\varepsilon_1}{3}\right)u(s)\psi_2(s)\,\mathrm{d}s\\
&\le A\big(h^0 + g_1^0 + g_2^0 + \varepsilon_1\big)\|u\|\\
&\le \|u\|.
\end{aligned}$$

On the other hand, since $B(f_{+\infty} + g_{1,+\infty} + g_{2,+\infty}) > 1$, there exist constants $\overline{r}_2 > 0$ and M_i, $i = 0, 1, 2$, with $f_{+\infty} > M_0 > 0$, $g_{1,+\infty} > M_1 > 0$, $g_{2,+\infty} > M_2 > 0$ such that, for $t \in [l_1, l_2]$,

$$f(t,u) \ge \left(M_0 - \frac{\varepsilon_2}{3}\right)u, \qquad g_i(u) \ge \left(M_i - \frac{\varepsilon_2}{3}\right)u, \quad i = 1, 2, \tag{3.2}$$

where ε_2 satisfies $B(M_0 + M_1 + M_2 - \varepsilon_2) \ge 1$.

Let $r_2 = \max\{r_1, \frac{\bar{r}_2}{\gamma_0}\}$. In view of the definition of P,

$$\inf_{t\in[l_1,l_2]} u(t) \geq \gamma_0 \|u\| \quad \text{for } u \in P. \tag{3.3}$$

According to Lemma 2.3, for $u \in \partial K_{r_2}$, we have

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,+\infty)} \biggl| \int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \biggr| \\ &\geq \int_{l_1}^{l_2} G_m p(s) f\bigl(s,u(s)\bigr)\,\mathrm{d}s + F_{1m}\int_{l_1}^{l_2} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\ &\quad + F_{2m}\int_{l_1}^{l_2} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s.\end{aligned}$$

By (3.2) and (3.3) we have

$$\begin{aligned}\|Tu\| &\geq \int_{l_1}^{l_2} G_m p(s)\biggl(M_0 - \frac{\varepsilon_2}{3}\biggr)u(s)\,\mathrm{d}s + F_{1m}\int_{l_1}^{l_2}\biggl(M_1 - \frac{\varepsilon_2}{3}\biggr)u(s)\psi_1(s)\,\mathrm{d}s \\ &\quad + F_{2m}\int_{l_1}^{l_2}\biggl(M_2 - \frac{\varepsilon_2}{3}\biggr)u(s)\psi_2(s)\,\mathrm{d}s \\ &\geq B(M_0 + M_1 + M_2 - \varepsilon_2)\|u\| \\ &\geq \|u\|.\end{aligned}$$

Therefore, by (i) of Lemma 2.7 and Lemma 2.3, the boundary value problem (1.1) has at least one positive solution $u \in \overline{K}_{r_2} \setminus K_{r_1}$. □

Remark 3.1 It follows from the proof of Theorem 3.1 that the boundary value problem (1.1) has at least one positive solution $u \in P$ if one of the conditions $f_{+\infty} = +\infty$, $g_{1,+\infty} = +\infty$, and $g_{2,+\infty} = +\infty$ holds.

Theorem 3.2 *Suppose that* (H0) *and* (H1) *hold. If*

$$A\bigl(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty}\bigr) < 1 < B(f_0 + g_{1,0} + g_{2,0}),$$

then the boundary value problem (1.1) *has at least one positive solution.*

Proof It follows from $B(f_0 + g_{1,0} + g_{2,0}) > 1$ that there exist constants $r_3 > 0$, M_i', $i = 0,1,2$, with $f_0 > M_0' > 0$, $g_{1,0} > M_1' > 0$, $g_{2,0} > M_2' > 0$ such that, for $t \in [l_1, l_2]$ and $0 < u \leq r_3$, we have

$$f(t,u) \geq \biggl(M_0' - \frac{\varepsilon_3}{3}\biggr)u, \qquad g_i(u) \geq \biggl(M_i' - \frac{\varepsilon_3}{3}\biggr)u, \tag{3.4}$$

where $i = 1,2$, and ε_3 satisfies $B(M_0' + M_1' + M_2' - \varepsilon_3) \geq 1$.

Thus, for any $t \in [0,+\infty)$ and $u \in \partial K_{r_3}$, we have $\inf_{t\in[l_1,l_2]} u(t) \geq \gamma_0\|u\|$ and

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,+\infty)}\left|\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,ds + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,ds \right.\\ &\quad \left. + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,ds\right| \\ &\geq \int_{l_1}^{l_2} G_m p(s)f\bigl(s,u(s)\bigr)\,ds + F_{1m}\int_{l_1}^{l_2} g_1\bigl(u(s)\bigr)\psi_1(s)\,ds \\ &\quad + F_{2m}\int_{l_1}^{l_2} g_2\bigl(u(s)\bigr)\psi_2(s)\,ds.\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}\|Tu\| &\geq \int_{l_1}^{l_2} G_m p(s)\left(M_0' - \frac{\varepsilon_3}{3}\right)u(s)\,ds + F_{1m}\int_{l_1}^{l_2}\left(M_1' - \frac{\varepsilon_3}{3}\right)u(s)\psi_1(s)\,ds \\ &\quad + F_{2m}\int_{l_1}^{l_2}\left(M_2' - \frac{\varepsilon_3}{3}\right)u(s)\psi_2(s)\,ds \\ &\geq B\bigl(M_0' + M_1' + M_2' - \varepsilon_3\bigr)\|u\| \\ &\geq \|u\|.\end{aligned}$$

On the other hand, since $A(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty}) < 1$, there exists a constant $\overline{r}_4 > 0$ such that, for $u \geq \overline{r}_4$, we have

$$h(u) \leq \left(h^{+\infty} + \frac{\varepsilon_4}{3}\right)u, \qquad g_i(u) \leq \left(g_i^{+\infty} + \frac{\varepsilon_4}{3}\right)u \quad \text{for } i = 1,2,$$

where ε_4 with $A(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty} + \varepsilon_4) < 1$.

Let

$$r_4 > \max\left\{r_3, \overline{r}_4, \frac{G_M S_{\overline{r}_4}\int_0^{+\infty} p(s)v(s)\,ds + F_{1M}S'_{\overline{r}_4}\int_0^{+\infty}\psi_1(s)\,ds + F_{2M}S''_{\overline{r}_4}\int_0^{+\infty}\psi_2(s)\,ds}{1 - A(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty} + \varepsilon_4)}\right\}.$$

For any $t \in [0,+\infty)$, $u \in \partial K_{r_4}$, we denote

$$\begin{aligned}D_1 &= \bigl\{t \in [0,+\infty) : u(t) \geq \overline{r}_4, u \in \partial K_{r_4}\bigr\}, \\ D_2 &= \bigl\{t \in [0,+\infty) : 0 \leq u(t) \leq \overline{r}_4 u \in \partial K_{r_4}\bigr\}.\end{aligned}$$

We have

$$\begin{aligned}\bigl|Tu(t)\bigr| &= \left|\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,ds + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,ds\right. \\ &\quad \left. + F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,ds\right| \\ &\leq \int_{D_1} G_M p(s)v(s)h\bigl(u(s)\bigr)\,ds + F_1(t)\int_{D_1} g_1\bigl(u(s)\bigr)\psi_1(s)\,ds \\ &\quad + F_2(t)\int_{D_1} g_2\bigl(u(s)\bigr)\psi_2(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{D_2} G_M p(s)v(s)h\bigl(u(s)\bigr)\,\mathrm{d}s + F_1(t)\int_{D_2} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\
&+ F_2(t)\int_{D_2} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \\
\le{}& A\bigl(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty} + \varepsilon_4\bigr)\|u\| + G_M S_{\bar{r}_4}\int_0^{+\infty} p(s)v(s)\,\mathrm{d}s \\
&+ F_{1M}S'_{\bar{r}_4}\int_0^{+\infty}\psi_1(s)\,\mathrm{d}s + F_{2M}S''_{\bar{r}_4}\int_0^{+\infty}\psi_2(s)\,\mathrm{d}s \\
\le{}& \|u\|.
\end{aligned}
$$

Hence, by using (ii) of Lemma 2.7 and Lemma 2.3, the boundary value problem (1.1) has at least one positive solution $u \in \overline{K}_{r_4} \setminus K_{r_3}$. □

Remark 3.2 It follows from the proof of Theorem 3.2 that the boundary value problem (1.1) has one positive solution $u \in P$ if at least one of the conditions $f_0 = +\infty$, $g_{1,0} = +\infty$, and $g_{2,0} = +\infty$ holds.

Theorem 3.3 *Suppose that* (H0) *and* (H1) *hold. If*

(1) $B(f_0 + g_{1,0} + g_{2,0}) > 1$, $B(f_{+\infty} + g_{1,+\infty} + g_{2,+\infty}) > 1$ *and*

(2) *there exists a constant* $c > 0$ *such that* $\max\{S_c, S'_c, S''_c\} := S^* < A^{-1}c$,

then the boundary value problem (1.1) *has at least two positive solutions.*

Proof Since $B(f_0 + g_{1,0} + g_{2,0}) > 1$, similarly to the proof of Theorem 3.2, there exists a constant $0 < r < c$ with

$$\|Tu\| \ge \|u\|, \quad \|u\| = r.$$

Since $B(f_{+\infty} + g_{1,+\infty} + g_{2,+\infty}) > 1$, there also exists a constant $R > c$ such that

$$\|Tu\| \ge \|u\|, \quad \|u\| = R.$$

On the other hand, by condition (2), for any $u \in \partial K_c$,

$$
\begin{aligned}
\|Tu\| ={}& \sup_{t\in[0,+\infty)}\biggl|\int_0^{+\infty} G(t,s)p(s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s + F_1(t)\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\
&+ F_2(t)\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s\biggr| \\
\le{}& \int_0^{+\infty} G_M p(s)v(s)h\bigl(u(s)\bigr)\,\mathrm{d}s + F_{1M}\int_0^{+\infty} g_1\bigl(u(s)\bigr)\psi_1(s)\,\mathrm{d}s \\
&+ F_{2M}\int_0^{+\infty} g_2\bigl(u(s)\bigr)\psi_2(s)\,\mathrm{d}s \\
\le{}& G_M S_c\int_0^{+\infty} p(s)v(s)\,\mathrm{d}s + F_{1M}S'_c\int_0^{+\infty}\psi_1(s)\,\mathrm{d}s + F_{2M}S''_c\int_0^{+\infty}\psi_2(s)\,\mathrm{d}s \\
\le{}& S^*\biggl(G_M\int_0^{+\infty} p(s)v(s)\,\mathrm{d}s + F_{1M}\int_0^{+\infty}\psi_1(s)\,\mathrm{d}s + F_{2M}\int_0^{+\infty}\psi_2(s)\,\mathrm{d}s\biggr) \\
<{}& c.
\end{aligned}
$$

Namely,

$$\|Tu\| < c = \|u\|, \quad u \in \partial K_c.$$

According to Lemma 2.7, the boundary value problem (1.1) has at least two positive solutions u_1, u_2 with $0 < \|u_1\| < c < \|u_2\|$. □

Remark 3.3 It follows from the proof of Theorem 3.3 that the boundary value problem (1.1) has at least two positive solutions $u \in P$ if one of the conditions $f_0 = +\infty$, $g_{1,0} = +\infty$, $g_{2,0} = +\infty$, $f_{+\infty} = +\infty$, $g_{1,+\infty} = +\infty$, and $g_{2,+\infty} = +\infty$ holds.

Theorem 3.4 *Suppose that* (H0) *and* (H1) *hold. If*

(1) $A(h^0 + g_1^0 + g_2^0) < 1$, $A(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty}) < 1$, *and*

(2) *there exists a constant* $C > 0$ *such that, for any* $t \in [l_1, l_2]$ *and* $u \in [\gamma_0 C, C]$, *we have*

$$\min\{f(t,u), g_1(u), g_2(u)\} > \gamma_0 B^{-1} C,$$

then the boundary value problem (1.1) *has at least two positive solutions.*

Proof The proof is similar to that of Theorem 3.3.

It is easy to get the two positive solutions u_3, u_4 with $0 < \|u_3\| < C < \|u_4\|$. □

4 Illustration

Example We consider the following boundary value problem:

$$\begin{cases} \frac{1}{e^t}(e^t\, {}^{C}D_{0+}^{\frac{1}{2}} u(t))' + f(t,u(t)) = 0, \quad t \in (0,+\infty), \\ u(0) - \lim_{t\to 0^+} p(t)\, {}^{C}D_{0+}^{\alpha} u(t) = \int_0^{+\infty} \frac{g_1(u(s))}{1+s^2}\, ds, \\ \lim_{t\to+\infty} u(t) + 2\lim_{t\to+\infty} p(t)\, {}^{C}D_{0+}^{\alpha} u(t) = \int_0^{+\infty} \frac{g_2(u(s))}{1+s^2}\, ds, \end{cases} \tag{4.1}$$

where $f(t,u) = \frac{e^{-3t}(1+t)(u+e^{-u})}{10\sqrt{t}}$, $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, $p(t) = e^t$, $\psi_1(s) = \psi_2(s) = \frac{1}{1+s^2}$, and

$$g_1(u) = g_2(u) = \begin{cases} \frac{u}{10}, & 0 \le u \le 100, \\ 60\sqrt{u} - 590, & 100 < u < +\infty. \end{cases}$$

It is obvious that $f : (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function and singular at $t = 0$.

Let

$$v(t) = \frac{e^{-3t}(1+t)}{\sqrt{t}}, \qquad h(u) = \frac{u + e^{-u}}{10}.$$

Then we have $M \approx 0.610503$, $A = 2.19551$, $h^{+\infty} = \frac{1}{10}$, $g_1^{+\infty} = g_2^{+\infty} = 0$, and $A(h^{+\infty} + g_1^{+\infty} + g_2^{+\infty}) = 0.219551 < 1$.

On the other hand, let $l_1 = \frac{1}{2}$, $l_2 = 1$. So we have $B \approx 0.03$, $f_0 = +\infty$, $g_{i,0} = \frac{1}{10}$, $i = 1,2$. Namely, $B(f_0 + g_{1,0} + g_{2,0}) = +\infty > 1$.

By using Theorem 3.2 the boundary value problem (4.1) has at least one positive solution.

Competing interests
The authors declare that no competing interests exist.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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