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Some existence results on boundary value problems for fractional p-Laplacian equation at resonance

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Abstract
Two boundary value problems of the fractional p-Laplacian equation at resonance are considered in this paper. By using the continuation theorem due to Ge, we obtain some existence results for such boundary value problems.

MSC: 34A08; 34B15

Keywords: fractional differential equation; p-Laplacian operator; boundary value problem; continuation theorem; resonance

1 Introduction

Consider the following fractional p-Laplacian equation:

$$D_{0^+}^{\beta}\phi_p\bigl(D_{0^+}^{\alpha}x(t)\bigr)=f\bigl(t,x(t),D_{0^+}^{\alpha}x(t)\bigr),\quad t\in[0,1], \tag{1.1}$$

with the boundary value conditions either

$$x(0)=x(1),\qquad D_{0^+}^{\alpha}x(0)=0, \tag{1.2}$$

or

$$x(0)=x(1),\qquad D_{0^+}^{\alpha}x(1)=0, \tag{1.3}$$

where $0<\alpha, \beta\le 1$, $\phi_p(s)=|s|^{p-2}s$ $(p>1)$, $D_{0^+}^{\alpha}$ is a Caputo fractional derivative, and $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

In the last two decades, the theory of fractional calculus has gained popularity due to its wide applications in various fields of engineering and the sciences [1–8]. Moreover, the p-Laplacian equations often exist in non-Newtonian fluid theory, nonlinear elastic mechanics, and so on.

Recently, many important results on the p-Laplacian equations or the fractional differential equations have been given. We refer the reader to [9–31]. However, as far as we know, there is little work about boundary value problems (BVPs for short) for the fractional differential equations with p-Laplacian operator at resonance.

Note that BVP (1.1)-(1.2) (or BVP (1.1)-(1.3)) happens to be at resonance because its associated homogeneous BVP

$$\begin{cases} D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}x(t)) = 0, \quad t \in [0,1], \\ x(0) = x(1), \qquad D_{0^+}^{\alpha}x(0) = 0 \quad (\text{or } x(0) = x(1), D_{0^+}^{\alpha}x(1) = 0), \end{cases}$$

has a solution $x(t) = c$, $\forall c \in \mathbb{R}$.

The rest of this paper is organized as follows. Section 2 contains some definitions, lemmas and notations. In Section 3, some related lemmas are stated and proved which are useful in the proof of our main results. In Section 4 and Section 5, in view of the continuation theorem due to Ge, we establish two theorems about the existence of solutions for BVP (1.1)-(1.2) (Theorem 4.1) and BVP (1.1)-(1.3) (Theorem 5.1).

2 Preliminaries

We give here some definitions and lemmas about the fractional calculus.

Definition 2.1 [32] The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $x : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2 [32] The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $x : (0,+\infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D_{0^+}^{\alpha}x(t) &= I_{0^+}^{n-\alpha}\frac{d^n x(t)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)\,ds, \end{aligned}$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.1 [8] *Let $\alpha > 0$. Assume that $x, D_{0^+}^{\alpha}x \in L([0,1],\mathbb{R})$. Then the following equality holds*:

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(t) = x(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$, and n is the smallest integer greater than or equal to α.

Lemma 2.2 [33] *For any $u, v \geq 0$,*

$$\begin{aligned} &\phi_p(u+v) \leq \phi_p(u) + \phi_p(v), \quad \text{if } p < 2; \\ &\phi_p(u+v) \leq 2^{p-2}\big(\phi_p(u) + \phi_p(v)\big), \quad \text{if } p \geq 2. \end{aligned}$$

Next we introduce an extension of Mawhin's continuation theorem [34, 35] which allows us to deal with the more general abstract operator equations, such as BVPs of p-Laplacian equations.

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively.

Definition 2.3 [35] A continuous operator $M:\operatorname{dom} M\cap X\to Z$ is said to be a quasi-linear operator if

(1) $\operatorname{Im} M = M(\operatorname{dom} M\cap X)$ is a closed subset of Z,
(2) $\operatorname{Ker} M = \{x\in \operatorname{dom} M\cap X | Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$ with $n<\infty$.

Definition 2.4 [35] Let Z_1 be a subspace of Z. An operator $Q:Z\to Z_1$ is said to be a semi-projector provided that

(1) $Q^2 z = Qz$, $\forall z\in Z$,
(2) $Q(\lambda z) = \lambda Qz$, $\forall z\in Z$, $\lambda\in\mathbb{R}$.

Set $X_1 = \operatorname{Ker} M$ and let X_2 be the complement space of X_1 in X, then $X = X_1\oplus X_2$. Suppose Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z such that $Z = Z_1\oplus Z_2$. Let $P:X\to X_1$ be a projector and $Q:Z\to Z_1$ a semi-projector, and $\Omega\subset X$ an open bounded set with the origin $\theta\in\Omega$.

Definition 2.5 [35] A continuous operator $N_\lambda:\overline{\Omega}\to Z$, $\lambda\in[0,1]$ is said to be M-compact in $\overline{\Omega}$ if there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$, and an operator $R:\overline{\Omega}\times[0,1]\to X_2$ being continuous and compact such that

$$(I-Q)N_\lambda(\overline{\Omega})\subset \operatorname{Im} M\subset (I-Q)Z, \tag{2.1}$$

$$QN_\lambda x = \theta,\quad \lambda\in(0,1)\quad\Leftrightarrow\quad QNx = \theta, \tag{2.2}$$

$$R(\cdot,0)\text{ is the zero operator}\quad\text{and}\quad R(\cdot,\lambda)|_{\sum_\lambda} = (I-P)|_{\sum_\lambda}, \tag{2.3}$$

$$M\bigl(P+R(\cdot,\lambda)\bigr) = (I-Q)N_\lambda, \tag{2.4}$$

where $\lambda\in[0,1]$, $N = N_1$, and $\sum_\lambda = \{x\in\overline{\Omega} | Mx = N_\lambda x\}$.

Lemma 2.3 [35] *Suppose $M:\operatorname{dom} M\cap X\to Z$ is a quasi-linear operator and $N_\lambda:\overline{\Omega}\to Z$, $\lambda\in[0,1]$ is M-compact in $\overline{\Omega}$. In addition, if*

(C_1) *$Mx\neq N_\lambda x$ for every $(x,\lambda)\in[(\operatorname{dom} M\setminus\operatorname{Ker} M)\cap\partial\Omega]\times(0,1)$;*
(C_2) *$QNx\neq 0$ for every $x\in\operatorname{Ker} M\cap\partial\Omega$;*
(C_3) *$\deg\{JQN,\Omega\cap\operatorname{Ker} M,0\}\neq 0$,*

where $N = N_1$ and $J:Z_1\to X_1$ is a homeomorphism with $J(\theta)=\theta$, then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M\cap\overline{\Omega}$.

We set $Z = C([0,1],\mathbb{R})$ with the norm $\|z\|_0 = \max_{t\in[0,1]}|z(t)|$, and $X = \{x\in Z | D^\alpha_{0^+}x\in Z, x(0) = x(1), D^\alpha_{0^+}x(0) = 0\}$, $X^1 = \{x\in Z | D^\alpha_{0^+}x\in Z, x(0) = x(1), D^\alpha_{0^+}x(1) = 0\}$ with the norm $\|x\|_X = \max\{\|x\|_0, \|D^\alpha_{0^+}x\|_0\}$. By using linear functional analysis theory, we can prove X, X^1 are Banach spaces.

3 Related lemmas

We will give some lemmas that are useful in the proof of our main results.

Define the operator $M : \operatorname{dom} M \cap X \to Z$ by

$$Mx = D_{0^+}^{\beta} \phi_p \big(D_{0^+}^{\alpha} x \big), \tag{3.1}$$

where $\operatorname{dom} M = \{x \in X | D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x) \in Z\}$. For $\lambda \in [0,1]$, we define $N_\lambda : X \to Z$ by

$$N_\lambda x(t) = \lambda f\big(t, x(t), D_{0^+}^{\alpha} x(t)\big), \quad \forall t \in [0,1]. \tag{3.2}$$

Then BVP (1.1)-(1.2) is equivalent to the equation

$$Mx = Nx, \quad x \in \operatorname{dom} M,$$

where $N = N_1$.

Lemma 3.1 *The operator M, defined by* (3.1), *is a quasi-linear operator.*

Proof The proof will be given in the following two steps.

Step 1. $\operatorname{Ker} M$ is linearly homeomorphic to $\mathbb{R}$.

From Lemma 2.1, the homogeneous equation $D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x(t)) = 0$ has the following solutions:

$$x(t) = d_2 + \frac{\phi_q(d_1)}{\Gamma(\alpha+1)} t^{\alpha}, \quad d_1, d_2 \in \mathbb{R}.$$

Thus, by the boundary value condition $D_{0^+}^{\alpha} x(0) = 0$, one has

$$\operatorname{Ker} M = \big\{ x \in X | x(t) = d, \forall t \in [0,1], d \in \mathbb{R} \big\}.$$

Obviously, $\operatorname{Ker} M \simeq \mathbb{R}$.

Step 2. $\operatorname{Im} M$ is a closed subset of Z.

Take $x \in \operatorname{dom} M$ and consider the equation $D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x(t)) = z(t)$. Then we have $z \in Z$ and

$$\phi_p\big(D_{0^+}^{\alpha} x(t) \big) = d_1 + I_{0^+}^{\beta} z(t), \quad d_1 \in \mathbb{R}.$$

By the condition $D_{0^+}^{\alpha} x(0) = 0$, one has $d_1 = 0$. Thus we get

$$x(t) = d_2 + I_{0^+}^{\alpha} \phi_q \big(I_{0^+}^{\beta} z \big)(t), \quad d_2 \in \mathbb{R},$$

where ϕ_q is understood as the operator $\phi_q : Z \to Z$ defined by $\phi_q(x)(t) = \phi_q(x(t))$. Hence, from the condition $x(0) = x(1)$, we obtain

$$I_{0^+}^{\alpha} \phi_q \big(I_{0^+}^{\beta} z \big)(1) = 0. \tag{3.3}$$

Suppose $z \in Z$ and satisfies (3.3). Let $x(t) = I_{0^+}^{\alpha}\phi_q(I_{0^+}^{\beta}z)(t)$, then we have $x \in \operatorname{dom} M$ and

$$Mx(t) = D_{0^+}^{\beta}\phi_p\big[D_{0^+}^{\alpha}I_{0^+}^{\alpha}\phi_q\big(I_{0^+}^{\beta}z\big)\big](t) = z(t).$$

Hence we obtain

$$\operatorname{Im} M = \left\{ z \in Z \,\middle|\, \int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1}z(\tau)\,d\tau\right)ds = 0 \right\}.$$

Obviously, $\operatorname{Im} M \subset Z$ is closed.

Therefore, by Definition 2.3, M is a quasi-linear operator. □

Let $X_1 = \operatorname{Ker} M$ and define the continuous operators $P : X \to X$, $Q : Z \to Z$ by

$$Px(t) = x(0), \quad \forall t \in [0,1],$$
$$Qz(t) = \phi_p\left[\frac{1}{\rho}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1}z(\tau)\,d\tau\right)ds\right], \quad \forall t \in [0,1],$$

where $\rho = \frac{1}{\beta^{q-1}}\int_0^1 (1-s)^{\alpha-1}s^{\beta(q-1)}\,ds > 0$. It is easy to see that P is a projector and $Q^2z = Qz$, $Q(\lambda z) = \lambda Qz$, $\forall z \in Z$, $\lambda \in \mathbb{R}$, that is, Q is a semi-projector. Moreover, $X_1 = \operatorname{Im} P$ and $\operatorname{Im} M = \operatorname{Ker} Q$.

Lemma 3.2 *Let $\Omega \subset X$ be an open bounded set. Then the operator N_λ, defined by* (3.2), *is M-compact in $\overline{\Omega}$.*

Proof Choose $X_2 = \operatorname{Ker} P$, $Z_1 = \operatorname{Im} Q$ and define the operator $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$\begin{aligned}R(x,\lambda)(t) &= I_{0^+}^{\alpha}\phi_q\big[I_{0^+}^{\beta}(I-Q)N_\lambda x\big](t)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\right.\\ &\qquad \left.\cdot\int_0^s (s-\tau)^{\beta-1}\big(\lambda f\big(\tau, x(\tau), D_{0^+}^{\alpha}x(\tau)\big) - QN_\lambda x(\tau)\big)\,d\tau\right]ds.\end{aligned}$$

Obviously, $\dim Z_1 = \dim X_1 = 1$. The remainder of the proof will be given in the following two steps.

Step 1. $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact.

By the definition of R, we obtain

$$D_{0^+}^{\alpha}Rx(t) = \phi_q\big[I_{0^+}^{\beta}(I-Q)N_\lambda x\big](t).$$

Clearly, the operators R, $D_{0^+}^{\alpha}R$ are compositions of the continuous operators. So R, $D_{0^+}^{\alpha}R$ are continuous in Z. Hence R is a continuous operator, and $R(\overline{\Omega})$, $D_{0^+}^{\alpha}R(\overline{\Omega})$ are bounded in Z. Furthermore, there exists a constant $T > 0$ such that $|I_{0^+}^{\beta}(I-Q)N_\lambda x(t)| \leq T$, $\forall x \in \overline{\Omega}$, $t \in [0,1]$. Thus, based on the Arzelà-Ascoli theorem, we need only to show $R(\overline{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&\left|Rx(t_2) - Rx(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} (t_2 - s)^{\alpha-1}\phi_q\left[I_{0^+}^{\beta}(I-Q)N_\lambda x(s)\right] ds\right. \\
&\qquad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1}\phi_q\left[I_{0^+}^{\beta}(I-Q)N_\lambda x(s)\right] ds\right| \\
&\quad \leq \frac{T^{q-1}}{\Gamma(\alpha)}\left\{\int_0^{t_1}\left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right] ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds\right\} \\
&\quad = \frac{T^{q-1}}{\Gamma(\alpha+1)}\left[t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}\right].
\end{aligned}$$

As t^{α} is uniformly continuous in $[0,1]$, we obtain $R(\overline{\Omega}) \subset Z$ is equicontinuous. A similar proof can show that $I_{0^+}^{\beta}(I-Q)N_\lambda(\overline{\Omega}) \subset Z$ is equicontinuous. This, together with the uniformly continuity of $\phi_q(s)$ on $[-T,T]$, shows that $D_{0^+}^{\alpha}R(\overline{\Omega}) \subset Z$ is equicontinuous. Thus we find R is compact.

Step 2. Equations (2.1)-(2.4) are satisfied.

For $x \in \overline{\Omega}$, it is easy to show that $Q(I-Q)N_\lambda x = QN_\lambda x - Q^2 N_\lambda x = 0$. So $(I-Q)N_\lambda x \in \operatorname{Ker} Q = \operatorname{Im} M$. Moreover, for $z \in \operatorname{Im} M \subset Z$, one has $Qz = 0$. Thus $z = z - Qz = (I-Q)z \in (I-Q)Z$. Hence (2.1) holds. Since $QN_\lambda x = \lambda QNx$, (2.2) holds too.

For $x \in \sum_\lambda$, we have $Mx = N_\lambda x \in \operatorname{Im} M = \operatorname{Ker} Q$. So $QN_\lambda x = 0$. From the condition $D_{0^+}^{\alpha}x(0) = 0$, one has $I_{0^+}^{\beta}D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}x) = \phi_p(D_{0^+}^{\alpha}x)$. Thus we obtain

$$\begin{aligned}
R(x,\lambda)(t) &= I_{0^+}^{\alpha}\phi_q\left(I_{0^+}^{\beta}N_\lambda x\right)(t) \\
&= I_{0^+}^{\alpha}\phi_q\left[I_{0^+}^{\beta}D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}x\right)\right](t) \\
&= x(t) - x(0) \\
&= (I-P)x(t).
\end{aligned}$$

Furthermore, when $\lambda = 0$, we have $N_\lambda x(t) \equiv 0$, which yields $R(x,0)(t) \equiv 0$, $\forall x \in \overline{\Omega}$. Hence (2.3) holds.

For $x \in \overline{\Omega}$, one has

$$\begin{aligned}
M\left(Px + R(x,\lambda)\right)(t) &= D_{0^+}^{\beta}\phi_p\left[D_{0^+}^{\alpha}\left(Px + R(x,\lambda)\right)\right](t) \\
&= D_{0^+}^{\beta}\phi_p\left[D_{0^+}^{\alpha}I_{0^+}^{\alpha}\phi_q\left(I_{0^+}^{\beta}(I-Q)N_\lambda x\right)\right](t) \\
&= (I-Q)N_\lambda x(t),
\end{aligned}$$

which implies that (2.4) holds.

Therefore, by Definition 2.5, N_λ is M-compact in $\overline{\Omega}$. □

4 Solutions of BVP (1.1)-(1.2)

We will give a theorem on the existence of solutions for BVP (1.1)-(1.2).

Theorem 4.1 *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. Assume that*:

(H_1) *there exist nonnegative functions* $a, b, c \in Z$ *such that*

$$\left|f(t,x,y)\right| \le a(t) + b(t)|x|^{p-1} + c(t)|y|^{p-1}, \quad \forall t \in [0,1], (x,y) \in \mathbb{R}^2;$$

(H_2) *there exists a constant* $A > 0$ *such that, for* $\forall x \in \operatorname{dom} M \setminus \operatorname{Ker} M$ *satisfying* $|x(t)| > A$ *for* $\forall t \in [0,1]$, *we have*

$$\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f\left(\tau, x(\tau), D_{0^+}^{\alpha}x(\tau)\right) d\tau\right) ds \neq 0;$$

(H_3) *there exists a constant* $B > 0$ *such that, for* $\forall r \in \mathbb{R}$ *with* $|r| > B$, *we have either*

$$\phi_q(r)\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, r, 0)\, d\tau\right) ds > 0 \tag{4.1}$$

or

$$\phi_q(r)\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, r, 0)\, d\tau\right) ds < 0. \tag{4.2}$$

Then BVP (1.1)-(1.2) *has at least one solution, provided that*

$$\begin{aligned} \gamma_1 &:= \frac{1}{\Gamma(\beta+1)}\left[\frac{2^{p-1}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_0\right] < 1, \quad \text{if } p < 2; \\ \gamma_2 &:= \frac{1}{\Gamma(\beta+1)}\left[\frac{2^{2p-3}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_0\right] < 1, \quad \text{if } p \ge 2. \end{aligned} \tag{4.3}$$

Proof The proof will be given in the following four steps.

Step 1. $\Omega_1 = \{x \in \operatorname{dom} M \setminus \operatorname{Ker} M | Mx = N_\lambda x, \lambda \in (0,1)\}$ is bounded.

For $x \in \Omega_1$, one has $Nx \in \operatorname{Im} M = \operatorname{Ker} Q$. Thus we have

$$\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f\left(\tau, x(\tau), D_{0^+}^{\alpha}x(\tau)\right) d\tau\right) ds = 0.$$

From (H_2), there exists a constant $\xi \in [0,1]$ such that $|x(\xi)| \le A$. By Lemma 2.1, one has

$$x(t) = x(\xi) - I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(\xi) + I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(t),$$

which together with

$$\begin{aligned} \left|I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(t)\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}D_{0^+}^{\alpha}x(s)\, ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left\|D_{0^+}^{\alpha}x\right\|_0 \cdot \frac{1}{\alpha}t^{\alpha} \\ &\le \frac{1}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_0, \quad \forall t \in [0,1], \end{aligned} \tag{4.4}$$

and $|x(\xi)| \le A$ yields

$$\|x\|_0 \le A + \frac{2}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_0. \tag{4.5}$$

Then, from (H_1), we have

$$\begin{aligned}\left|I_{0^+}^{\beta}Nx(t)\right| &= \frac{1}{\Gamma(\beta)}\left|\int_0^t (t-s)^{\beta-1} f\big(s,x(s),D_{0^+}^{\alpha}x(s)\big)\,ds\right| \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\big(a(s)+b(s)\left|x(s)\right|^{p-1} \\ &\quad + c(s)\left|D_{0^+}^{\alpha}x(s)\right|^{p-1}\big)\,ds \\ &\leq \frac{1}{\Gamma(\beta)}\big(\|a\|_0+\|b\|_0\|x\|_0^{p-1}+\|c\|_0\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1}\big)\cdot\frac{1}{\beta}t^{\beta} \\ &\leq \frac{1}{\Gamma(\beta+1)}\bigg[\|a\|_0+\|c\|_0\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1} \\ &\quad + \|b\|_0\bigg(A+\frac{2}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_0\bigg)^{p-1}\bigg],\quad \forall t\in[0,1]. \end{aligned} \tag{4.6}$$

By $Mx = N_\lambda x$, $D_{0^+}^{\alpha}x(0)=0$, and Lemma 2.1, one has

$$\phi_p\big(D_{0^+}^{\alpha}x(t)\big)=\lambda I_{0^+}^{\beta}Nx(t),$$

which, together with $|\phi_p(D_{0^+}^{\alpha}x(t))| = |D_{0^+}^{\alpha}x(t)|^{p-1}$ and (4.6), implies

$$\begin{aligned}\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1} &\leq \frac{1}{\Gamma(\beta+1)}\bigg[\|a\|_0+\|c\|_0\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1} \\ &\quad + \|b\|_0\bigg(A+\frac{2}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_0\bigg)^{p-1}\bigg]. \end{aligned} \tag{4.7}$$

If $p<2$, from (4.7) and Lemma 2.2, we have

$$\begin{aligned}\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1} &\leq \frac{1}{\Gamma(\beta+1)}\bigg[\|a\|_0+A^{p-1}\|b\|_0 \\ &\quad + \bigg(\frac{2^{p-1}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}}+\|c\|_0\bigg)\left\|D_{0^+}^{\alpha}x\right\|_0^{p-1}\bigg]. \end{aligned}$$

Then, based on (4.3), one has

$$\left\|D_{0^+}^{\alpha}x\right\|_0 \leq \left[\frac{\|a\|_0+A^{p-1}\|b\|_0}{(1-\gamma_1)\Gamma(\beta+1)}\right]^{q-1} := K_1. \tag{4.8}$$

Thus, from (4.5), we have

$$\|x\|_0 \leq A+\frac{2K_1}{\Gamma(\alpha+1)}. \tag{4.9}$$

Similarly, if $p\geq 2$, we obtain

$$\left\|D_{0^+}^{\alpha}x\right\|_0 \leq \left[\frac{\|a\|_0+2^{p-2}A^{p-1}\|b\|_0}{(1-\gamma_2)\Gamma(\beta+1)}\right]^{q-1} := K_2, \tag{4.10}$$

$$\|x\|_0 \leq A+\frac{2K_2}{\Gamma(\alpha+1)}. \tag{4.11}$$

Therefore, combining (4.8), (4.10) with (4.9), (4.11), we have

$$\begin{aligned}\|x\|_X &= \max\left\{\|x\|_0, \left\|D_{0^+}^\alpha x\right\|_0\right\}\\ &\le \max\left\{K_1, K_2, A + \frac{2K_1}{\Gamma(\alpha+1)}, A + \frac{2K_2}{\Gamma(\alpha+1)}\right\} := K.\end{aligned}$$

That is, Ω_1 is bounded.

Step 2. $\Omega_2 = \{x \in \operatorname{Ker} M | QNx = 0\}$ is bounded.

For $x \in \Omega_2$, one has $x(t) = d$, $\forall d \in \mathbb{R}$. Then we have

$$\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, d, 0)\, d\tau\right) ds = 0,$$

which together with (H_3) implies $|d| \le B$. Thus we obtain

$$\|x\|_X \le \max\{B, 0\} = B.$$

Hence Ω_2 is bounded.

Step 3. If (4.1) holds, then

$$\Omega_3 = \left\{x \in \operatorname{Ker} M | \lambda Ix + (1-\lambda)JQNx = 0, \lambda \in [0,1]\right\}$$

is bounded, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism such that $J(d) = d$, $\forall d \in \mathbb{R}$. If (4.2) holds, then

$$\Omega_3' = \left\{x \in \operatorname{Ker} M | -\lambda Ix + (1-\lambda)JQNx = 0, \lambda \in [0,1]\right\}$$

is bounded.

For $x \in \Omega_3$, we have $x(t) = d$, $\forall d \in \mathbb{R}$, and

$$\lambda d = -(1-\lambda)\phi_p\left[\frac{1}{\rho}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, d, 0)\, d\tau\right) ds\right].$$

If $\lambda = 1$, then $d = 0$. If $\lambda \in [0, 1)$, we can show $|d| \le B$. Otherwise, if $|d| > B$, in view of (4.1), one has

$$\begin{aligned}0 \le \lambda d^2 = -(1-\lambda)\phi_p\Bigg[&\frac{\phi_q(d)}{\rho}\int_0^1 (1-s)^{\alpha-1}\\ &\cdot \phi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, d, 0)\, d\tau\right) ds\Bigg] < 0,\end{aligned}$$

which is a contradiction. Hence Ω_3 is bounded.

Similar to the above argument, we can show Ω_3' is also bounded.

Step 4. All conditions of Lemma 2.3 are satisfied.

Define

$$\Omega = \left\{x \in X | \|x\|_X < \max\{K, B\} + 1\right\}.$$

Clearly, $(\Omega_1 \cup \Omega_2 \cup \Omega_3) \subset \Omega$ (or $(\Omega_1 \cup \Omega_2 \cup \Omega_3') \subset \Omega$). From Lemma 3.1 and Lemma 3.2, M is a quasi-linear operator and N_λ is M-compact in $\overline{\Omega}$. Moreover, by the above arguments, we see that the following two conditions are satisfied:

(C_1) $Mx \neq N_\lambda x$ for every $(x,\lambda) \in [(\operatorname{dom} M \setminus \operatorname{Ker} M) \cap \partial\Omega] \times (0,1)$;
(C_2) $QNx \neq 0$ for every $x \in \operatorname{Ker} M \cap \partial\Omega$.

Now we verify the condition (C_3) of Lemma 2.3. Let us define the homotopy

$$H(x,\lambda) = \pm\lambda Ix + (1-\lambda)JQNx.$$

According to the above argument, we know

$$H(x,\lambda) \neq 0, \quad \forall x \in \partial\Omega \cap \operatorname{Ker} M.$$

Thus we have

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} M, \theta\} &= \deg\bigl\{H(\cdot,0), \Omega \cap \operatorname{Ker} M, \theta\bigr\} \\ &= \deg\bigl\{H(\cdot,1), \Omega \cap \operatorname{Ker} M, \theta\bigr\} \\ &= \deg\{\pm I, \Omega \cap \operatorname{Ker} M, \theta\} \neq 0. \end{aligned}$$

So the condition (C_3) of Lemma 2.3 is satisfied.

Therefore, the operator equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$. That is, BVP (1.1)-(1.2) has at least one solution in X. □

5 Solutions of BVP (1.1)-(1.3)

We will give a theorem on the existence of solutions for BVP (1.1)-(1.3).

Define the operator $M_1 : \operatorname{dom} M_1 \cap X^1 \to Z$ by

$$M_1 x = D_{0^+}^{\beta} \phi_p\bigl(D_{0^+}^{\alpha} x\bigr), \tag{5.1}$$

where $\operatorname{dom} M_1 = \{x \in X^1 | D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} x) \in Z\}$. Then BVP (1.1)-(1.3) is equivalent to the operator equation

$$M_1 x = Nx, \quad x \in \operatorname{dom} M_1,$$

where $N = N_1$ and $N_\lambda : X^1 \to Z$, $\lambda \in [0,1]$ is defined by (3.2).

By similar arguments to Section 3, we obtain

$$\begin{aligned} \operatorname{Ker} M_1 &= \bigl\{x \in X^1 | x(t) = d, \forall t \in [0,1], d \in \mathbb{R}\bigr\}, \\ \operatorname{Im} M_1 &= \biggl\{z \in Z \biggm| \int_0^1 (1-s)^{\alpha-1} \phi_q\biggl(-\int_0^1 (1-\tau)^{\beta-1} z(\tau)\,d\tau \\ &\quad + \int_0^s (s-\tau)^{\beta-1} z(\tau)\,d\tau\biggr)\,ds = 0\biggr\}. \end{aligned}$$

Lemma 5.1 *The operator M_1, defined by* (5.1), *is a quasi-linear operator.*

Let $X_1^1 = \operatorname{Ker} M_1$, define the projector $P_1 : X^1 \to X^1$ and the semi-projector $Q_1 : Z \to Z$ by

$$P_1 x(t) = x(0), \quad \forall t \in [0,1],$$

$$Q_1 z(t) = \phi_p\left[\frac{1}{\rho_1}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(-\int_0^1 (1-\tau)^{\beta-1} z(\tau)\,d\tau + \int_0^s (s-\tau)^{\beta-1} z(\tau)\,d\tau\right)ds\right], \quad \forall t \in [0,1],$$

where $\rho_1 = \frac{1}{\beta^{q-1}}\int_0^1 (1-s)^{\alpha-1}\phi_q(-1+s^\beta)\,ds < 0$. Furthermore, let $\Omega^1 \subset X^1$ be an open bounded set, choose $X_2^1 = \operatorname{Ker} P_1$, $Z_1^1 = \operatorname{Im} Q_1$ and define the operator $R_1 : \overline{\Omega^1} \times [0,1] \to X_2^1$ by

$$\begin{aligned} R_1(x,\lambda)(t) &= I_{0^+}^{\alpha}\phi_q\big[I_{0^+}^{\beta}(I-Q)N_\lambda x + \tilde{d}\big((I-Q)N_\lambda x\big)\big](t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\right. \\ &\quad \cdot \int_0^s (s-\tau)^{\beta-1}\big(\lambda f\big(\tau, x(\tau), D_{0^+}^{\alpha} x(\tau)\big) - QN_\lambda x(\tau)\big)\,d\tau \\ &\quad \left. - \frac{1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}\big((I-Q)N_\lambda x(\tau)\big)\,d\tau\right]ds, \end{aligned}$$

where $\tilde{d} : Z \to \mathbb{R}$ is defined by

$$\begin{aligned} \tilde{d}(z) &= -I_{0^+}^{\beta} z(1) \\ &= -\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} z(s)\,ds. \end{aligned}$$

Lemma 5.2 *The operator $N_\lambda : X^1 \to Z$, $\lambda \in [0,1]$, defined by* (3.2), *is M-compact in $\overline{\Omega^1}$.*

Our second result, based on Lemma 5.1 and Lemma 5.2, is stated as follows.

Theorem 5.1 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*:

(H_4) *there exists a constant $A_1 > 0$ such that, for $\forall x \in \operatorname{dom} M_1 \setminus \operatorname{Ker} M_1$ satisfying $|x(t)| > A_1$ for $\forall t \in [0,1]$, we have*

$$\int_0^1 (1-s)^{\alpha-1}\phi_q\left(-\int_0^1 (1-\tau)^{\beta-1} f\big(\tau, x(\tau), D_{0^+}^{\alpha} x(\tau)\big)\,d\tau + \int_0^s (s-\tau)^{\beta-1} f\big(\tau, x(\tau), D_{0^+}^{\alpha} x(\tau)\big)\,d\tau\right)ds \neq 0;$$

(H_5) *there exists a constant $B_1 > 0$ such that, for $\forall r_1 \in \mathbb{R}$ with $|r_1| > B_1$, we have either*

$$\phi_q(r_1)\int_0^1 (1-s)^{\alpha-1}\phi_q\left(-\int_0^1 (1-\tau)^{\beta-1} f(\tau, r_1, 0)\,d\tau + \int_0^s (s-\tau)^{\beta-1} f(\tau, r_1, 0)\,d\tau\right)ds > 0$$

or

$$\phi_q(r_1)\int_0^1 (1-s)^{\alpha-1}\phi_q\left(-\int_0^1 (1-\tau)^{\beta-1}f(\tau,r_1,0)\,d\tau + \int_0^s (s-\tau)^{\beta-1}f(\tau,r_1,0)\,d\tau\right)ds<0,$$

and (H_1) *is true. Then BVP* (1.1)-(1.3) *has at least one solution, provided that*

$$\begin{aligned}\delta_1 &:= \frac{2}{\Gamma(\beta+1)}\left[\frac{2^{p-1}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}}+\|c\|_0\right]<1, \quad \textit{if } p<2;\\ \delta_2 &:= \frac{2}{\Gamma(\beta+1)}\left[\frac{2^{2p-3}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}}+\|c\|_0\right]<1, \quad \textit{if } p\geq 2.\end{aligned} \tag{5.2}$$

Proof Let

$$\Omega_1^1=\left\{x\in \operatorname{dom} M_1\setminus \operatorname{Ker} M_1 | M_1x=N_\lambda x, \lambda\in(0,1)\right\}.$$

Now we prove Ω_1^1 is bounded.

For $x\in\Omega_1^1$, one has $Nx\in \operatorname{Im} M_1=\operatorname{Ker} Q_1$. Thus we have

$$\int_0^1 (1-s)^{\alpha-1}\phi_q\left(-\int_0^1 (1-\tau)^{\beta-1}f\big(\tau,x(\tau),D_{0^+}^\alpha x(\tau)\big)\,d\tau + \int_0^s (s-\tau)^{\beta-1}f\big(\tau,x(\tau),D_{0^+}^\alpha x(\tau)\big)\,d\tau\right)ds=0.$$

From (H_4), there exists a constant $\eta\in[0,1]$ such that $|x(\eta)|\leq A_1$. Hence, by (4.4), one has

$$\|x\|_0\leq A_1+\frac{2}{\Gamma(\alpha+1)}\left\|D_{0^+}^\alpha x\right\|_0. \tag{5.3}$$

Since $M_1x=N_\lambda x$, $D_{0^+}^\alpha x(1)=0$, one has

$$\phi_p\big(D_{0^+}^\alpha x(t)\big)=-\lambda I_{0^+}^\beta Nx(1)+\lambda I_{0^+}^\beta Nx(t),$$

which together with (4.6) and (5.3) implies

$$\left\|D_{0^+}^\alpha x\right\|_0^{p-1}\leq \frac{2}{\Gamma(\beta+1)}\left[\|a\|_0+\|c\|_0\left\|D_{0^+}^\alpha x\right\|_0^{p-1} + \|b\|_0\left(A_1+\frac{2}{\Gamma(\alpha+1)}\left\|D_{0^+}^\alpha x\right\|_0\right)^{p-1}\right]. \tag{5.4}$$

If $p<2$, from (5.4) and Lemma 2.2, we have

$$\left\|D_{0^+}^\alpha x\right\|_0^{p-1}\leq \frac{2}{\Gamma(\beta+1)}\left[\|a\|_0+A_1^{p-1}\|b\|_0 + \left(\frac{2^{p-1}\|b\|_0}{(\Gamma(\alpha+1))^{p-1}}+\|c\|_0\right)\left\|D_{0^+}^\alpha x\right\|_0^{p-1}\right].$$

Then, in view of (5.2), one has

$$\|D_{0^+}^{\alpha}x\|_0 \leq \left[\frac{2(\|a\|_0 + A_1^{p-1}\|b\|_0)}{(1-\delta_1)\Gamma(\beta+1)}\right]^{q-1} := T_1. \tag{5.5}$$

Similarly, if $p \geq 2$, we obtain

$$\|D_{0^+}^{\alpha}x\|_0 \leq \left[\frac{2(\|a\|_0 + 2^{p-2}A_1^{p-1}\|b\|_0)}{(1-\delta_2)\Gamma(\beta+1)}\right]^{q-1} := T_2. \tag{5.6}$$

Therefore, from (5.3), (5.5), and (5.6), we have

$$\|x\|_X \leq \max\left\{T_1, T_2, A_1 + \frac{2T_1}{\Gamma(\alpha+1)}, A_1 + \frac{2T_2}{\Gamma(\alpha+1)}\right\}.$$

That is, Ω_1^1 is bounded.

The remainder of proof are similar to the proof of Theorem 4.1, so we omit the details. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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