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Uniqueness of entire functions sharing two values with their difference operators

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Abstract

In this paper, we mainly discuss the uniqueness problem when an entire function shares 0 CM and nonzero complex constant a IM with its difference operator. We also consider the general case where they share two distinct complex constants a^* CM and a IM under some additional condition and give some further discussions.

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1 Introduction and main results

In this paper, meromorphic means meromorphic in the whole complex plane. We assume that the reader is familiar with the standard notation and results of the Nevanlinna theory (see, for instance, [1–3]).

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let a be an arbitrary complex constant. If $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities (ignoring multiplicities), we say that $f(z)$ and $g(z)$ share a CM (IM). Especially, if $f(z)$ and $g(z)$ share a IM, then we denote by $N_{(p,q)}(r, \frac{1}{f(z)-a})$ ($\bar{N}_{(p,q)}(r, \frac{1}{f(z)-a})$) the counting function (the reduced counting function) of zeros of $f(z) - a$ with respect to all the points such that they are zeros of $f(z) - a$ with multiplicity p and zeros of $g(z) - a$ with multiplicity q . In addition, by $S(r, f)$ we denote any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside of an exceptional set of finite logarithmic measure.

Furthermore, we need some notation on differences. Let c be a nonzero complex constant, and let $f(z)$ be a meromorphic function. We use the notation $\Delta_c^n f(z)$ to denote the difference operators of $f(z)$, which are defined by

$$\Delta_c f(z) = f(z+c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N}, n \geq 2.$$

In particular, if $c = 1$, then we denote $\Delta_c f(z) = \Delta f(z)$.

The uniqueness of meromorphic functions sharing values with their derivatives has always been an important topic of uniqueness of meromorphic functions. Many good and general results have been obtained (see [2]).

In 2000, Li and Yang [4] proved the following result.

Theorem A ([4]) *Let $f(z)$ be a nonconstant entire function, and let a and b be two distinct complex numbers. If $f(z)$ and $f^{(k)}(z)$ ($k \geq 1$) share a, b IM, then $f(z) \equiv f^{(k)}(z)$.*

The uniqueness of meromorphic functions sharing values with their shifts or difference operators has become a subject of great interest recently. In 2009, Heittokangas et al. [5] started to consider the value sharing problems for shifts of meromorphic functions and obtained some important results. After that, many authors considered some related problems (see, for instance, [6–14]).

In 2013, Chen and Yi [6] considered the case where entire functions $f(z)$ and $\Delta_c f(z)$ share two values CM under the condition that the order of $f(z)$ is not an integer or infinite and obtained the following theorem.

Theorem B ([6]) *Let $f(z)$ be a transcendental entire function such that its order $\sigma(f)$ is not an integer or infinite, and let c be a constant such that $f(z+c) \not\equiv f(z)$. If $f(z)$ and $\Delta_c f(z)$ share two distinct finite values a, b CM, then $f(z) \equiv \Delta_c f(z)$.*

In 2014, Zhang and Liao [14] discussed the case where the condition ‘its order $\sigma(f)$ is not an integer or infinite’ is omitted and proved the following result.

Theorem C ([14]) *Let $f(z)$ be a transcendental entire function of finite order, and let a and b be two distinct constants. If $\Delta f(z) (\not\equiv 0)$ and $f(z)$ share a, b CM, then $\Delta f(z) \equiv f(z)$. Furthermore, $f(z)$ must be of the form $f(z) = 2^z h(z)$, where h is a periodic entire function with period 1.*

In 2016, Li and Yi [8] considered the case where entire functions $f(z)$ and $\Delta_c f(z)$ share three values IM and obtained the following theorem.

Theorem D ([8]) *Let $f(z)$ be a nonconstant entire function such that $\rho_2(f) < 1$, and let c be a nonzero complex number. Suppose that $f(z)$ and $\Delta_c f(z)$ share a_1, a_2, a_3 IM, where a_1, a_2, a_3 are three distinct finite values. Then $2f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

From Theorems A–D a natural question is what results we can get if the condition that $f(z)$ and $\Delta_c f(z)$ share two values CM or three values IM is relaxed to one value CM and another one IM or even two values IM, and if $\Delta_c f(z)$ is replaced by $\Delta_c^n f(z)$? Corresponding to this question, we first consider the case where $f(z)$ and $\Delta_c^n f(z)$ share 0 CM and nonzero complex constant a IM and get the following result.

Theorem 1.1 *Let $c \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, and let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$ and $\Delta_c^n f(z)$ share 0 CM and a nonzero complex constant a IM, then $f(z) \equiv \Delta_c^n f(z)$.*

Remark 1 Some idea of the proof of Theorem 1.1 is due to [4]. We have not found any example such that $f(z) \not\equiv \Delta_c^n f(z)$ under the condition that $f(z)$ and $\Delta_c^n f(z)$ share two distinct complex constants a^* CM and a IM. We wonder whether 0 CM in Theorem 1.1 can be replaced by an arbitrary complex constant $a^* (\neq a)$ CM or not.

Then we continue to investigate the case where 0 CM is replaced by arbitrary complex constant $a^* (\neq a)$ CM under some additional condition. Using a similar method as in the proof of Theorem 1.1, we have the next result.

Theorem 1.2 Let $c \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}$, and let $f(z)$ be a nonconstant entire function of finite order. If $f(z)$ and $\Delta_c^n f(z)$ share two distinct complex constants a^* CM and a IM and if

$$N\left(r, \frac{1}{f(z) - a^*}\right) = T(r, f) + S(r, f), \quad (1.1)$$

then $f(z) \equiv \Delta_c^n f(z)$.

Remark 2 We omit the proof of Theorem 1.2, as it can be proved with a similar idea as in the proof of Theorem 1.1. In fact, we can consider the functions

$$\gamma(z) = \frac{f'(z)(\Delta_c^n f(z) - f(z))}{(f(z) - a)(f(z) - a^*)}, \quad \eta(z) = \frac{(\Delta_c^n f(z))'(\Delta_c^n f(z) - f(z))}{(\Delta_c^n f(z) - a)(\Delta_c^n f(z) - a^*)}.$$

Especially, it follows from (1.1) that $m(r, \frac{1}{f(z) - a^*}) = S(r, f)$, which ensures that still $T(r, \Delta_c^n f(z)) = T(r, f(z)) + S(r, f)$ when $f(z)$ and $\Delta_c^n f(z)$ share a^* CM instead of 0 CM.

2 Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we need to introduce some lemmas. In particular, the following lemma can be derived from the difference logarithmic derivative lemma (see [15]), which was obtained independently by Chiang and Feng [16] and Halburd and Korhonen [17] and plays a very important role in studying the difference analogues of Nevanlinna theory.

Lemma 2.1 ([15]) Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c , with respect to $f(z)$,

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f),$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Lemma 2.2 ([2]) Suppose that $f(z)$ is a nonconstant meromorphic function and $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p$ ($a_0 \neq 0$) is a polynomial in f of degree p with constant coefficients a_j ($j = 0, 1, \dots, p$). Suppose furthermore that b_j ($j = 1, 2, \dots, q$) ($q > p$) are distinct values. Then

$$m\left(r, \frac{P(f)f'}{(f - b_1)(f - b_2) \cdots (f - b_q)}\right) = S(r, f).$$

Lemma 2.3 ([3]) If $f_1(z)$ and $f_2(z)$ are meromorphic functions in $|z| < R$ ($R \leq \infty$), then

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right),$$

where $0 < r < R$.

Lemma 2.4 ([2]) Let $f(z)$ be a nonconstant meromorphic function in the complex plane, and let $R(f) = \frac{P(f)}{Q(f)}$, where

$$P(f) = \sum_{k=0}^p a_k f^k \quad \text{and} \quad Q(f) = \sum_{j=0}^q b_j f^j$$

are two mutually prime polynomials in f . If the coefficients $\{a_k(z)\}$ and $\{b_j(z)\}$ are small functions of f and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\} T(r, f).$$

Proof of Theorem 1.1 Suppose $f(z) \not\equiv \Delta_c^n f(z)$. Set

$$\begin{aligned}\gamma(z) &= \frac{f'(z)(\Delta_c^n f(z) - f(z))}{f(z)(f(z) - a)}, \\ \eta(z) &= \frac{(\Delta_c^n f(z))'(\Delta_c^n f(z) - f(z))}{\Delta_c^n f(z)(\Delta_c^n f(z) - a)}.\end{aligned}\quad (2.1)$$

Note that $f(z)$ is a nonconstant entire function of finite order and that 0 and a are CM and IM values shared by $f(z)$ and $\Delta_c^n f(z)$, respectively. We see that $\gamma(z)$ and $\eta(z)$ are entire. By the lemma of the logarithmic derivative and Lemma 2.1 it is obvious that

$$\begin{aligned}T(r, \gamma(z)) &= m(r, \gamma(z)) = m\left(r, \frac{f'(z)(\Delta_c^n f(z) - f(z))}{f(z)(f(z) - a)}\right) \\ &\leq m\left(r, \frac{f'(z)}{f(z) - a}\right) + m\left(r, \frac{\Delta_c^n f(z)}{f(z)} - 1\right) + S(r, f) = S(r, f).\end{aligned}\quad (2.2)$$

Since 0 is a CM value shared by $f(z)$ and $\Delta_c^n f(z)$, it gives

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{h(z)},$$

where $h(z)$ is an entire function. Using Lemma 2.1 again, we get

$$T(r, e^{h(z)}) = m(r, e^{h(z)}) = m\left(r, \frac{\Delta_c^n f(z)}{f(z)}\right) = S(r, f),$$

and then

$$T(r, \Delta_c^n f(z)) = T(r, e^{h(z)} f(z)) = T(r, f(z)) + S(r, f). \quad (2.3)$$

For any $b \in \mathbb{C} \setminus \{0, a\}$, by Lemmas 2.1 and 2.2 we obtain

$$\begin{aligned}m\left(r, \frac{1}{f(z) - b}\right) &= m\left(r, \frac{f'(z)(\Delta_c^n f(z) - f(z))}{f(z)(f(z) - a)(f(z) - b)\gamma(z)}\right) \\ &\leq m\left(r, \frac{\Delta_c^n f(z)}{f(z)} - 1\right) + m\left(r, \frac{f'(z)}{(f(z) - a)(f(z) - b)}\right) + S(r, f) \\ &= S(r, f).\end{aligned}\quad (2.4)$$

According to the second fundamental theorem, we get

$$\begin{aligned}T(r, f(z)) &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z) - a}\right) + \overline{N}(r, f(z)) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z) - a}\right) + S(r, f).\end{aligned}$$

In addition, by (2.1), (2.2), and Lemma 2.1 we obtain

$$\begin{aligned}
 & \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-a}\right) \\
 &= N\left(r, \frac{f'(z)}{f(z)(f(z)-a)}\right) = N\left(r, \frac{\gamma(z)}{\Delta_c^n f(z) - f(z)}\right) \\
 &\leq T(r, \Delta_c^n f(z) - f(z)) + S(r, f) = m(r, \Delta_c^n f(z) - f(z)) + S(r, f) \\
 &\leq m\left(r, \frac{\Delta_c^n f(z)}{f(z)} - 1\right) + m(r, f(z)) + S(r, f) \\
 &= T(r, f(z)) + S(r, f).
 \end{aligned}$$

Combining the above two inequalities, we get

$$\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-a}\right) = T(r, f(z)) + S(r, f). \quad (2.5)$$

Since $f(z)$ is a nonconstant entire function of finite order, it follows from the second fundamental theorem and (2.3) that

$$\begin{aligned}
 2T(r, f(z)) &= 2T(r, \Delta_c^n f(z)) + S(r, f) \\
 &\leq \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) \\
 &\quad + S(r, f).
 \end{aligned}$$

Note that 0 and a are CM and IM values shared by $f(z)$ and $\Delta_c^n f(z)$, respectively. From (2.3) and (2.5) we have

$$\begin{aligned}
 & \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-a}\right) + \overline{N}\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) \\
 &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-a}\right) + T\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) - m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) \\
 &\leq T(r, f(z)) + T(r, \Delta_c^n f(z)) - m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) + S(r, f) \\
 &= 2T(r, f(z)) - m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) + S(r, f).
 \end{aligned}$$

From the above two inequalities we obtain that

$$m\left(r, \frac{1}{\Delta_c^n f(z)-b}\right) = S(r, f). \quad (2.6)$$

Obviously, we have

$$\begin{aligned}
 & m\left(r, \frac{f(z)-b}{\Delta_c^n f(z)-b}\right) - m\left(r, \frac{\Delta_c^n f(z)-b}{f(z)-b}\right) \\
 &= T\left(r, \frac{f(z)-b}{\Delta_c^n f(z)-b}\right) - N\left(r, \frac{f(z)-b}{\Delta_c^n f(z)-b}\right) - T\left(r, \frac{\Delta_c^n f(z)-b}{f(z)-b}\right)
 \end{aligned}$$

$$\begin{aligned}
& + N\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - b}{f(z) - b}\right) \\
& = N\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - b}{f(z) - b}\right) - N\left(r, \frac{f(z) - b}{\Delta_{\mathcal{C}}^n f(z) - b}\right) + O(1).
\end{aligned}$$

Considering functions $f_1(z) = \Delta_{\mathcal{C}}^n f(z) - b$, $f_2(z) = \frac{1}{f(z) - b}$ and applying Lemma 2.3, we have

$$\begin{aligned}
& N\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - b}{f(z) - b}\right) - N\left(r, \frac{f(z) - b}{\Delta_{\mathcal{C}}^n f(z) - b}\right) \\
& = N(r, \Delta_{\mathcal{C}}^n f(z) - b) + N\left(r, \frac{1}{f(z) - b}\right) - N(r, f(z) - b) \\
& \quad - N\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) + O(1) \\
& = N\left(r, \frac{1}{f(z) - b}\right) - N\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) + O(1).
\end{aligned}$$

Then by (2.3), (2.4), and (2.6) we get

$$\begin{aligned}
& N\left(r, \frac{1}{f(z) - b}\right) - N\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) \\
& = T\left(r, \frac{1}{f(z) - b}\right) - m\left(r, \frac{1}{f(z) - b}\right) - T\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) \\
& \quad + m\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) \\
& = T\left(r, \frac{1}{f(z) - b}\right) - T\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - b}\right) + S(r, f) \\
& = T(r, f(z)) - T(r, \Delta_{\mathcal{C}}^n f(z)) + S(r, f) = S(r, f).
\end{aligned}$$

Clearly, the last three equations give

$$m\left(r, \frac{f(z) - b}{\Delta_{\mathcal{C}}^n f(z) - b}\right) - m\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - b}{f(z) - b}\right) = S(r, f).$$

By Lemma 2.1 and (2.4) this equation yields

$$\begin{aligned}
m\left(r, \frac{f(z) - b}{\Delta_{\mathcal{C}}^n f(z) - b}\right) & = m\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - b}{f(z) - b}\right) + S(r, f) \\
& \leq m\left(r, \frac{\Delta_{\mathcal{C}}^n f(z)}{f(z) - b}\right) + m\left(r, \frac{b}{f(z) - b}\right) + S(r, f) = S(r, f).
\end{aligned} \tag{2.7}$$

According to Lemma 2.2, from (2.1) and (2.7) we deduce that

$$\begin{aligned}
T(r, \eta(z)) & = m(r, \eta(z)) = m\left(r, \frac{(\Delta_{\mathcal{C}}^n f(z))'(\Delta_{\mathcal{C}}^n f(z) - f(z))}{\Delta_{\mathcal{C}}^n f(z)(\Delta_{\mathcal{C}}^n f(z) - a)}\right) \\
& \leq m\left(r, \frac{(\Delta_{\mathcal{C}}^n f(z))'(\Delta_{\mathcal{C}}^n f(z) - b)}{\Delta_{\mathcal{C}}^n f(z)(\Delta_{\mathcal{C}}^n f(z) - a)}\right) + m\left(r, \frac{\Delta_{\mathcal{C}}^n f(z) - f(z)}{\Delta_{\mathcal{C}}^n f(z) - b}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq m\left(r, \frac{(\Delta_{\mathcal{C}}^n f(z))'(\Delta_{\mathcal{C}}^n f(z) - b)}{\Delta_{\mathcal{C}}^n f(z)(\Delta_{\mathcal{C}}^n f(z) - a)}\right) + m\left(r, 1 - \frac{f(z) - b}{\Delta_{\mathcal{C}}^n f(z) - b}\right) \\
&= S(r, \Delta_{\mathcal{C}}^n f(z)) + S(r, f) = S(r, f).
\end{aligned} \tag{2.8}$$

Let z_0 be any zero of $f(z) - a$ and $\Delta_{\mathcal{C}}^n f(z) - a$ with multiplicities p and q , respectively. From (2.1) we see that

$$\begin{aligned}
\gamma(z_0)a &= p \cdot \left(\frac{\Delta_{\mathcal{C}}^n f(z) - f(z)}{z - z_0} \right) \Big|_{z=z_0}, \\
\eta(z_0)a &= q \cdot \left(\frac{\Delta_{\mathcal{C}}^n f(z) - f(z)}{z - z_0} \right) \Big|_{z=z_0}.
\end{aligned}$$

This leads to $q\gamma(z_0) = p\eta(z_0)$. Similarly, for any zero of $f(z)$ and $\Delta_{\mathcal{C}}^n f(z)$ with multiplicities p and q , denoted by z_1 , we can prove that $q\gamma(z_1) = p\eta(z_1)$. We distinguish two cases.

Case 1. Suppose that $q\gamma(z) \neq p\eta(z)$. From the above discussion we see that any zero of $f(z) - a$ and $\Delta_{\mathcal{C}}^n f(z) - a$ (or any zero of $f(z)$ and $\Delta_{\mathcal{C}}^n f(z)$) with multiplicities p and q must be the zero of $q\gamma(z) - p\eta(z)$, and this, together with (2.2) and (2.8), yields

$$\begin{aligned}
&\overline{N}_{(p,q)}\left(r, \frac{1}{f(z)}\right) + \overline{N}_{(p,q)}\left(r, \frac{1}{f(z) - a}\right) \\
&\leq \overline{N}\left(r, \frac{1}{q\gamma(z) - p\eta(z)}\right) \leq T(r, q\gamma(z) - p\eta(z)) \\
&\leq T(r, \gamma(z)) + T(r, \eta(z)) + O(1) = S(r, f).
\end{aligned} \tag{2.9}$$

Thus (2.3) and (2.9) show that

$$\begin{aligned}
&\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z) - a}\right) \\
&= \sum_{p,q} \left(\overline{N}_{(p,q)}\left(r, \frac{1}{f(z)}\right) + \overline{N}_{(p,q)}\left(r, \frac{1}{f(z) - a}\right) \right) \\
&\leq \sum_{p+q < 8} \left(\overline{N}_{(p,q)}\left(r, \frac{1}{f(z)}\right) + \overline{N}_{(p,q)}\left(r, \frac{1}{f(z) - a}\right) \right) \\
&\quad + \sum_{p+q \geq 8} \left(\overline{N}_{(p,q)}\left(r, \frac{1}{f(z)}\right) + \overline{N}_{(p,q)}\left(r, \frac{1}{f(z) - a}\right) \right) \\
&\leq \frac{1}{8} \sum_{p+q \geq 8} \left(N_{(p,q)}\left(r, \frac{1}{f(z)}\right) + N_{(p,q)}\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z)}\right) \right) \\
&\quad + \frac{1}{8} \sum_{p+q \geq 8} \left(N_{(p,q)}\left(r, \frac{1}{f(z) - a}\right) + N_{(p,q)}\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - a}\right) \right) + S(r, f) \\
&\leq \frac{1}{8} \left(N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z)}\right) \right) \\
&\quad + \frac{1}{8} \left(N\left(r, \frac{1}{f(z) - a}\right) + N\left(r, \frac{1}{\Delta_{\mathcal{C}}^n f(z) - a}\right) \right) + S(r, f) \\
&\leq \frac{1}{4} T(r, f(z)) + \frac{1}{4} T(r, \Delta_{\mathcal{C}}^n f(z)) + S(r, f) = \frac{1}{2} T(r, f(z)) + S(r, f),
\end{aligned}$$

which contradicts (2.5).

Case 2. Suppose that $q\gamma(z) \equiv p\eta(z)$.

Since $\Delta_c^n f(z) \not\equiv f(z)$, according to (2.1), we have

$$q \cdot \frac{f'(z)}{f(z)(f(z)-a)} \equiv p \cdot \frac{(\Delta_c^n f(z))'}{\Delta_c^n f(z)(\Delta_c^n f(z)-a)}.$$

Integration gives

$$\left(\frac{f(z)}{f(z)-a} \right)^q \equiv A \left(\frac{\Delta_c^n f(z)}{\Delta_c^n f(z)-a} \right)^p, \quad (2.10)$$

where $A (\neq 0)$ is a constant.

By Lemma 2.4 and (2.3) the last equation implies

$$qT(r, f(z)) = pT(r, \Delta_c^n f(z)) + O(1) = pT(r, f(z)) + S(r, f),$$

which leads to $p = q$. By (2.10) it follows that there exists a nonzero constant B such that

$$\frac{f(z)}{f(z)-a} \equiv B \frac{\Delta_c^n f(z)}{\Delta_c^n f(z)-a}.$$

Since $\Delta_c^n f(z) \not\equiv f(z)$, we get that $B \neq 1$.

Rewrite the last equation as

$$f(z) - \frac{aB}{B-1} = \frac{aB}{1-B} \cdot \frac{f(z)-a}{\Delta_c^n f(z)-a}. \quad (2.11)$$

Since $B \neq 0, 1$, we get $\frac{aB}{B-1} \neq 0, a$. According to (2.11), it is obvious that any zero of $f(z) - \frac{aB}{B-1}$ must be a zero of $f(z) - a$, which is impossible. Hence $f(z) - \frac{aB}{B-1}$ has no zeros. According to the second fundamental theorem and (2.5), we deduce the following contradiction:

$$\begin{aligned} T(r, f(z)) &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z) - \frac{aB}{B-1}}\right) \\ &\quad + \overline{N}(r, f(z)) - T(r, f(z)) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)-a}\right) + S(r, f) - T(r, f(z)) = S(r, f). \end{aligned}$$

Therefore we immediately get $\Delta_c^n f(z) \equiv f(z)$. The proof of Theorem 1.1 is completed. \square

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have drafted the manuscript, read, and approved the final manuscript.

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