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Hahn difference equations in Banach algebras

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Abstract

Hahn introduced the difference operator $D_{q,\omega}f(t) = (f(qt+\omega) - f(t))/(t(q-1)+\omega)$ in 1949, where $0 < q < 1$ and $\omega > 0$ are fixed real numbers. This operator extends the classical difference operator $\Delta_\omega f(t) = (f(t+\omega) - f(t))/\omega$ as well as the Jackson q-difference operator $D_q f(t) = (f(qt) - f(t))/(t(q-1))$.

In this paper, we study the theory of abstract linear Hahn difference equations of the form

$$A_0(t)D^n_{q,\omega}x(t) + A_1(t)D^{n-1}_{q,\omega}x(t) + \cdots + A_n(t)x(t) = B(t),$$

where B and A_i are mappings from an interval I into a Banach algebra $\mathbb{X}$, $i = 1, \ldots, n$. We define the abstract exponential functions and the abstract trigonometric (hyperbolic) functions. We prove they are solutions of first and second order Hahn difference equations, respectively. Also, we obtain an integral equation corresponding to the second order linear Hahn difference equations which is known as the Volterra integral equation. Finally, we present the analogs of the variation of parameter technique and the annihilator method for the non-homogeneous case.

MSC: 39A13; 39A70

Keywords: Hahn difference operator; Jackson q-difference operator

1 Introduction and preliminaries

Hahn introduced his difference operator, which is defined by

$$D_{q,\omega}f(t) = \begin{cases} \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, & \text{if } t \neq \theta, \\ f'(\theta), & \text{if } t = \theta, \end{cases}$$

where $0 < q < 1$ and $\omega > 0$ are fixed real numbers, $\theta = \omega/(1-q)$; see [1, 2]. This operator unifies and generalizes two well-known difference operators. The first is the Jackson q-difference operator defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)}, \quad t \neq 0.$$

Here f is supposed to be defined on a q-geometric set $A \subset \mathbb{R}$ for which $qt \in A$ whenever $t \in A$; see [3–11]. The second operator is the forward difference operator

$$\Delta_\omega f(t) = \frac{f(t+\omega) - f(t)}{\omega};$$

see [12–15]. Hahn's operator was applied and used in a lot of fields, especially in the construction of families of orthogonal polynomials and in investigating some approximation problems. For more details, see [16–18]. Contrary to the q-difference operator and the forward difference operator, the Hahn difference operator did not generate any interest until Annaby *et al.* gave a rigorous analysis of the calculus associated with $D_{q,\omega}$ in [19]. Thereafter, Hamza and Ahmed proved the existence and uniqueness of solutions of Hahn difference equations and studied the theory of linear Hahn difference equations; see [20, 21].

This article is devoted to the study of the theory of Hahn difference equations in Banach algebras. We define the abstract exponential functions and the abstract trigonometric (hyperbolic) functions. We prove they are solutions of first and second order Hahn difference equations, respectively. Every choice of the Banach algebra gives a wide class of Hahn difference equations. For instance, this study allows us to consider equations with solutions with values in the Banach algebra $B(X)$, the Banach space of all bounded linear operators from a Banach space X into itself. As special cases, our study includes finite and infinite systems of Hahn difference equations.

In our study we need the function $h(t) = qt + \omega$, which is normally taken to be defined on an interval I, which contains the number θ. The sequence

$$h^k(t) = q^k t + \omega[k]_q, \quad t \in I,$$

is the kth order iteration of $h(t)$, which uniformly converges to θ on I, and $[k]_q$ is defined by

$$[k]_q = \frac{1-q^k}{1-q}.$$

Throughout this paper, X is a Banach space, $\mathbb{X}$ is a Banach algebra with a norm $\|\ \|$, and I is an interval including θ. Now, we will introduce some basic definitions and theorems that will be needed in our study.

Definition 1.1 Assume that $f : I \to X$ is a function and let $a, b \in I$. The q,ω-integral of f from a to b is defined by

$$\int_a^b f(t)\, d_{q,\omega}t = \int_\theta^b f(t)\, d_{q,\omega}t - \int_\theta^a f(t)\, d_{q,\omega}t,$$

where

$$\int_\theta^x f(t)\, d_{q,\omega}t = \bigl(x(1-q)-\omega\bigr)\sum_{k=0}^{\infty} q^k f\bigl(h^k(x)\bigr), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$.

Definition 1.2 For certain $z \in \mathbb{C}$, the q,ω-exponential functions $e_z(t)$ and $E_z(t)$ are defined by

$$e_z(t) = \sum_{k=0}^{\infty} \frac{(z(t(1-q)-\omega))^k}{(q;q)_k} = \frac{1}{\prod_{k=0}^{\infty}(1-zq^k(t(1-q)-\omega))} \tag{1.1}$$

and

$$E_z(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}k(k-1)}(z(t(1-q)-\omega))^k}{(q;q)_k} = \prod_{k=0}^{\infty}\bigl(1 + zq^k\bigl(t(1-q)-\omega\bigr)\bigr), \tag{1.2}$$

where $e_z(t)$ and $E_z(t)$ are the solutions of the first order Hahn difference problems

$$D_{q,\omega}y(t) = zy(t), \qquad y(\theta) = 1, \tag{1.3}$$

and

$$D_{q,\omega}y(t) = -zy(qt+\omega), \qquad y(\theta) = 1, \quad z,t \in \mathbb{C}, \tag{1.4}$$

respectively; see [19]. For a fixed $z \in \mathbb{C}$, (1.2) converges for all $t \in \mathbb{C}$, defining an entire function of order zero. For the proofs of the equalities in (1.1) and (1.2), see Section 1.3 in [22] and [23]. Also, we can prove these equalities using the method of successive approximation; see Section 4. Here the q-shifted factorial $(b;q)_n$ for a complex number b and $n \in \mathbb{N}_0$ is defined to be

$$(b;q)_n = \begin{cases} \prod_{j=1}^{n}(1-bq^{j-1}), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

By replacing the complex fixed number z by a complex function $p(t)$ which is continuous at θ in (1.3), we obtain the exponential functions $e_p(t)$ and $E_p(t)$,

$$e_p(t) = \frac{1}{\prod_{k=0}^{\infty}(1 - p(h^k(t))q^k(t(1-q)-\omega))}, \tag{1.5}$$

$$E_p(t) = \prod_{k=0}^{\infty}\bigl(1 + p\bigl(h^k(t)\bigr)q^k\bigl(t(1-q)-\omega\bigr)\bigr), \tag{1.6}$$

whenever the two products are convergent to a nonzero number for every $t \in I$; see [21]. It is worth noting that the two products are convergent since $\sum_{k=0}^{\infty}|p(h^k(t))|q^k(t(1-q)-\omega)$ is convergent; see [6].

The following lemma gives the q, ω derivative of sum, product, and quotients of q,ω-differentiable functions, with values in $\mathbb{X}$.

Lemma 1.3 *Let $A : I \to \mathbb{X}$ and $B : I \to \mathbb{X}$ be q,ω-differentiable at $t \in I$. Then*:

(i) $D_{q,\omega}(A+B)(t) = D_{q,\omega}A(t) + D_{q,\omega}B(t)$,

(ii) $D_{q,\omega}(AB)(t) = D_{q,\omega}(A(t))B(h(t)) + A(t)D_{q,\omega}B(t) = D_{q,\omega}(A(t))B(t) + A(h(t))D_{q,\omega}B(t)$,

(iii) *for any constant* $c \in \mathbb{X}$, $D_{q,\omega}(cA)(t) = cD_{q,\omega}(A(t))$,

(iv) $D_{q,\omega}(A^{-1})(t) = -(A(h(t)))^{-1}(D_{q,\omega}A(t))(A(t))^{-1}$ *provided that for every* $t \in I$, $(A(t))^{-1}$ *exists*,

(v) $D_{q,\omega}(AB^{-1})(t) = D_{q,\omega}A(t)(B(h(t)))^{-1} - A(t)(B(h(t)))^{-1}D_{q,\omega}B(t)(B(t))^{-1}$ *provided that for every* $t \in I$, $(B(t))^{-1}$ *exists*.

The following theorem is important and will be used later on.

Theorem 1.4 [19] *Assume $f : I \to \mathbb{R}$ is continuous at θ. Then the following statements are true.*

(i) $\{f((sq^k) + \omega[k]_q)\}_{k\in\mathbb{N}}$ *converges uniformly to* $f(\theta)$ *on* I.

(ii) $\sum_{k=0}^{\infty} q^k|f(sq^k + \omega[k]_q)|$ *is uniformly convergent on* I *and consequently* f *is* q,ω*-integrable over* I.

(iii) *Define*

$$F(x) := \int_\theta^x f(t)\,d_{q,\omega}t, \quad x \in I.$$

Then F *is continuous at* θ. *Furthermore,* $D_{q,\omega}F(x)$ *exists for every* $x \in I$ *and*

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}f(t)\,d_{q,\omega}t = f(b) - f(a) \quad \textit{for all } a, b \in I.$$

This paper is organized as follows:

In Section 2, we introduce some existence and uniqueness results from [20]. At the end of this section, we apply these results to obtain the required conditions for the existence and uniqueness of solutions of linear Hahn difference equations

$$\begin{aligned} &A_0(t)D_{q,\omega}^n x(t) + A_1(t)D_{q,\omega}^{n-1}x(t) + \cdots + A_n(t)x(t) = B(t), \\ &D_{q,\omega}^{i-1}x(\theta) = y_i \in \mathbb{X}, \quad i = 1, \ldots, n, \end{aligned}$$

where $A_i, B : I \to \mathbb{X}$. In Section 3, we present the Hahn Wronskian in Banach algebras. We establish its properties. It is an effective tool to determine whether the set of solutions is a fundamental set or not. Finally we give Liouville's formula for Hahn difference equations of the second order. In Section 4, we define the abstract exponential functions. We prove they are the solutions of the first order linear Hahn difference equations. At the end of this section, we establish their properties. Section 5 is devoted to the abstract trigonometric (hyperbolic) functions. We prove that they are solutions of second order linear Hahn difference equations. Some of their properties are established. In Section 6, we exhibit the variation of parameters method and the annihilator method for non-homogeneous Hahn difference equations.

2 Existence and uniqueness results

Inspired by the work of Hamza and Ahmed [20, 21], we can obtain the required conditions for the existence and uniqueness of solutions of linear Hahn difference equations of the form

$$\left.\begin{aligned} &A_0(t)D_{q,\omega}^n x(t) + A_1(t)D_{q,\omega}^{n-1}x(t) + \cdots + A_n(t)x(t) = B(t), \\ &D_{q,\omega}^{i-1}x(\theta) = y_i \in \mathbb{X}, \quad i = 1, \ldots, n, \end{aligned}\right\} \tag{2.1}$$

where $A_i : I \to \mathbb{X}$, $i = 0, 1, \ldots, n$, and $B : I \to \mathbb{X}$.

As usual, we denote

$$S(x_0,b) = \{x \in X : \|x - x_0\| \le b\}$$

and

$$R = [\theta, \theta + a] \times S(x_0, b),$$

where a, b are fixed positive numbers.

First, we mention the following results from [20], which will be needed to establish the main results of this section.

Theorem 2.1 *Assume that $f : R \to X$ satisfies the following conditions*:

(i) *$f(t,x)$ is continuous at $t = \theta$ for every $x \in S(x_0,b)$.*

(ii) *There is a positive constant V such that the Lipschitz condition $\|f(t,x) - f(t,y)\| \le V\|x - y\|$ for all $x, y \in X$ is satisfied.*

Then there is $h > 0$ such that the following Cauchy problem:

$$\left.\begin{array}{l} D_{q,\omega}x(t) = f(t,x), \\ x(\theta) = x_0, \end{array}\right\} \tag{2.2}$$

has a unique solution $x(t)$ on $[\theta, \theta + h]$.

As a direct consequence of Theorem 2.1, they proved the following result.

Corollary 2.2 *Let I be an interval containing θ, $f_i(t,x_1,x_2,\dots,x_n) : I \times \prod_{i=1}^n S_i(y_i,b_i) \to X$ such that the following conditions are satisfied*:

(i) *For $x_i \in S_i(y_i,b_i)$, $1 \le i \le n$, $f_i(t,x_1,x_2,\dots,x_n)$ is continuous at $t = \theta$.*

(ii) *There is a positive constant V such that, for $t \in I$, $x_i, x_i' \in S_i(y_i,b_i)$, $1 \le i \le n$, the following Lipschitz condition is satisfied*:

$$\|f_i(t,x_1,x_2,\dots,x_n) - f_i(t,x_1',x_2',\dots,x_n')\| \le V\sum_{i=1}^n \|x_i - x_i'\|.$$

Then there exists a unique solution of the initial value problem

$$\left.\begin{array}{l} D_{q,\omega}x_i(t) = f_i(t,x_1(t),x_2(t),\dots,x_n(t)), \quad 1 \le i \le n, t \in I, \\ x_i(\theta) = y_i \in X. \end{array}\right\} \tag{2.3}$$

The Cauchy problem

$$\left.\begin{array}{l} D^n_{q,\omega}x(t) = f(t,x(t),D_{q,\omega}x(t),\dots,D^{n-1}_{q,\omega}x(t)), \\ D^{i-1}_{q,\omega}x(\theta) = y_i, \quad 1 \le i \le n, \end{array}\right\} \tag{2.4}$$

is equivalent to the first order system (2.3) in the sense that $\{\phi_i(t)\}_{i=1}^n$ is a solution of (2.3) if and only if $\phi_1(t)$ is a solution of (2.4). Here,

$$f_i(t,x_1,\dots,x_n) = \begin{cases} x_{i+1}, & 1 \le i \le n-1, \\ f(t,x_1,\dots,x_n), & i = n. \end{cases}$$

As a consequence of the above results, they deduced the following theorem.

Theorem 2.3 *Let $f(t,x_1,\dots,x_n)$ be a function defined on $I \times \prod_{i=1}^{n} S_i(y_i,b_i)$ such that the following conditions are satisfied*:

(i) *For any values of $x_r \in S_r(y_r,b_r)$, f is continuous at $t=\theta$.*

(ii) *f satisfies Lipschitz condition*

$$\bigl\|f(t,x_1,\dots,x_n) - f\bigl(t,x_1',\dots,x_n'\bigr)\bigr\| \le V\sum_{i=1}^{n}\bigl\|x_i - x_i'\bigr\| \quad \forall x_r,x_r' \in S_r(y_r,b_r), t \in I,$$

where $V>0$.

Then the Cauchy problem (2.4) *has a unique solution which is valid on $[\theta,\theta+h]$.*

Now, we are ready to establish the required conditions for the existence and uniqueness of solutions of the Cauchy problem (2.1).

Theorem 2.4 *Assume that $A_j : I \to \mathbb{X}$, $0 \le j \le n$, and $B : I \to \mathbb{X}$ satisfy the following conditions*:

(i) *$A_j(t)$, $0 \le j \le n$ and $B(t)$ are continuous at θ and $A_0(t)$ is invertible, $t \in I$.*

(ii) *$A_0^{-1}(t)A_j(t)$ is bounded on I.*

Then, for any elements y_r, equation (2.1) *has a unique solution on a closed subinterval $J \subset I$ containing θ.*

Theorem 2.1 is called a local existence theorem because it guarantees the existence of a solution $x(t)$ defined for $t \in I$ which is 'close to' the initial point θ. On the other hand, a useful result which is concerned with global results was given in [20], which means that the solution exists on the entire interval $I = [\theta,\theta+a]$. One can see that a Lipschitz condition of f satisfied on a strip

$$S = [\theta,\theta+a] \times X$$

rather than on the rectangle R which is given in the beginning of this section implies the existence and uniqueness of the solutions on the entire interval $I = [\theta,\theta+a]$; see [20].

This can be stated in the next two theorems.

Theorem 2.5 *Let f be continuous on the strip S and suppose there exists a constant $V>0$ such that $\|f(t,x) - f(t,y)\| \le V\|x-y\|$ for all $(t,x),(t,y) \in S$. Then the successive approximations given in*

$$\left.\begin{aligned} &\phi_0(t) = x_0, \\ &\phi_{k+1}(t) = x_0 + \textstyle\int_\theta^t f(s,\phi_k(s))\, d_{q,\omega}s, \quad k \ge 0, \end{aligned}\right\} \tag{2.5}$$

exist on the entire interval $[\theta,\theta+a]$ and converge there uniformly to the unique solution of (2.2).

Theorem 2.6 *Let f be continuous on the half-plane*

$$[\theta,\infty) \times X.$$

Assume that f satisfies a Lipschitz condition

$$\bigl\|f(t,x)-f(t,y)\bigr\| \le L_{\theta,a}\|x-y\|$$

on each strip

$$S_{\theta,a}=\bigl\{(t,x)\in I\times X : t\in[\theta,\theta+a], \|x\|<\infty\bigr\},$$

where $L_{\theta,a}$ is a constant that may depend on θ and a. Then the initial value problem (2.2) *has a unique solution that exists on the whole half-line* $[\theta,\infty)$.

3 Fundamental set of solutions and Hahn Wronskian in Banach algebras

In this section, we consider the homogeneous linear Hahn difference equation in a Banach algebra,

$$A_0(t)D^n_{q,\omega}x(t)+A_1(t)D^{n-1}_{q,\omega}x(t)+\cdots+A_n(t)x(t)=0. \tag{3.1}$$

The coefficients $A_j(t)\in\mathbb{X}$, $0\le j\le n$ are assumed to satisfy the conditions of Theorem 2.4. Here $\mathbb{X}$ is a commutative Banach algebra with a unit element $\mathfrak{e}$. We present the Hahn Wronskian in Banach algebras. We establish its properties. We determine whether the set of solutions is a fundamental set or not according to the Wronskian being invertible or not. Finally we give Liouville's formula for Hahn difference equations of the second order.

Definition 3.1 $\mathbb{X}$ is called a Banach algebra with unit $\mathfrak{e}$ if:

(i) $\mathbb{X}$ is a Banach space.

(ii) There is a multiplication $\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ that has the following properties:

$$(xy)z=x(yz),\qquad (x+y)z=xz+yz,\qquad x(y+z)=xy+xz;$$
$$c(xy)=(cx)y=x(cy)$$

for all $x,y,z\in\mathbb{X}$, $c\in\mathbb{C}$. Moreover, there is a unit element $\mathfrak{e}$, *i.e.*

$$\mathfrak{e}x=x\mathfrak{e}=x\quad\text{for all } x\in\mathbb{X}.$$

(iii) $\|\mathfrak{e}\|=1$.

(iv) $\|xy\|\le\|x\|\|y\|$ for all $x,y\in\mathbb{X}$.

Lemma 3.2 *If $x_1(t)$ and $x_2(t)$ are two solutions of equation* (3.1), *then $c_1x_1(t)+c_2x_2(t)$ is also a solution where c_1 and c_2 are constants in $\mathbb{X}$.*

Definition 3.3 We say that the solutions $\psi_1(t),\dots,\psi_n(t)$ are linearly independent if

$$\sum_{j=1}^n x_j\psi_j=0,\quad\text{then } x_j=0 \text{ for every } j=1,2,\dots,n.$$

As usual $[\psi_1(t),\dots,\psi_n(t)]$ is called a fundamental set of solutions of equation (3.1) if they are linearly independent and every solution $\psi(t)$ has the representation

$$\psi(t)=\sum_{j=1}^{n}x_j\psi_j(t),\quad x_j\in\mathbb{X}.$$

Definition 3.4 A matrix $B\in M_{n\times n}(\mathbb{X})$ is said to have an inverse $C\in M_{n\times n}(\mathbb{X})$ if

$$BC=CB=\mathcal{I},$$

where

$$\mathcal{I}=\begin{bmatrix}\mathfrak{e} & 0 & \cdots & 0\\ 0 & \mathfrak{e} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathfrak{e}\end{bmatrix}.$$

We follow the proof of [21] to get the following result.

Theorem 3.5 *Let $b_{ij}\in\mathbb{X}$, $i,j\in\{1,\dots,n\}$, and for each j, $\psi_j(t)$ is the unique solution of equation* (3.1) *which satisfies the initial conditions*

$$D_{q,\omega}^{i-1}\psi_j(\theta)=b_{ij},\quad i,j=1,\dots,n.$$

Then $\{\psi_j(t)\}_{j=1}^{n}$ is a fundamental set of equation (3.1) *if and only if the matrix $[b_{ij}]$ is invertible.*

Theorem 3.6 *Let $\psi(t)$ be any solution of equation* (3.1) *and $\{\psi_j, 1\le j\le n\}$ forms a fundamental set for equation* (3.1) *valid in J. Then there are unique constants $c_j\in\mathbb{X}$ such that*

$$\psi(t)=c_1\psi_1(t)+\cdots+c_n\psi_n(t)\quad\forall t\in J. \tag{3.2}$$

Now we define the abstract Hahn Wronskian and prove some of its properties. In the rest of this section, unless we mention otherwise, $\mathbb{X}$ is a commutative Banach algebra with a unit $\mathfrak{e}$.

Definition 3.7 We define the q,ω-Wronskian of the functions $x_1,\dots,x_n:I\to\mathbb{X}$, by

$$W_{q,\omega}(x_1,\dots,x_n)(t)=\begin{vmatrix}x_1(t) & \cdots & x_n(t)\\ D_{q,\omega}x_1(t) & \cdots & D_{q,\omega}x_n(t)\\ \vdots & \ddots & \vdots\\ D_{q,\omega}^{n-1}x_1(t) & \cdots & D_{q,\omega}^{n-1}x_n(t)\end{vmatrix}$$

provided that $x_1,\dots,x_n$ are q,ω-differentiable functions $n-1$ times.

We write $W_{q,\omega}$ instead of $W_{q,\omega}(x_1,\dots,x_n)$ unless there is ambiguity.

Lemma 3.8 *Let $x_1(t), x_2(t), \dots, x_n(t) : I \to \mathbb{X}$ be q,ω-differentiable n times. Then, for any $t \in I$, $t \neq \theta$,*

$$D_{q,\omega} Wq,\omega(t) = \begin{vmatrix} x_1(h(t)) & \cdots & x_n(h(t)) \\ D_{q,\omega}x_1(h(t)) & \cdots & D_{q,\omega}x_n(h(t)) \\ \vdots & \ddots & \vdots \\ D^{n-2}_{q,\omega}x_1(h(t)) & \cdots & D^{n-2}_{q,\omega}x_n(h(t)) \\ D^{n}_{q,\omega}x_1(t) & \cdots & D^{n}_{q,\omega}x_n(t) \end{vmatrix}. \tag{3.3}$$

Proof See [9]. □

In the rest of this section, J is a closed subinterval of the interval I containing θ.

Theorem 3.9 *If $x_1, \dots, x_n$ are solutions of equation* (3.1) *in J, then their q,ω-Wronskian satisfies the first order Hahn difference equation*

$$D_{q,\omega} W_{q,\omega}(t) = -R(t) W_{q,\omega}(t) \quad \forall t \in J \setminus \{\theta\}, \tag{3.4}$$

where $R(t) = \sum_{k=0}^{n-1} (t - h(t))^k A_0^{-1}(t) A_{k+1}(t)$.

Theorem 3.10 *The q,ω-Wronskian of any set of solutions $\{\psi_i(t)\}_{i=1}^n$, valid in J, is given by*

$$W_{q,\omega}(t) = \left[\prod_{k=0}^{\infty} \bigl(\mathfrak{e} + q^k \bigl(t(1-q) - \omega\bigr) R\bigl(h^k(t)\bigr)\bigr)\right]^{-1} W_{q,\omega}(\theta), \quad t \in J, \tag{3.5}$$

provided that the product has an inverse.

An interesting result which can be deduced directly from Theorems 3.9 and 3.10 is the following.

Corollary 3.11 *Assume that the product in* (3.5) *has an inverse. Let $\{\psi_i\}_{i=1}^n$ be a set of solutions of equation* (3.1) *in J. Then $W_{q,\omega}(t)$ has two possibilities*:

(i) *$W_{q,\omega}(t)$ is invertible in J if and only if $\{\psi_i\}_{i=1}^n$ is a fundamental set of equation* (3.1) *valid in J.*

(ii) *$W_{q,\omega}(t)$ is not invertible in J if and only if $\{\psi_i\}_{i=1}^n$ is not a fundamental set of equation* (3.1).

4 Abstract exponential functions and first order linear Hahn difference equations

Let $A : I \to \mathbb{X}$ be continuous at θ. We define the exponential functions $e_A(t)$ and $E_A(t)$ by

$$e_A(t) = \left[\prod_{k=0}^{\infty} \bigl(\mathfrak{e} - A\bigl(h^k(t)\bigr) q^k \bigl(t(1-q) - \omega\bigr)\bigr)\right]^{-1} \tag{4.1}$$

and

$$E_A(t) = \prod_{k=0}^{\infty} \bigl(\mathfrak{e} + A\bigl(h^k(t)\bigr) q^k \bigl(t(1-q) - \omega\bigr)\bigr), \tag{4.2}$$

provided that the products in (4.1) and (4.2) are convergent and the first product has an inverse. Our aim in this section is to prove that $e_A(t)$ and $E_{-A}(t)$ are the unique solutions of the first order Hahn difference equations

$$D_{q,\omega}x(t) = A(t)x(t), \qquad x(\theta) = \mathfrak{e}$$

and

$$D_{q,\omega}x(t) = -A(t)x(qt+\omega), \qquad x(\theta) = \mathfrak{e},$$

respectively.

We need the following lemma.

Lemma 4.1 *Let $\mathbb{X}$ be a Banach algebra with a unit $\mathfrak{e}$ and $\{B_k\}_{k=0}^{\infty} \subseteq \mathbb{X}$. If $\sum_{k=0}^{\infty} \|B_k\|$ is convergent, then $\prod_{k=0}^{\infty}(\mathfrak{e}+B_k)$ converges to an element $P \in \mathbb{X}$. Moreover, if P has an inverse and $(\mathfrak{e}+B_k)$ is invertible for every k, then $P^{-1} = \prod_{k=0}^{\infty}(\mathfrak{e}+B_k)^{-1}$.*

Proof Assume $\sum_{k=0}^{\infty} \|B_k\|$ is convergent to a number M.

Let $\tilde{B}_N$ be the sequence of partial products. One can see that

$$\tilde{B}_N = \bigl(\bigl(1+\|B_0\|\bigr)\cdots\bigl(1+\|B_N\|\bigr)\bigr) \leq e^{(\|B_0\|+\cdots+\|B_N\|)}$$

and

$$1 \leq \tilde{B}_N \leq e^M.$$

Hence $\prod_{k=0}^{\infty}(1+\|B_k\|)$ is convergent. This implies that $\sum_{k=0}^{\infty} \|B_k\|$ is convergent; see [24].

We define the two sequences P_j and $\tilde{P}_j$ by

$$P_J = \prod_{k=0}^{J}(\mathfrak{e}+B_k) \quad \text{and} \quad \tilde{P}_J = \prod_{k=0}^{J}\bigl(1+\|B_k\|\bigr).$$

For $M, N \in \mathbb{N}$, one can see that

$$\|P_M - P_N\| \leq \|\tilde{P}_M - \tilde{P}_N\|.$$

Thus the convergence of the sequence $\tilde{P}_N$ implies the convergence of the sequence of P_N. Assume that $P_N \to P$ as $N \to \infty$.

This leads to the desired result. The mapping $g : \mathfrak{I} \to \mathfrak{I}$, where $\mathfrak{I} = \{P \in \mathbb{X} : P^{-1} \text{ exists}\}$, defined by $g(P) = P^{-1}$, is continuous.

Since

$$\lim_{N\to\infty} P_N = P$$

we have

$$(P_N)^{-1} = \left(\prod_{k=0}^{N}(\mathfrak{e}+B_k)\right)^{-1} = \prod_{k=0}^{N}(\mathfrak{e}+B_k)^{-1} \to P^{-1},$$

which completes our proof. □

It is worth noting that the products in (4.1) and (4.2) are convergent, by Lemma 4.1, since

$$\sum_{k=0}^{\infty} \left\| A\big(h^k(t)\big) \right\| q^k\big(t(1-q)-\omega\big)$$

is convergent. In the rest of the paper, we assume that

$$\big(\mathfrak{e} - A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)$$

has an inverse, $k \in \mathbb{Z}^{\geq 0}$. Consequently, again by Lemma 4.1 we have

$$\left[\prod_{k=0}^{\infty} \big(\mathfrak{e} - A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)\right]^{-1} = \prod_{k=0}^{\infty} \big(\mathfrak{e} - A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)^{-1}.$$

We will need the following lemma in the next theorem.

Lemma 4.2 *Assume that*

$$A(t)A\big(h^k(t)\big) = A\big(h^k(t)\big)A(t), \quad k = 1, 2, \ldots. \tag{4.3}$$

Then A and E_{-A} commute.

Proof Assume that (4.3) holds. Then

$$\begin{aligned} A(t)\big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big) &= A(t) + A(t)A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big) \\ &= \big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)A(t). \end{aligned}$$

Consequently,

$$A(t)\left[\prod_{k=0}^{n} \big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)\right] = \left[\prod_{k=0}^{n} \big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)\right]A(t).$$

From the continuity of $A(t)$, we conclude that

$$A(t)\left[\prod_{k=0}^{\infty} \big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)\right] = \left[\prod_{k=0}^{\infty} \big(\mathfrak{e} + A\big(h^k(t)\big) q^k\big(t(1-q)-\omega\big)\big)\right]A(t).$$

Therefore, $A(t)E_{-A}(t) = E_{-A}(t)A(t)$. □

Theorem 4.3 *Assume $A(t)$ and $A(h^k(t))$ commute for every k. The q,ω-exponential functions $e_A(t)$ and $E_{-A}(t)$ are the unique solutions of the initial value problems*

$$D_{q,\omega}x(t) = A(t)x(t), \qquad x(\theta) = \mathfrak{e} \tag{4.4}$$

and

$$D_{q,\omega}x(t) = -A(t)x(qt+\omega), \qquad x(\theta) = \mathfrak{e}. \tag{4.5}$$

Proof First, $e_A(t)$ is a solution of equation (4.4). Indeed, we have, for $t \neq \theta$,

$$\begin{aligned} D_{q,\omega} e_A(t) &= \frac{1}{h(t)-t}\Bigg[\prod_{k=1}^{\infty}\big(\mathfrak{e} - A\big(h^k(t)\big)q^k\big(t(1-q)-\omega\big)\big)^{-1} \\ &\quad - \prod_{k=0}^{\infty}\big(\mathfrak{e} - A\big(h^k(t)\big)q^k\big(t(1-q)-\omega\big)\big)^{-1}\Bigg] \\ &= \frac{1}{h(t)-t}\Bigg[\prod_{k=0}^{\infty}\big(\mathfrak{e} - A(t)\big(t(1-q)-\omega\big)\big)\big(\mathfrak{e} - A\big(h^k(t)\big)q^k\big(t(1-q)-\omega\big)\big)^{-1} \\ &\quad - \prod_{k=0}^{\infty}\big(\mathfrak{e} - A\big(h^k(t)\big)q^k\big(t(1-q)-\omega\big)\big)^{-1}\Bigg] \\ &= \frac{1}{h(t)-t}\big(\mathfrak{e} - A(t)\big(t(1-q)-\omega\big) - \mathfrak{e}\big)\prod_{k=0}^{\infty}\big(\mathfrak{e} - A\big(h^k(t)\big)q^k\big(t(1-q)-\omega\big)\big)^{-1} \\ &= \frac{-A(t)(t(1-q)-\omega)}{h(t)-t} e_A(t) \\ &= A(t)e_A(t). \end{aligned}$$

Second, we see that $e_A(t)$ is unique. If $x(t)$ is another solution, then we have

$$\begin{aligned} D_{q,\omega}\big(e_A^{-1}(t)x(t)\big) &= D_{q,\omega}\big(E_{-A}(t)\big)x(t) + E_{-A}\big(h(t)\big)D_{q,\omega}\big(x(t)\big) \\ &= -A(t)E_{-A}\big(h(t)\big)x(t) + E_{-A}\big(h(t)\big)A(t)x(t) \\ &= 0. \end{aligned}$$

Hence, $e_A^{-1}(t)x(t)$ is constant, which implies $e_A^{-1}(t)x(t) = e_A^{-1}(\theta)x(\theta) = \mathfrak{e}$. Thus, $x(t) = e_A(t)$. Similarly, we can see that $E_{-A}(t)$ is a unique solution of equation (4.5). □

In the following, we derive the solution of the first order non-homogeneous Hahn difference equations of the form

$$D_{q,\omega}x(t) = A(t)x(t) + f(t), \qquad x(\theta) = x_\theta, \quad t \in I, x_\theta \in \mathbb{X}. \tag{4.6}$$

Theorem 4.4 *Assume that $f : I \to \mathbb{X}$ is continuous at θ, and $t, \tau \in I$, $t > \tau$. Then the solution of equation* (4.6) *has the form*

$$x(t) = e_A(t)\left(x_\theta + \int_\theta^t E_{-A}(q\tau + \omega)f(\tau)\,d_{q,\omega}\tau\right). \tag{4.7}$$

Proof The function $x(t)$ given in (4.7) solves equation (4.6). Indeed, we have

$$\begin{aligned} D_{q,\omega}x(t) &= A(t)e_A(t)x_\theta + A(t)e_A(t)\left(\int_\theta^t E_{-A}(q\tau + \omega)f(\tau)\,d_{q,\omega}\tau\right) \\ &\quad + e_A\big(h(t)\big)E_{-A}(qt + \omega)f(t) \\ &= A(t)x(t) + f(t). \end{aligned}$$

□

We prove some useful properties of the exponential function $e_A(t)$.

We define $\xi(t)$ by

$$\xi(t) = h(t) - t = t(q-1) + \omega.$$

Theorem 4.5 *Assume that $A : I \to \mathbb{X}$, and $B : I \to \mathbb{X}$ are continuous at θ. The following properties are true.*

(i) $e_A^{-1}(t) = e_{-A(\mathfrak{e}+\xi A)^{-1}}(t)$.

(ii) $e_A(t)e_B(t) = e_{A+B+\xi AB}(t)$, *where $e_A(t)$ and $B(t)$ commute.*

(iii) $e_A(t)e_B^{-1}(t) = e_{(A-B)(\mathfrak{e}+\xi B)^{-1}}(t)$, *where $e_A(t)$, $(\mathfrak{e}+\xi B)^{-1}$, and $B(t)$ are pairwise commutative.*

Proof (i) From equations (4.1) and (4.2) we have $e_A^{-1}(t) = E_{-A}(t)$. Then

$$\begin{aligned} D_{q,\omega}\bigl(E_{-A}(t)\bigr) &= -AE_{-A}\bigl(h(t)\bigr) \\ &= -A(\mathfrak{e}+\xi A)^{-1}E_{-A}(t) \\ &= -A(\mathfrak{e}+\xi A)^{-1}e_A^{-1}(t). \end{aligned}$$

Then by Theorem 4.3, (i) is true.

(ii) Let $Y(t) = e_A(t)e_B(t)$ and $e_A(t)$, $B(t)$ commute. Then

$$\begin{aligned} D_{q,\omega}\bigl(e_A(t)e_B(t)\bigr) &= Ae_A(t)(\mathfrak{e}+\xi B)e_B(t) + e_A(t)Be_B(t) \\ &= Ae_A(t)e_B(t) + \xi ABe_A(t)e_B(t) + Be_A(t)e_B(t) \\ &= (A+B+\xi AB)e_A(t)e_B(t). \end{aligned}$$

Therefore by Theorem 4.3, (ii) is true.

(iii) Let $Y(t) = e_A(t)e_B^{-1}(t)$ and suppose that $e_A(t)$, $(\mathfrak{e}+\xi B)^{-1}$, $B(t)$ are pairwise commutative. Hence,

$$\begin{aligned} D_{q,\omega}\bigl(e_A(t)e_B^{-1}(t)\bigr) &= Ae_A(t)e_B^{-1}\bigl(h(t)\bigr) - e_A(t)Be_B^{-1}\bigl(h(t)\bigr) \\ &= Ae_A(t)(\mathfrak{e}+\xi B)^{-1}e_B^{-1}(t) - Be_A(t)(\mathfrak{e}+\xi B)^{-1}Be_B^{-1}(t) \\ &= (A-B)(\mathfrak{e}+\xi B)^{-1}e_A(t)e_B^{-1}(t). \end{aligned}$$

Again by Theorem 4.3, (iii) is true. □

From now on, we define $e_A(t,\tau)$ by

$$e_A(t,\tau) = e_A(t)e_A^{-1}(\tau).$$

Theorem 4.6 *The following statements are true*:

(i) $e_A(\theta) = \mathfrak{e}$ *and* $e_0(t) = \mathfrak{e}$.

(ii) $D_{q,\omega}(e_A^{-1}(t)) = -e_A^{-1}(h(t))D_{q,\omega}(e_A(t))e_A^{-1}(t) = -e_A^{-1}(h(t))A(t) = (\mathfrak{e}+\xi A(t))^{-1}A(t)e_A^{-1}(t)$, *where $e_A(h(t))$, $e_A(t)$, $(\mathfrak{e}+\xi A(t))^{-1}$, and $A(t)$ are pairwise commutative.*

(iii) $e_A(t,\tau) = e_A^{-1}(\tau,t)$.

(iv) $e_A(t,s)e_A(s,\tau) = e_A(t,\tau)$.

Proof (i) is straightforward.

(ii) Let $e_A(h(t))$, $e_A(t)$, $(\mathfrak{e} + \xi A(t))^{-1}$, and $A(t)$ be pairwise commutative. We can see easily that $D_{q,\omega}(e_A(t)e_A^{-1}(t)) = 0$. Hence

$$D_{q,\omega}e_A(t)e_A^{-1}(t) + e_A\big(h(t)\big)D_{q,\omega}e_A^{-1}(t) = 0.$$

Then

$$e_A\big(h(t)\big)D_{q,\omega}e_A^{-1}(t) = -D_{q,\omega}e_A(t)e_A^{-1}(t).$$

This implies that

$$\begin{aligned} D_{q,\omega}\big(e_A^{-1}(t)\big) &= -e_A^{-1}\big(h(t)\big)D_{q,\omega}e_A(t)e_A^{-1}(t) \\ &= -e_A^{-1}\big(h(t)\big)A(t)e_A(t)e_A^{-1}(t) \\ &= -e_A^{-1}(t)\big(\mathfrak{e} + \xi A(t)\big)^{-1}A(t) \\ &= -E_{-A}(t)\big(\mathfrak{e} + \xi A(t)\big)^{-1}A(t). \end{aligned}$$

So,

$$D_{q,\omega}\big(e_A^{-1}(t)\big) = -E_{-A}(t)\big(\mathfrak{e} + \xi A(t)\big)^{-1}A(t).$$

Since

$$e_A^{-1}\big(h(t)\big) = e_A^{-1}(t) + \xi(t)D_{q,\omega}e_A^{-1}(t),$$

we have

$$\begin{aligned} e_A^{-1}\big(h(t)\big)A(t) &= e_A^{-1}(t)A(t) - \xi(t)A(t)e_A^{-1}\big(h(t)\big)A(t) \\ &= \big[\mathfrak{e} - \xi(t)A(t)\big(\mathfrak{e} + \xi A(t)\big)^{-1}\big]e_A^{-1}(t)A(t) \\ &= \big(\mathfrak{e} + \xi A(t)\big)^{-1}e_A^{-1}(t)A(t) \\ &= -D_{q,\omega}e_A^{-1}(t). \end{aligned}$$

(iii) We have

$$X(t) = e_A(t,\tau)e_A(\tau,t) = e_A(t)e_A^{-1}(\tau)e_A(\tau)e_A^{-1}(t) = \mathfrak{e}.$$

This implies

$$e_A(t,\tau) = e_A^{-1}(\tau,t).$$

(iv) Let $X(t) = e_A(t,s)e_A(s,\tau)$. So we conclude that

$$X(t) = e_A(t,s)e_A(s,\tau)e_A(t)e_A^{-1}(s)e_A(s)e_A^{-1}(\tau)e_A(t)e_A^{-1}(\tau) = e_A(t,\tau). \qquad \square$$

In the next lemma we assume $\mathbb{X} = L(X)$, the space of all bounded linear operators from a Banach space X into itself, with identity operator I_X. Now let $A^*(t) : X^* \to X^*$, where X^* is the dual of X, be the adjoint operator of $A(t) \in L(X)$, $t \in I$ defined by

$$\bigl(A^*(t)f\bigr)(x) = f\bigl(A(t)x\bigr) \quad \text{for all } f \in X^* \text{ and } x \in X.$$

Lemma 4.7
(i) $D_{q,\omega}(A^*) = (D_{q,\omega}A)^*$.
(ii) $e_A^{-1}(t) = e^*_{-A^*(I+\xi A^*)^{-1}}(t)$.

Proof (i) Since A is $D_{q,\omega}$ differentiable at $t \in I$, we have

$$\begin{aligned} D_{q,\omega}\bigl(A^*\bigr)(t) &= \left(\frac{A(h(t)) - A(t)}{h(t) - t}\right)^* \\ &= (D_{q,\omega}A)^*(t). \end{aligned}$$

(ii) Putting $X(t) = (e_A^{-1}(t))^*$, we obtain

$$\begin{aligned} D_{q,\omega}X(t) &= \bigl(D_{q,\omega}e_A^{-1}(t)\bigr)^* \\ &= \bigl(-e_A^{-1}\bigl(h(t)\bigr)D_{q,\omega}e_A(t)e_A^{-1}(t)\bigr)^* \\ &= \bigl(-A^*\bigl(I + \xi A^*\bigr)^{-1}e_{A^*}(t)\bigr) \\ &= -A^*\bigl(I + \xi A^*\bigr)^{-1}X(t), \end{aligned}$$

and at $t = \theta$, we have $X(\theta) = (e_A^{-1}(\theta))^* = I_X$. Hence, $X(t)$ is the solution of the IVP

$$D_{q,\omega}X(t) = -A^*\bigl(I + \xi A^*\bigr)^{-1}X(t), \qquad X(\theta) = I_X,$$

which exactly is $X(t) = e_{-A^*(I+\xi A^*)^{-1}}(t)$. Therefore, $e_A^{-1}(t) = e^*_{-A^*(I+\xi A^*)^{-1}}(t)$. □

Now, we return to the first order equation (4.4), when $A(t) = z \in \mathbb{X}$, $t \in I$. The unique solution of the Hahn difference equation

$$D_{q,\omega}x(t) = zx(t), \qquad x(\theta) = \mathfrak{e},$$

where $z \in \mathbb{X}$ is

$$x(t) = e_z(t) = \left[\prod_{k=0}^{\infty}\bigl(\mathfrak{e} - zq^k\bigl(t(1-q) - \omega\bigr)\bigr)\right]^{-1}.$$

In the following theorem, we deduce the summation expansion of $e_z(t)$.

Theorem 4.8 *Let $z \in \mathbb{X}$. The exponential function $e_z(t)$ is given by*

$$e_z(t) = \sum_{k=0}^{\infty}\frac{(z(t(1-q) - \omega))^k}{(q;q)_k}.$$

Proof Using the successive approximations (2.5) with $x_0 = \mathfrak{e}$,

$$x_{k+1}(t) = x_0 + z\int_\theta^t x_k(s)\,d_{q,\omega}s.$$

We prove by induction on n that

$$x_{n+1}(t) = \mathfrak{e}\sum_{k=0}^{n+1}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k}. \tag{4.8}$$

At $n = 0$,

$$\begin{aligned}
\text{L.H.S.} = x_1(t) &= \mathfrak{e} + z\mathfrak{e}(t-\theta) \\
&= \left(\mathfrak{e} + \frac{z(t(1-q)-\omega)}{(q;q)_1}\right) \\
&= \sum_{k=0}^{1}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k} \\
&= \text{R.H.S.}
\end{aligned}$$

Assume that (4.8) holds for $n = m$.
We want to prove

$$\begin{aligned}
x_{m+1}(t) &= \sum_{k=0}^{m+1}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k}, \\
\text{L.H.S.} &= \mathfrak{e} + z\int_\theta^t\left(\mathfrak{e} + z(t-\theta) + \frac{z^2(t-\theta)^2}{(1-q^2)} + \cdots + \frac{z^m(t-\theta)^m}{(1-q^2)\cdots(1-q^m)}\right)d_{q,\omega}s \\
&= \mathfrak{e} + z(t-\theta) + \frac{z^2(t-\theta)^2}{(1-q^2)} + \frac{z^3(t-\theta)^3}{(1-q^2)(1-q^3)} + \cdots \\
&\quad + \frac{z^{m+1}(t-\theta)^m}{(1-q^2)\cdots(1-q^m)}\sum_{k=0}^{\infty}q^k\bigl(tq^k + \omega[k]_q - \theta\bigr)^m \\
&= \mathfrak{e} + z(t-\theta) + \frac{z^2(t-\theta)^2}{(1-q^2)} + \frac{z^3(t-\theta)^3}{(1-q^2)(1-q^3)} \\
&\quad + \cdots + \frac{z^{m+1}(t-\theta)^{m+1}}{(1-q^2)\cdots(1-q^m)}\sum_{k=0}^{\infty}q^{k(m+1)} \\
&= \sum_{k=0}^{m+1}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k} \\
&= \text{R.H.S.}
\end{aligned}$$

Therefore,

$$x_{n+1}(t) = \sum_{k=0}^{n+1}\frac{(z(t(1-q)-\omega))^{k+1}}{(q;q)_{k+1}},$$

which leads directly to our desired result by taking $n \to \infty$. □

In the following theorem we can obtain a summation expansion of $e_A(t)$ for a general mapping $A : I \to \mathbb{X}$. Similarly, we use the successive approximation method to prove this theorem.

Theorem 4.9 *Let $A : I \to \mathbb{X}$ be continuous at θ. The exponential functions $e_A(t)$ can be written as follows*:

$$\begin{aligned} e_A(t) &= \mathfrak{e} + \int_\theta^t A(s_1)\, d_{q,\omega}s_1 + \int_\theta^t A(s_1) \int_\theta^{s_1} A(s_2)\, d_{q,\omega}s_2\, d_{q,\omega}s_1 \\ &\quad + \cdots + \int_\theta^t A(s_1) \int_\theta^{s_1} A(s_2) \cdots \int_\theta^{s_{i-1}} A(s_i)\, d_{q,\omega}s_i \cdots d_{q,\omega}s_2\, d_{q,\omega}s_1 \cdots . \end{aligned} \tag{4.9}$$

5 Abstract trigonometric functions and second order linear Hahn difference equations

Let $A : I \to \mathbb{X}$ be continuous at θ. We define the abstract trigonometric and hyperbolic functions and we study some of their properties.

Definition 5.1 We define the abstract trigonometric functions by

$$\sin_A(t) = \frac{e_{iA}(t) - e_{-iA}(t)}{2i} \tag{5.1}$$

and

$$\cos_A(t) = \frac{e_{iA}(t) + e_{-iA}(t)}{2}, \tag{5.2}$$

and we define the functions $\mathrm{Sin}_A(t)$ and $\mathrm{Cos}_A(t)$ by

$$\mathrm{Sin}_A(t) = \frac{E_{iA}(t) - E_{-iA}(t)}{2i} \tag{5.3}$$

and

$$\mathrm{Cos}_A(t) = \frac{E_{iA}(t) + E_{-iA}(t)}{2}. \tag{5.4}$$

The following formulas can be proved easily:

(i) $\cos_A(t) + i\sin_A(t) = e_{iA}(t)$,

(ii) $\sin_A(t)\,\mathrm{Sin}_A(t) + \cos_A(t)\,\mathrm{Cos}_A(t) = 1$,

(iii) $\sin_A(t)\,\mathrm{Cos}_A(t) - \cos_A(t)\,\mathrm{Sin}_A(t) = 0$,

(iv) $\sin_A^2(t) + \cos_A^2(t) = e_{iA}(t)e_{-iA}(t)$,

(v) $\mathrm{Sin}_A^2(t) + \mathrm{Cos}_A^2(t) = E_{iA}(t)E_{-iA}(t)$.

Simple computations show that

$$D_{q,\omega} \sin_A(t) = A\cos_A(t), \qquad D_{q,\omega} \cos_A(t) = -A\sin_A(t)$$

and

$$D_{q,\omega}\,\mathrm{Sin}_A(t) = A\,\mathrm{Cos}_A(qt + \omega), \qquad D_{q,\omega}\,\mathrm{Cos}_A(t) = -A\,\mathrm{Sin}_A(qt + \omega).$$

The next theorem shows that the trigonometric functions satisfy second order Hahn difference equations.

Theorem 5.2 *The functions* $\sin_A(t)$, $\cos_A(t)$, $\mathrm{Sin}_A(t)$, *and* $\mathrm{Cos}_A(t)$ *satisfy the following second order Hahn difference equations, respectively*:

$$D^2_{q,\omega}y(t) - A(qt+\omega)A(t)y(t) = D_{q,\omega}A(t)\cos_A(t), \qquad D_{q,\omega}y(\theta) = \mathfrak{e}, \quad y(\theta) = 0; \tag{5.5}$$

$$D^2_{q,\omega}y(t) + A(qt+\omega)A(t)y(t) = -D_{q,\omega}A(t)\sin_A(t), \qquad D_{q,\omega}y(\theta) = 0, \quad y(\theta) = \mathfrak{e}; \tag{5.6}$$

$$\begin{aligned} &D^2_{q,\omega}y(t) - A^2(qt+\omega)y\bigl(q^2t + (q+1)\omega\bigr) = D_{q,\omega}A(t)\,\mathrm{Cos}_A(qt+\omega), \\ &D_{q,\omega}y(\theta) = \mathfrak{e}, \quad y(\theta) = 0; \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} &D^2_{q,\omega}y(t) + A^2(qt+\omega)y\bigl(q^2t + (q+1)\omega\bigr) = -D_{q,\omega}A(t)\,\mathrm{Sin}_A(qt+\omega), \\ &D_{q,\omega}y(\theta) = 0, \quad y(\theta) = \mathfrak{e}. \end{aligned} \tag{5.8}$$

Definition 5.3 We define the abstract hyperbolic functions by

$$\sinh_A(t) = \frac{e_A(t) - e_{-A}(t)}{2}, \tag{5.9}$$

$$\cosh_A(t) = \frac{e_A(t) + e_{-A}(t)}{2}, \tag{5.10}$$

$$\mathrm{Sinh}_A(t) = \frac{E_A(t) - E_{-A}(t)}{2}, \tag{5.11}$$

and

$$\mathrm{Cosh}_A(t) = \frac{E_A(t) + E_{-A}(t)}{2}. \tag{5.12}$$

The following formulas can be proved easily:

(i) $\cosh_A(t) + \sinh_A(t) = e_A(t)$,
(ii) $\cosh_A(t) - \sinh_A(t) = e_{-A}(t)$,
(iii) $\cosh_A(t)\,\mathrm{Cosh}_A(t) - \sinh_A(t)\,\mathrm{Sinh}_A(t) = 1$,
(iv) $\sinh_A(t)\,\mathrm{Cosh}_A(t) - \cosh_A(t)\,\mathrm{Sinh}_A(t) = 0$,
(v) $\cosh^2_A(t) - \sinh^2_A(t) = e_A(t)e_{-A}(t)$,
(vi) $\mathrm{Cosh}^2_A(t) - \mathrm{Sinh}^2_A(t) = E_A(t)E_{-A}(t)$.

As we did before we obtain the following identities:

$$D_{q,\omega}\sinh_A(t) = A\cosh_A(t), \qquad D_{q,\omega}\cosh_A(t) = A\sinh_A(t)$$

and

$$D_{q,\omega}\,\mathrm{Sinh}_A(t) = A\,\mathrm{Cosh}_A(qt+\omega), \qquad D_{q,\omega}\,\mathrm{Cosh}_A(t) = A\,\mathrm{Sinh}_A(qt+\omega).$$

The following theorem shows that the abstract hyperbolic functions satisfy the second order Hahn difference equations.

Theorem 5.4 *The functions* $\sinh_A(t)$, $\cosh_A(t)$, $\operatorname{Sinh}_A(t)$, *and* $\operatorname{Cosh}_A(t)$ *satisfy the following second order Hahn difference equations*:

$$D^2_{q,\omega}y(t) - A(qt+\omega)A(t)y(t) = D_{q,\omega}A(t)\cosh_A(t), \qquad D_{q,\omega}y(\theta) = \mathfrak{e}, \quad y(\theta) = 0; \tag{5.13}$$

$$D^2_{q,\omega}y(t) - A(qt+\omega)A(t)y(t) = D_{q,\omega}A(t)\sinh_A(t), \qquad D_{q,\omega}y(\theta) = 0, \quad y(\theta) = \mathfrak{e}; \tag{5.14}$$

$$\begin{aligned} &D^2_{q,\omega}y(t) - A^2(qt+\omega)A(t)y\bigl(q^2t+(q+1)\omega\bigr) = D_{q,\omega}A(t)\operatorname{Cosh}_A(qt+\omega), \\ &D_{q,\omega}y(\theta) = \mathfrak{e}, \quad y(\theta) = 0; \end{aligned} \tag{5.15}$$

$$\begin{aligned} &D^2_{q,\omega}y(t) - A^2(qt+\omega)A(t)y\bigl(q^2t+(q+1)\omega\bigr) = D_{q,\omega}A(t)\operatorname{Sinh}_A(qt+\omega), \\ &D_{q,\omega}y(\theta) = 0, \quad y(\theta) = \mathfrak{e}, \end{aligned} \tag{5.16}$$

respectively.

In the following theorem, $[a,b]$ is a closed interval containing θ and $A:[a,b]\to\mathbb{X}$ is continuous at θ.

Theorem 5.5 *Any solution* ψ *of the equation*

$$D^2_{q,\omega}x(t) + A(t)x(t) = 0, \quad t\in[a,b],$$

satisfies the following relation:

$$\begin{aligned} \psi(t) &= c_1(b-t) + c_2(t-a) + \frac{b-t}{b-a}\int_a^t (\tau-a)A\left(\frac{\tau-\omega}{q}\right)\psi\left(\frac{\tau-\omega}{q}\right)d_{q,\omega}\tau \\ &\quad + \frac{t-a}{b-a}\int_t^b (b-\tau)A\left(\frac{\tau-\omega}{q}\right)\psi\left(\frac{\tau-\omega}{q}\right)d_{q,\omega}\tau. \end{aligned}$$

Proof By direct computations, we get

$$\begin{aligned} D_{q,\omega}\psi(t) &= -c_1 + c_2 - \frac{1}{b-a}\int_a^{h(t)} (\tau-a)A\left(\frac{\tau-\omega}{q}\right)\psi\left(\frac{\tau-\omega}{q}\right)d_{q,\omega}\tau \\ &\quad - \frac{1}{b-a}\int_{h(t)}^b (b-\tau)A\left(\frac{\tau-\omega}{q}\right)\psi\left(\frac{\tau-\omega}{q}\right)d_{q,\omega}\tau. \end{aligned}$$

Then

$$\begin{aligned} D^2_{q,\omega}\psi(t) &= -\frac{1}{b-a}\bigl(h(t)-a\bigr)A\left(\frac{h(t)-\omega}{q}\right)\psi\left(\frac{h(t)-\omega}{q}\right) \\ &\quad - \frac{1}{b-a}\bigl(b-h(t)\bigr)A\left(\frac{h(t)-\omega}{q}\right)\psi\left(\frac{h(t)-\omega}{q}\right) \\ &= -A(t)\psi(t). \end{aligned}$$

□

Example 5.6 The second order Hahn difference equation

$$D^2_{q,\omega}x(t) - A\bigl(h(t)\bigr)A(t)x(t) = D_{q,\omega}A(t)\sin_A(t), \quad t\in I, \qquad D_{q,\omega}x(\theta) = 0, \quad x(\theta) = \mathfrak{e},$$

has the solution $x(t) = \cos_A(t)$, which can be written in the form

$$x(t) = \frac{\mathcal{I}}{2}\left[\sum_{k=0}^{\infty} \frac{((1+(-i)^k)c(t(1-q)-\omega))^k}{(q;q)_k}\right].$$

6 Non-homogeneous Hahn difference equations

In this section, we consider the non-homogeneous difference equation of the form

$$A_0 D_{q,\omega}^n x(t) + A_1 D_{q,\omega}^{n-1} x(t) + \cdots + A_n x(t) = B(t), \tag{6.1}$$

where $A_j : I \to \mathbb{X}$, $0 \le j \le n$, and $\mathbb{X}$ is a commutative Banach algebra containing a unit element $\mathfrak{e}$ and A_j are coefficients that satisfy the conditions of Theorem 2.4. We study this equation and find the general solution of the non-homogeneous Hahn difference equation (6.1). As in the theory of differential equations, one can see that if $\psi_1(t)$ and $\psi_2(t)$ are two solutions of (6.1), then $\psi_1(t) - \psi_2(t)$ is a solution of the corresponding homogeneous equation (3.1). Based on the above-mentioned note and Theorem 3.6, we get the following: if $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ form a fundamental set for (3.1) and $\psi_0(t)$ is a solution of equation (6.1), then, for any solution of equation (6.1), there are unique constants $c_1, \ldots, c_n \in \mathbb{X}$ such that

$$\psi(t) = c_1\psi_1(t) + \cdots + c_n\psi_n(t) + \psi_0(t). \tag{6.2}$$

Therefore, if we can find any particular solution $\psi_0(t)$ of equation (6.1), then (6.2) gives a general formula for all solutions of equation (6.1).

6.1 Method of variation of parameters

The method of variation of parameters is a method that helps us to obtain a particular solution. This solution takes the form

$$\psi_0(t) = \sum_{j=1}^{n} c_j(t)\psi_j(t). \tag{6.3}$$

To determine the functions $c_r(t)$, we have to operate by $D_{q,\omega}$ after replacing $i = n$ in

$$D_{q,\omega}^{i-1}\psi_0(t) = \sum_{j=1}^{n} c_j(t) D_{q,\omega}^{i-1}\psi_j(t), \quad 1 \le i \le n, \tag{6.4}$$

provided that

$$\sum_{j=1}^{n} D_{q,\omega} c_j(t) D_{q,\omega}^{i-1}\psi_j\bigl(h(t)\bigr) = 0, \quad 1 \le i \le n-1. \tag{6.5}$$

We obtain

$$D_{q,\omega}^n\psi_0(t) = \sum_{j=1}^{n}\bigl(c_j(t) D_{q,\omega}^n\psi_j(t) + D_{q,\omega} c_j(t) D_{q,\omega}^{n-1}\psi_j\bigl(h(t)\bigr)\bigr). \tag{6.6}$$

Since $\psi_0(t)$ satisfies equation (6.1). It follows that

$$A_0(t)D^n_{q,\omega}\psi_0(t) + A_1(t)D^{n-1}_{q,\omega}\psi_0(t) + \cdots + A_n(t)\psi_0(t) = B(t). \tag{6.7}$$

Substitute by (6.4) and (6.6) in (6.7) and in view of equation (6.1), we obtain

$$\sum_{j=1}^{n} D_{q,\omega}c_j(t)D^{n-1}_{q,\omega}\psi_j\bigl(h(t)\bigr) = A_0^{-1}(t)B(t). \tag{6.8}$$

Equation (6.5) with (6.8) yields the following system:

$$\left.\begin{array}{l} D_{q,\omega}c_1(t)\psi_1(h(t)) + \cdots + D_{q,\omega}c_n(t)\psi_n(h(t)) = 0, \\ \ldots, \\ D_{q,\omega}c_1(t)D^{n-2}_{q,\omega}\psi_1(h(t)) + \cdots + D_{q,\omega}c_n(t)D^{n-2}_{q,\omega}\psi_n(h(t)) = 0, \\ D_{q,\omega}c_1(t)D^{n-1}_{q,\omega}\psi_1(h(t)) + \cdots + D_{q,\omega}c_n(t)D^{n-1}_{q,\omega}\psi_n(h(t)) = A_0^{-1}(t)B(t). \end{array}\right\} \tag{6.9}$$

Consequently,

$$D_{q,\omega}c_r(t) = W^r_{q,\omega}\bigl(h(t)\bigr)W^{-1}_{q,\omega}\bigl(h(t)\bigr) \times A_0^{-1}(t)B(t), \quad t \in I,$$

where $1 \le r \le n$ and $W^r_{q,\omega}(h(t))$ is the determinant obtained from $W_{q,\omega}(h(t))$ by replacing the rth column by $(0,\ldots,0,\mathfrak{e})$. It follows that

$$c_r(t) = \int_\theta^t W^r_{q,\omega}\bigl(h(\tau)\bigr)W^{-1}_{q,\omega}\bigl(h(\tau)\bigr) \times A_0^{-1}(t)B(t)\,d_{q,\omega}\tau, \quad r = 1,\ldots,n.$$

Example 6.1 We calculate the Hahn Wronskian for the following Hahn difference equation:

$$D^3_{q,\omega}x(t) - 6D^2_{q,\omega}x(t) + 11D_{q,\omega}x(t) - 6x(t) = 0, \tag{6.10}$$

where $x(t) \in \mathbb{X}$ commutative Banach algebra

The functions $x_1(t) = e_{\mathfrak{e}}(t)$, $x_2(t) = e_{2\mathfrak{e}}(t)$ and $x_3(t) = e_{3\mathfrak{e}}(t)$ are solutions of equation (6.10), with the initial conditions $x_1(\theta) = \mathfrak{e}$, $D_{q,\omega}x_1(\theta) = \mathfrak{e}$, and $D^2_{q,\omega}x_1(\theta) = \mathfrak{e}$, $x_2(\theta) = \mathfrak{e}$, $D_{q,\omega}x_2(\theta) = 2\mathfrak{e}$, and $D^2_{q,\omega}x_2(\theta) = 4\mathfrak{e}$, and for the third solution $x_3(\theta) = \mathfrak{e}$, $D_{q,\omega}x_3(\theta) = 3\mathfrak{e}$, and $D^2_{q,\omega}x_3(\theta) = 9\mathfrak{e}$, respectively.

Here, $R(t) = (-6\mathfrak{e}\xi^2 + 11\mathfrak{e}\xi - 6\mathfrak{e})$, where $\xi = h(t) - t$.

The Wronskian at the initial point θ,

$$\begin{aligned} W_{q,\omega}(\theta) &= \begin{vmatrix} e_{\mathfrak{e}}(\theta) & e_{2\mathfrak{e}}(\theta) & e_{3\mathfrak{e}}(\theta) \\ e_{\mathfrak{e}}(\theta) & 2e_{2\mathfrak{e}}(\theta) & 3e_{3\mathfrak{e}}(\theta) \\ e_{\mathfrak{e}}(\theta) & 4e_{2\mathfrak{e}}(\theta) & 9e_{3\mathfrak{e}}(\theta) \end{vmatrix} \\ &= 2\mathfrak{e}. \end{aligned}$$

Finally, from equation (3.5) we get

$$W_{q,\omega}(t) = 2\mathfrak{e}\left[\prod_{k=0}^{\infty}\bigl(-6 - 6q^{3k}\bigl(t(1-q)-\omega\bigr)^3 + 5q^{2k}\bigl(t(1-q)-\omega\bigr)^2 - 6q^k\bigl(t(1-q)-\omega\bigr)\bigr)\right]^{-1}.$$

Example 6.2 Consider the equation

$$D^2_{q,\omega}x(t) + A\big(h(t)\big)A(t)x(t) = B(t), \tag{6.11}$$

where $A(h(t)), A(t) \in \mathbb{X}$, are invertible. $\cos_A(t)$ and $\sin_A(t)$ are the solutions of the corresponding homogeneous equation of (6.11).

Consequently,

$$\begin{aligned} W_{q,\omega}(\psi_1,\psi_2)\big(h(t)\big) &= \begin{vmatrix} \cos_A(h(t)) & \sin_A(h(t)) \\ -A(h(t))\sin_A(h(t)) & A(h(t))\cos_A(h(t)) \end{vmatrix} \\ &= A\big(h(t)\big)\big(\cos^2_A\big(h(t)\big) + \sin^2_A\big(h(t)\big)\big) \\ &= A\big(h(t)\big)e_{iA}\big(h(t)\big)e_{-iA}\big(h(t)\big). \end{aligned}$$

Hence,

$$W^{-1}_{q,\omega}(\psi_1,\psi_2)\big(h(t)\big) = A^{-1}\big(h(t)\big)e^{-1}_{-iA}\big(h(t)\big)e^{-1}_{iA}\big(h(t)\big).$$

Also, we have

$$W_1\big(h(t)\big) = \begin{vmatrix} 0 & \sin_A(h(t)) \\ 1 & A(h(t))\cos_A(h(t)) \end{vmatrix} = -\sin_A\big(h(t)\big)$$

and

$$W_2\big(h(t)\big) = \begin{vmatrix} \cos_A(h(t)) & 0 \\ -A(h(t))\sin_A(h(t)) & 1 \end{vmatrix} = \cos_A\big(h(t)\big).$$

We get

$$\begin{aligned} \psi_0(t) = {} & \cos_A(t)\int_\theta^t -B(\tau)\sin_A\big(h(\tau)\big)A^{-1}\big(h(\tau)\big)e^{-1}_{-iA}\big(h(\tau)\big)e^{-1}_{iA}\big(h(\tau)\big)\,d_{q,\omega}\tau \\ & + \sin_A(t)\int_\theta^t B(\tau)\cos_A\big(h(\tau)\big)A^{-1}\big(h(\tau)\big)e^{-1}_{-iA}\big(h(\tau)\big)e^{-1}_{iA}\big(h(\tau)\big)\,d_{q,\omega}\tau \\ = {} & \frac{-\cos_A(t)}{2i}\int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big)\big(e_{iA}\big(h(\tau)\big) - e_{-iA}\big(h(\tau)\big)\big) \\ & \times e^{-1}_{-iA}\big(h(\tau)\big)e^{-1}_{iA}\big(h(\tau)\big)\,d_{q,\omega}\tau \\ & + \frac{\sin_A(t)}{2}\int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big)\big(e_{iA}\big(h(\tau)\big) + e_{-iA}\big(h(\tau)\big)\big) \\ & \times e^{-1}_{-iA}\big(h(\tau)\big)e^{-1}_{iA}\big(h(\tau)\big)\,d_{q,\omega}\tau, \end{aligned}$$

i.e.,

$$\begin{aligned} \psi_0(t) = {} & -\cos_A(t)\int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big)\operatorname{Sin}_A\big(h(\tau)\big)\,d_{q,\omega}\tau \\ & + \sin_A(t)\int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big)\operatorname{Cos}_A\big(h(\tau)\big)\,d_{q,\omega}\tau. \end{aligned}$$

It follows that every solution of equation (6.11) has the form

$$\begin{aligned}\psi(t) &= c_1 \cos_A(t) + c_2 \sin_A(t) \\ &\quad - \cos_A(t) \int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big) \operatorname{Sin}_A\big(h(\tau)\big)\, d_{q,\omega}\tau \\ &\quad + \sin_A(t) \int_\theta^t B(\tau)A^{-1}\big(h(\tau)\big) \operatorname{Cos}_A\big(h(\tau)\big)\, d_{q,\omega}\tau.\end{aligned}$$

6.2 Annihilator method

The annihilator method is based on annihilating the non-homogeneous part by applying a special differential operator. It is very easy to apply if we have found this operator, but a lot of cases cannot be solved by it.

Definition 6.3 We say that $f : I \to \mathbb{X}$ can be annihilated provided if we can find an operator of the form

$$L(D) = A_n(t)D_{q,\omega}^n + A_{n-1}(t)D_{q,\omega}^{n-1} + \cdots + A_0(t)$$

such that

$$L(D)f(t) = 0, \quad t \in I,$$

where $A_i(t)$, $0 \le i \le n$, are elements in $\mathbb{X}$, not all zero.

Example 6.4 Since $(D_{q,\omega} - A(t))e_A(t) = 0$, $D_{q,\omega} - A(t)$ is an annihilator of $e_A(t)$.

Example 6.5 The operator $D_{q,\omega}^3$ is the annihilator of t^2x, since

$$\begin{aligned}D_{q,\omega}^3 xt^2 &= xD_{q,\omega}^2(t + qt + \omega) \\ &= xD_{q,\omega}(1 + q) \\ &= 0.\end{aligned}$$

Table 1 indicates a list of some functions and their annihilators.

Example 6.6 We solve the following equation by using the annihilator method:

$$D_{q,\omega}^2 x(t) - 6D_{q,\omega}x(t) + 8x(t) = e_{5\mathfrak{e}}(t). \tag{6.12}$$

Table 1 Some functions and their annihilators

Function	Annihilator
$x \in \mathbb{X}$	$D_{q,\omega}$
xt^n	$D_{q,\omega}^{n+1}$
$e_A(t)$	$D_{q,\omega} - A(t)$
$\cos_{q,\omega}(A,t)$	$D_{q,\omega}^2 + A^2$
$\sin_{q,\omega}(A,t)$	$D_{q,\omega}^2 + A^2$

Equation (6.12) can be rewritten in the form

$$(D_{q,\omega} - 4\mathfrak{c})(D_{q,\omega} - 2\mathfrak{c})x(t) = e_{5\mathfrak{c}}(t).$$

Multiplying both sides by the annihilator $D_{q,\omega} - 5\mathfrak{c}$, we see that if $x(t)$ is a solution of (6.12), then $x(t)$ satisfies

$$(D_{q,\omega} - 5\mathfrak{c})(D_{q,\omega} - 4\mathfrak{c})(D_{q,\omega} - 2\mathfrak{c})x(t) = 0.$$

Hence,

$$x(t) = c_1 e_{5\mathfrak{c}}(t) + c_2 e_{4\mathfrak{c}}(t) + c_3 e_{2\mathfrak{c}}(t).$$

In the following example we will assume that $\mathbb{X} = L(X)$, the space of all bounded linear operators from a Banach space X into itself.

Example 6.7 Consider the following q,ω-difference equations:

$$D_{q,\omega}X(t) = A(t)X(t) \quad \text{and} \quad X(\theta) = I_X \tag{6.13}$$

and

$$D_{q,\omega}x(t) = A(t)x(t) \quad \text{and} \quad x(\theta) = x_\theta \in X, \tag{6.14}$$

where $\{A(t)\} \subseteq L(X)$ satisfies $A(t)A(h^k(t)) = A(h^k(t))A(t)$.

By Theorem 4.3, equation (6.13) has the unique solution

$$X(t) = e_A(t).$$

Consequently equation (6.14) has the unique solution

$$x(t) = e_A(t)x_\theta.$$

As a special case, if $A(t)$ is the constant operator $A(t) = zI_X$, where $z \in \mathbb{C}$, then the unique solution of equation (6.13) will be

$$X(t) = e_z(t)I_X,$$

where

$$e_z(t) = \left(\prod_{k=0}^{\infty}\bigl(1 - zq^k\bigl(t(1-q) - \omega\bigr)\bigr)\right)^{-1},$$

and the solution of equation (6.14) will be

$$x(t) = e_z(t)x_\theta.$$

Competing interests
The authors declare that they have no competing interests.

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