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A note on the shooting method and its applications in the Stieltjes integral boundary value problems

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Abstract
In this paper, the existence results of positive solutions for three-point Riemann-Stieltjes integral BVPs (boundary value problems) is considered. By applying shooting method and comparison principle, we obtain some new results which extend the known ones. At the same time, the theorems in one of our published articles are corrected by another theorem in this paper.

MSC: 34B10; 34B18; 34C10

Keywords: integral boundary value problem; existence; positive solution; shooting method; comparison principle

1 Introduction

By applying the shooting method, we establish the criteria for the existence of positive solutions to the following Riemann-Stieltjes integral BVPs:

$$u''(t) + a(t)f\big(u(t)\big) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \alpha \int_0^\eta u(s)\,ds, \tag{1.2}$$

where $f \in C([0,\infty); [0,\infty))$ and $0 < \eta < 1$, $\alpha \geq 0$ are given constants, and $0 < \alpha\eta^2 < 2$.

Set

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u},$$

$$\bar{f}_x = \limsup_{u\to x} \frac{f(u)}{u}, \qquad \underline{f}_x = \liminf_{u\to x} \frac{f(u)}{u}, \qquad x \in \{0, +\infty\}.$$

By Krasnoselskii's fixed point theorem in a cone, Tariboon and Sitthiwirattham [1] proved that BVP (1.1)-(1.2) has a positive solution in the case $f_0 = 0$ and $f_\infty = \infty$ (super-linear case) or in the case $f_0 = \infty$ and $f_\infty = 0$ (sub-linear case) when $0 < \alpha\eta^2 < 2$.

Some meaningful results of nonlinear second-order integral BVPs have already been obtained by Kong [2], Webb and Infante [3, 4], *etc.* The following BVP:

$$u''(t) + f\big(u(t)\big) = 0, \quad 0 < t < 1; \qquad u(0) = 0, \qquad u(1) = \alpha \int_0^\eta u(s)\,ds, \tag{1.3}$$

is a special case of Webb and Infante's [4], where we can deduce the result. Suppose $0 < \alpha\eta^2 < 1$; BVP (1.3) has at least one positive solution if one of the following conditions holds:

(i) $\bar{f}_0 < \mu$ and $\underline{f}_\infty > \mu$;

(ii) $\bar{f}_0 > \mu$ and $\underline{f}_\infty < \mu$,

where $\mu = 1/r(L)$ and $r(L)$ is the spectral radius of the associated linear operator. In [4], the authors used fixed point index theory.

As a numerical method, the shooting method is efficient to find the solution of BVPs [5–7]. Kwong and Wong [7] obtained some results for the Robin boundary condition of the form

$$\sin\theta u(0) - \cos\theta u'(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} \alpha_i u_i(\eta_i) = 0, \tag{1.4}$$

where $\theta \in [0, 3\pi/4]$ and $\theta \neq \pi/2$. Kwong and Wong [7] showed that BVP (1.1) with (1.4) has at least one positive solution if $\bar{f}_0 < L_\theta$ and $\underline{f}_\infty > L_\theta$, where L_θ is a certain but not specified constant related to the associated linear operator.

When $\theta = \pi/2$ and $0 \leq \sum_{i=1}^{m-2} \alpha_i\eta_i < 1$, Ma [8] has studied BVP (1.1) with (1.4) by using Krasnoselskii's fixed point theorem in a cone. The sufficient condition for the existence of positive solutions is also the super-linear case or the sub-linear case.

When $a(t) \equiv 1$, $m = 3$, $\eta = 1/2$, as a special case of [8], the BVP

$$u''(t) + f\bigl(u(t)\bigr) = 0, \quad 0 < t < 1; \qquad u(0) = 0, \qquad u(1) = \mu u(\eta), \tag{1.5}$$

was studied by Kwong in [6], where the existence condition is

$$\bar{f}_0 < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2 < \underline{f}_\infty, \quad \text{or} \quad \bar{f}_\infty < \left(2\cos^{-1}\left(\frac{\mu}{2}\right)\right)^2 < \underline{f}_0, \tag{1.6}$$

which is obtained by the shooting method.

Following the main idea in [6, 7], we considered the generalized multi-point integral BCs [9]

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{n} \alpha_i \int_0^{\eta_i} u(s)\,ds, \tag{1.7}$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\alpha_i \geq 0$ for $i = 1, \ldots, n-1$, and $\alpha_n > 0$ are given constants.

However, Theorem 1.1 and some proofs in [9] need to be corrected, which is one of the reasons why we write this paper. Furthermore, more general existence criteria are presented in this article as well as the application of the shooting method in the study of BVPs. For simplicity and without loss of generality, we start from BVP (1.1)-(1.2).

2 Preliminaries: some notation and lemmas

The principle of the shooting method is converting the BVP into an IVP (initial value problem) by finding suitable initial slopes $m > 0$ such that the solution of (1.1) comes with the initial value condition

$$u(0) = 0, \qquad u'(0) = m. \tag{2.1}$$

Denote by $u(t,m)$ the solution of the IVP (1.1) with (2.1) provided it exists, and define

$$k(m) = \frac{\alpha \int_0^\eta u(s,m)\,ds}{u(1,m)}, \qquad \varphi(m) = \alpha \int_0^\eta u(s,m)\,ds - u(1,m). \tag{2.2}$$

Then solving the boundary value problem is equivalent to finding a m^* such that $k(m^*) = 1$ or $\varphi(m^*) = 0$.

For the sake of convenience, we denote

$$\max_{0\le t\le 1}\{a(t)\} = a^L, \qquad \min_{0\le t\le 1}\{a(t)\} = a^l.$$

In this paper, we always assume

(H_1) $f \in C([0,\infty);[0,\infty))$, $a \in C([0,1];[0,\infty))$.

Furthermore, we assume that f is strong continuous enough to guarantee that $u(t,m)$ is uniquely defined and that it depends continuously on both t and m. As for the discussion of this problem, see [6].

Next, we present some comparison theorems which help us to establish the main results.

Lemma 2.1 (Sturm comparison theorem) *Let φ_1 and φ_2 be non-trivial solutions of the equations*

$$y'' + q_1(x)y = 0, \qquad y'' + q_2(x)y = 0,$$

respectively, on an interval I; here q_1 and q_2 are continuous functions such that $q_1(x) \le q_2(x)$ on I. Then between any two consecutive zeros x_1 and x_2 of φ_1, there exists at least one zero of φ_2 unless $q_1(x) \equiv q_2(x)$ on (x_1,x_2).

Lemma 2.2 *Let $y(t,m)$, $z(t,m)$, $Z(t,m)$ be the positive solution of the initial value problems, respectively,*

$$\begin{aligned}
&y''(t) + f\bigl(y(t)\bigr) = 0, \qquad y(0) = 0, \qquad y'(0) = m,\\
&Z''(t) + G(t)Z(t) = 0, \qquad Z(0) = 0, \qquad Z'(0) = m,\\
&z''(t) + g(t)z(t) = 0, \qquad z(0) = 0, \qquad z'(0) = m.
\end{aligned}$$

Suppose $g(t) \le G(t)$ be two piecewise continuous functions defined on $[0,1]$. *If*

$$0 \le g(t) \le \frac{f(y(t))}{y(t)} \le G(t)$$

and suppose that $Z(t)$ does not vanish in $(0,1]$, *then for any* $0 \le s \le \xi \le 1$, *it yields*

$$\frac{z(s,m)}{z(\xi,m)} \le \frac{y(s,m)}{y(\xi,m)} \le \frac{Z(s,m)}{Z(\xi,m)}, \tag{2.3}$$

and hence, for any $0 \le \eta \le \xi \le 1$, *we have*

$$\frac{\int_0^\eta z(s,m)\,ds}{z(\xi,m)} \le \frac{\int_0^\eta y(s,m)\,ds}{y(\xi,m)} \le \frac{\int_0^\eta Z(s,m)\,ds}{Z(\xi,m)}. \tag{2.4}$$

Proof Since $0 \le g(t) \le f(y(t))/y(t) \le G(t)$ and $Z(t)$ does not vanish in $(0,1]$, from Lemma 2.1, it follows that $y(t)$ and $z(t)$ will not vanish in $(0,1]$. The proof for (2.3) can be seen in [6]. The continuity of the integrands implies the existence of the Riemann integral. In view of the definition of Riemann integral, by using the inequality of the limit, we have (2.4). □

Remark 2.1 Lemma 2.2 is also the correction for Theorem 1.1 in [9].

Lemma 2.3 *Consider the BVP*

$$y''(t) + Ay(t) = 0, \quad 0 < t < 1, \tag{2.5}$$

$$y(0) = 0, \qquad y(1) = b. \tag{2.6}$$

(i) *If* $A = \pi^2$, *then* $y(t)$ *vanishes at* $t = 1$ *for the first time on interval* $(0,1]$ *and* $b = 0$;
(ii) *if* $0 < A < \pi^2$, *then* $y(t)$ *does not vanish on the interval* $(0,1]$ *and* $b > 0$;
(iii) *if* $A > \pi^2$, *then* $y(t)$ *vanishes before* $t = 1$ *on interval* $(0,1]$.

Proof Obviously, $y(t) = \sin(\pi^2 t)$ satisfies the conditions $y(0) = 0$, $y(1) = 0$, and $y(t) > 0$ for $t \in (0,1)$, hence (i) is established. According to the Sturm comparison theorem, we can draw the conclusions (ii) and (iii). □

Lemma 2.4 ([1]) *Assume that* (H_1) *holds and* $\alpha\eta^2 > 2$, *then BVP* (1.1)-(1.2) *has no positive solution.*

In [1] and [9], the proofs are conducted by contradiction to the concavity of solution (also see [4]). In fact, for $m > 0$, we compare the solution $u(t,m)$ of the IVP given by (1.1) and (2.1) with the solution $y(t) = mt$ of

$$y''(t) + 0y(t) = 0, \qquad y(0) = 0, \qquad y'(0) = m. \tag{2.7}$$

If BVP (1.1)-(1.2) has a positive solution $u(t,m)$, then by Lemma 2.2 and the concavity of $u(t,m)$, we have

$$\frac{1}{\eta} \ge \frac{u(1,m)}{u(\eta,m)} = \frac{\alpha \int_0^\eta u(s,m)\,ds}{u(\eta,m)} \ge \frac{\alpha \int_0^\eta y(s,m)\,ds}{y(\eta,m)} = \frac{\alpha \int_0^\eta ms\,ds}{m\eta} = \frac{\alpha\eta}{2}, \tag{2.8}$$

that is, $\alpha\eta^2 \le 2$.

In the following, we always assume that

(H_2) $0 < \alpha\eta^2 < 2$.

3 Main results

Lemma 3.1 *Assume that* (H_1)-(H_2) *holds. Then there exist a solution* $x = A_1 \in (0,\pi)$ *such that*

$$g_1(x) := \frac{\alpha[1 - \cos(\eta x)]}{x \sin x} = 1 \tag{3.1}$$

and a solution $x = A_2 \in (0,\pi)$ *such that*

$$g_2(x) := \frac{\alpha\eta\sin(\eta x)}{2\sin x} = 1. \tag{3.2}$$

Proof It is not difficult to show that

$$\lim_{x\to 0^+} g_1(x) = \frac{\alpha\eta^2}{2} < 1, \qquad \lim_{x\to\pi^-} g_1(x) = \infty > 1.$$

Since the function $g_1(x)$ is continuous on $(0,\pi)$, there must exist a constant $A_1 \in (0,\pi)$ such that $g_1(A_1) = 1$.

Similarly,

$$\lim_{x\to 0^+} g_2(x) = \frac{\alpha\eta^2}{2} < 1, \qquad \lim_{x\to\pi^-} g_2(x) = \infty > 1.$$

Thus, there exists a positive constant $A_2 \in (0,\pi)$ such that $g_2(A_2) = 1$. □

Theorem 3.1 *Assume that* (H_1)-(H_2) *holds. Suppose one of the following conditions holds*:

$$\text{(i)}\quad 0 \le \bar{f}_0 < \frac{\underline{A}^2}{a^L}, \qquad \underline{f}_\infty > \frac{\bar{A}^2}{a^l}; \qquad \text{(ii)}\quad 0 \le \bar{f}_\infty < \frac{\underline{A}^2}{a^L}, \qquad \underline{f}_0 > \frac{\bar{A}^2}{a^l}.$$

Then problem (1.1)-(1.2) *has at least one positive solution, where*

$$\underline{A} = \min\{A_1, A_2\}, \qquad \bar{A} = \max\{A_1, A_2\},$$

and A_1, A_2 *is defined in* (3.1) *and* (3.2), *respectively.*

Proof (i) Since $0 \le \bar{f}_0 < \frac{\underline{A}^2}{a^L}$, there exists a positive number r such that

$$\frac{f(u)}{u} < \frac{\underline{A}^2}{a^L} \le \frac{{A_1}^2}{a^L}, \quad 0 < u \le r. \tag{3.3}$$

Let $0 < m_1^* < r$, then from the Sturm comparison theorem and the concavity of $u(t, m_1^*)$, it follows that $0 \le u(t, m_1^*) \le m_1^* t \le m_1^* < r$ for $t \in [0,1]$. Thus

$$0 \le a(t)f\bigl(u\bigl(t, m_1^*\bigr)\bigr) < a^L \frac{A_1^2}{a^L} u\bigl(t, m_1^*\bigr) = A_1^2 u\bigl(t, m_1^*\bigr) < \pi^2 u\bigl(t, m_1^*\bigr), \quad t \in (0,1].$$

By Lemma 2.3, it gives $u(t, m_1^*) > 0$ for $t \in (0,1]$.

Let $Z(t) = (m_1^*/A_1)\sin(A_1 t)$ for $t \in [0,1]$, then

$$Z''(t) + A_1^2 Z(t) = 0, \qquad Z(0) = 0, \qquad Z'(0) = m_1^*. \tag{3.4}$$

From Lemma 2.2 and Lemma 3.1, we have

$$k\bigl(m_1^*\bigr) = \frac{\alpha\int_0^\eta u(s, m_1^*)\,ds}{u(1, m_1^*)} < \frac{\alpha\int_0^\eta m_1^*\sin(A_1 s)\,ds}{m_1^*\sin A_1} = \frac{\alpha[1-\cos(\eta A_1)]}{A_1\sin A_1} = 1, \tag{3.5}$$

that is, $\varphi(m_1^*) \le 0$.

On the other hand, the second inequality in (i) implies that there exists a number L large enough such that

$$\frac{f(u)}{u} > \frac{\bar{A}^2}{a^l} \geq \frac{A_2^2}{a^l}, \quad u \geq L, \tag{3.6}$$

and there exists a positive number $\epsilon < A_2(1-\eta)/\eta$ small enough that

$$\frac{f(u)}{u} \geq \frac{(A_2+\epsilon)^2}{a^l}, \quad u \geq L. \tag{3.7}$$

Next, we will find a positive number m_2^* such that $\varphi(m_2^*) \geq 0$.

Claim. There exist a slope m_2^* and two positive numbers ρ and σ such that

$$0 < \rho \leq \eta \leq \frac{A_2}{A_2+\epsilon} \leq \sigma \leq 1 \quad \text{and} \quad u\bigl(t, m_2^*\bigr) \geq L \quad \text{for } t \in [\rho, \sigma].$$

Since the solution $u(t,m)$ is concave, it hits the line $u = L$ at most two times for the constant L defined in (3.6) and $t \in (0,1]$. We denote the left intersecting time by $\underline{\delta}_m$ and the right one by $\overline{\delta}_m$ provided they exist. Henceforth, denote $I_m = [\underline{\delta}_m, \overline{\delta}_m] \subseteq (0,1]$. If $u(1,m) \geq L$, then $\overline{\delta}_m = 1$.

The discussion is divided into three steps.

Step 1. We claim that there exists a slope m_0 large enough such that $0 \leq u(t,m_0) \leq L$ for $t \in [0, \underline{\delta}_{m_0}]$ and $u(t,m_0) \geq L$ for $t \in I_{m_0}$.

Otherwise, provided $u(t,m) \leq L$ for all $t \in [0,1]$ as $m \to \infty$, then by integrating both sides of (1.1) from 0 to t, we have

$$u(t,m) = mt - \int_0^t (t-s)a(s)f\bigl(u(s,m)\bigr)\,ds. \tag{3.8}$$

Hence, from (3.3) and the continuity of $f(u)$, we have

$$m = u(1,m) + \int_0^1 (1-s)a(s)f\bigl(u(s,m)\bigr)\,ds \leq L + L_f a^L, \tag{3.9}$$

where $L_f = \max_{u\in[0,L]} f(u)$. If we choose $m > L + L_f a^L$, (3.9) will lead to a contradiction.

Since $u(t,m)$ is continuous and concave, there exists a number m_0 large enough such that $u(t,m_0) \geq L$ for $t \in I_{m_0}$.

Step 2. There exists a monotonically increasing sequence $\{m_k\}$ such that the sequence $\underline{\delta}_{m_k}$ is decreasing on m_k and $\overline{\delta}_{m_k}$ is increasing on m_k. That is,

$$I_{m_0} \subset I_{m_1} \subset \cdots \subset I_{m_k} \subset \cdots \subseteq (0,1]$$

and $u(t,m_k) \geq L$ for $t \in I_{m_k}$.

First, we prove that

$$\underline{\delta}_{m_k} < \underline{\delta}_{m_{k-1}}, \quad k = 1,2\ldots \text{ for } m_k > m_{k-1}. \tag{3.10}$$

When $k = 1$, we have

$$u(\underline{\delta}_{m_0}, m_1) > u(\underline{\delta}_{m_0}, m_0)$$

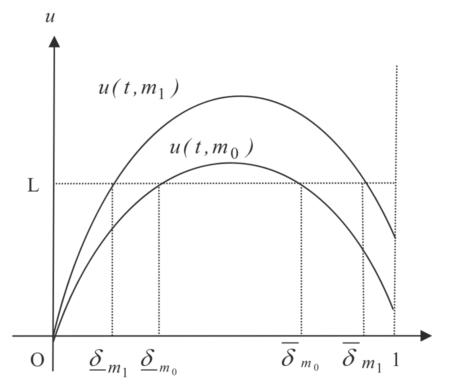

Figure 1 The relationship of m and I_m. $m_1 > m_0$, $I_{m_0} \subset I_{m_1}$.

in the case

$$m_1 > m_0 + 2a^L L_f \underline{\delta}_{m_0}. \tag{3.11}$$

Otherwise, provided

$$u(\underline{\delta}_{m_0}, m_1) \le u(\underline{\delta}_{m_0}, m_0) = L, \tag{3.12}$$

then from (3.8) and (3.11), we have

$$\begin{aligned} & u(\underline{\delta}_{m_0}, m_1) - u(\underline{\delta}_{m_0}, m_0) \\ &\quad = (m_1 - m_0)\underline{\delta}_{m_0} - \int_0^{\underline{\delta}_{m_0}} (\underline{\delta}_{m_0} - s)a(s)\bigl[f\bigl(u(s,m_1)\bigr) - f\bigl(u(s,m_0)\bigr)\bigr]\,ds \\ &\quad > (m_1 - m_0)\underline{\delta}_{m_0} - 2a^L L_f \underline{\delta}_{m_0}^2 \\ &\quad = \underline{\delta}_{m_0}\bigl[(m_1 - m_0) - 2a^L L_f \underline{\delta}_{m_0}\bigr] > 0, \end{aligned}$$

which contradicts (3.12).

Hence, for a slope $m_1 > m_0 + 2a^L L_f \underline{\delta}_{m_0}$, there exists a number $0 < \underline{\delta}_{m_1} < \underline{\delta}_{m_0}$ such that

$$u(\underline{\delta}_{m_1}, m_1) = L, \quad \text{and} \quad u(t, m_1) \le L \quad \text{for } t \in (0, \underline{\delta}_{m_1}].$$

See Figure 1.

By mathematical induction, it is not difficult to show that $\underline{\delta}_{m_k} < \underline{\delta}_{m_{k-1}}$, $k = 1, 2, \ldots$.

Further, we turn to the right hand of the interval I_{m_k}. Since f guarantees that $u(t,m)$ is uniquely defined, the solutions $u(t, m_{k-1})$ and $u(t, m_k)$ have no intersection in the interval $[\underline{\delta}_{m_{k-1}}, 1)$. It follows from

$$u(\underline{\delta}_{m_{k-1}}, m_k) > u(\underline{\delta}_{m_{k-1}}, m_{k-1})$$

that

$$u(\overline{\delta}_{m_{k-1}}, m_k) > u(\overline{\delta}_{m_{k-1}}, m_{k-1}).$$

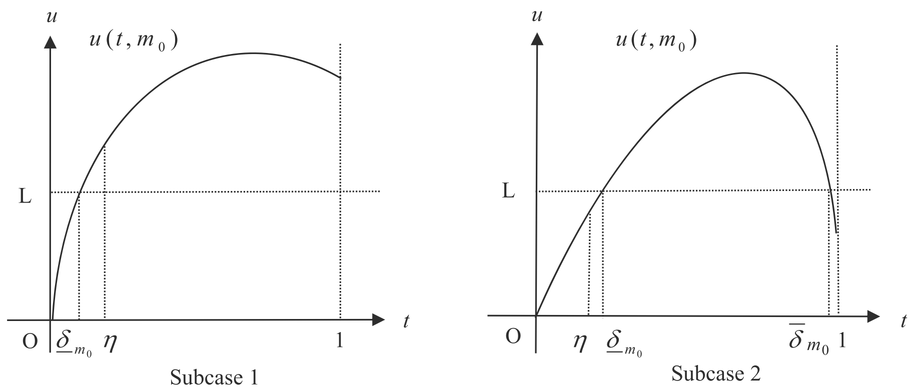

Figure 2 Two of the possible cases of I_{m_0}. Subcase 1: $\eta \in [\underline{\delta}_{m_0}, \overline{\delta}_{m_0}]$ and $u(1, m_0) \geq L$; Subcase 2: $\eta \not\subseteq [\underline{\delta}_{m_0}, \overline{\delta}_{m_0}]$ or $u(1, m_0) < L$.

Thus we have

$$\overline{\delta}_{m_k} > \overline{\delta}_{m_{k-1}}, \quad k = 1, 2, \ldots \text{ for } m_k > m_{k-1}. \tag{3.13}$$

When $k = 1$, also see Figure 1.

Step 3. Seek out a slope m_2^* and two positive numbers ρ and σ such that $0 < \rho \leq \eta \leq \frac{A_2}{A_2+\epsilon} \leq \sigma \leq 1$ and $u(t, m_2^*) \geq L$ for $t \in [\rho, \sigma]$.

Subcase 1. $\eta \in [\underline{\delta}_{m_0}, \overline{\delta}_{m_0}]$ and $u(1, m_0) \geq L$. In this case, we take $m_2^* = m_0$ and $\rho = \underline{\delta}_{m_0}$, $\sigma = \overline{\delta}_{m_0} = 1$.

Subcase 2. $\eta \not\subseteq [\underline{\delta}_{m_0}, \overline{\delta}_{m_0}]$ or $u(1, m_0) < L$. Following the step 1, step 2, and the extension principle of solutions, there exists a positive integer n large enough such that

$$\underline{\delta}_{m_n} < \eta, \qquad \overline{\delta}_{m_n} \geq \frac{A_2}{A_2 + \epsilon}. \tag{3.14}$$

If we take $m_2^* = m_n$ and $\rho = \underline{\delta}_{m_n}$, $\sigma = \overline{\delta}_{m_n}$, then

$$\sigma(A_2 + \epsilon) \geq A_2. \tag{3.15}$$

Two of the possible cases of I_{m_0} can be seen in Figure 2.

In the following, we prove that $k(m_2^*) \geq 1$ or $\varphi(m_2^*) > 0$ for the selected m_2^* and ρ, σ.

Set $z(t) = (m_2^*/\sigma(A_2 + \epsilon)) \sin(\sigma(A_2 + \epsilon)t)$, then

$$z''(t) + \sigma^2(A_2 + \epsilon)^2 z(t) = 0, \qquad z(0) = 0, \qquad z'(0) = m_2^*, \quad t \in [\rho, \sigma], \tag{3.16}$$

where $\rho \leq \eta < \sigma \leq 1$. From (3.7), we have

$$\frac{f(u)}{u} \geq \frac{\sigma^2(A_2 + \epsilon)^2}{a^l}, \quad u \geq L.$$

Further, noting that $u(1, m_2^*) > L$ (this time $\sigma = 1$) or $u(1, m_2^*) \leq u(\sigma, m_2^*) = L$ and the function

$$S(x) = \frac{\sin \eta x}{\sin x}$$

is increasing for $x \in (0,\pi)$, then by Lemma 2.2, Lemma 3.1, and inequality (3.15), we have

$$\begin{aligned} k(m_2^*) &= \frac{\alpha \int_0^\eta u(s,m_2^*)\,ds}{u(1,m_2^*)} \geq \frac{\alpha\eta u(\eta,m_2^*)}{2u(1,m_2^*)} \geq \frac{\alpha\eta u(\eta,m_2^*)}{2u(\sigma,m_2^*)} \\ &\geq \frac{\alpha\eta \sin\eta\sigma(A_2+\epsilon)}{2\sin\sigma(A_2+\epsilon)} \geq \frac{\alpha\eta\sin(\eta A_2)}{2\sin A_2} = 1, \end{aligned} \tag{3.17}$$

which implies $\varphi(m_2^*) \geq 0$.

From (3.5) and (3.17), we can find a m^* between m_1^* and m_2^* such that $u(t,m^*)$ is the solution of (1.1)-(1.2). The theorem is complete.

The proof for (ii) is similar, so we omit it. □

Now, we present the result for BVP (1.1) with (1.7), which is also the correction of Theorem 3.1 and Theorem 3.2 in [9].

Theorem 3.2 *Assume that* (H_1)-(H_2) *hold. Suppose one of the following conditions holds*:

$$\text{(i)}\quad 0 \leq \bar{f}_0 < \frac{\underline{A}^2}{a^L}, \qquad \underline{f}_\infty > \frac{\bar{A}^2}{a^l}; \qquad \text{(ii)}\quad 0 \leq \bar{f}_\infty < \frac{\underline{A}^2}{a^L}, \qquad \underline{f}_0 > \frac{\bar{A}^2}{a^l}.$$

Then problem (1.1) *with* (1.7) *has at least one positive solution, where*

$$\underline{A} = \min\{A_1, A_2\}, \qquad \bar{A} = \max\{A_1, A_2\}$$

and A_1, A_2 *is defined by*

$$\frac{\sum_{i=1}^n \alpha_i[1-\cos(A_1\eta_i)]}{A_1\sin A_1} = 1 \tag{3.18}$$

and

$$\frac{\sum_{i=1}^n \alpha_i\eta_i\sin(A_2\eta_i)}{2\sin A_2} = 1. \tag{3.19}$$

Proof Similar to (3.5) and (3.17), it follows from (1.7) and (3.18)-(3.19) that

$$\begin{aligned} k(m_1^*) &= \frac{\sum_{i=1}^n \alpha_i \int_0^{\eta_i} u(s,m_1^*)\,ds}{u(1,m_1^*)} < \frac{\sum_{i=1}^n \alpha_i \int_0^{\eta_i} m_1^*\sin(A_1 s)\,ds}{m_1^*\sin A_1} \\ &= \frac{\sum_{i=1}^n \alpha_i[1-\cos(A_1\eta_i)]}{A_1\sin A_1} = 1 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} k(m_2^*) &= \frac{\sum_{i=1}^n \alpha_i \int_0^{\eta_i} u(s,m_2^*)\,ds}{u(1,m_2^*)} \geq \frac{\sum_{i=1}^n \alpha_i\eta_i u(\eta_i,m_2^*)}{2u(1,m_2^*)} \geq \frac{\sum_{i=1}^n \alpha_i\eta_i u(\eta_i,m_2^*)}{2u(\sigma,m_2^*)} \\ &\geq \frac{\sum_{i=1}^n \alpha_i\eta_i\sin(\eta_i\sigma(A_2+\epsilon))}{2\sin\sigma(A_2+\epsilon)} \geq \frac{\sum_{i=1}^n \alpha_i\eta_i\sin(A_2\eta_i)}{2\sin A_2} = 1, \end{aligned} \tag{3.21}$$

where $\eta_n < \sigma \leq 1$ and (3.15) holds.

The remainder of the proof is similar, so we omit it. □

4 Conclusion and discussion

The conditions in [8] and [1] are easy to verify; however, they are not as general as ours, because the sup-linear case or the sub-linear case is sufficient for the conditions in Theorem 3.1. As an example of [4], where the constant μ is related to the Green's function and the spectral radius of associated linear operator, our calculation is more direct. The idea of this paper was illuminated by [6, 7]; however, the certain constant L_θ could not be given explicitly in [7] and η only equals 1/2 in [6]. From this point of view, this paper extends the work of [6, 7] and presents another way to find the 'eigenvalue' by numerical calculation, though it is related to a transcendental equation which has at least one numerical solution.

In fact, we can extent our results to [8]. The proof is fit, where

$$k(m) = \frac{\sum_{i=1}^{m-2} \alpha_i u(\eta_i, m)}{u(1,m)}$$

and the constant $A = A_1 = A_2 \in (0, \pi)$ is explicitly determined by

$$\frac{\sum_{i=1}^{m-2} \alpha_i \sin(A\eta_i)}{\sin A} = 1. \tag{4.1}$$

In other words, we can substitute the condition

(i) $f_0 = 0$ and $f_\infty = \infty$, or
(ii) $f_0 = \infty$ and $f_\infty = 0$,

with

(i′) $0 \le \bar{f}_0 < A^2 < \underline{f}_\infty$; or
(ii′) $0 \le \bar{f}_\infty < A^2 < \underline{f}_0$,

where A is defined in (4.1).

Next, we apply the result to the special case BVP (1.5), where $a^L = a^l = 1$, $m = 3$, $\alpha = \mu$, $\eta = 1/2$. From (4.1), we have

$$A = 2\cos^{-1}\left(\frac{\mu}{2}\right).$$

By plugging it into (i′) and (ii′), we have the same result as (1.6).

Further, when $\alpha\eta^2 = 2$, BVP (1.1)-(1.2) is at resonant. There may not exist a solution $x = A_1 \in (0, \pi)$ and $x = A_2 \in (0, \pi)$ to (3.1) and (3.2), respectively. If (3.1) and (3.2) has a solution $x = A_1 \in (0, \pi)$ and $x = A_2 \in (0, \pi)$, respectively, then we can also obtain the existence result for (1.1)-(1.2), similarly for (1.1) with (1.7).

When $\theta = \pi/2$ and $\sum_{i=1}^{m-2} \alpha_i \eta_i = 1$, BVP (1.1) with (1.4) is resonant. If there exists a number $A \in (0, \pi)$ such that (4.1), then the existence result for BVP (1.1) with (1.4) can be obtained, similarly for BVP (1.5).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work was carried out in collaboration between all authors. HW and ZO are responsible for the majority of the work. HT contributed to the proof of Theorem 3.1.

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