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Non-simultaneous blow-up for a parabolic system with nonlinear boundary flux which obey different laws

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Abstract
In this paper, we consider a system of two heat equations with nonlinear boundary flux which obey different laws, one is exponential nonlinearity and another is power nonlinearity. Under certain hypotheses on the initial data, we get the sufficient and necessary conditions, on which there exist initial data such that non-simultaneous blow-up occurs. Moreover, we get some conditions on which simultaneous blow-up must occur. Furthermore, we also get a result on the coexistence of both simultaneous and non-simultaneous blow-ups.
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Keywords: simultaneous blow-up; non-simultaneous blow-up; parabolic system; nonlinear boundary flux

1 Introduction and main results

In this paper, we study the following system of two heat equations coupled by nonlinear boundary conditions,

$$\begin{cases} u_t = \Delta u, & v_t = \Delta v, \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{pv} u^{\alpha}, & \frac{\partial v}{\partial \eta} = u^q e^{\beta v}, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^N$, parameters $\alpha, q \geq 1$, $p, \beta \geq 0$. Assume the non-zero, non-negative initial data u_0, v_0 are radially symmetric non-increasing continuous functions, vanishing on $\partial\Omega$, as well as satisfy the compatibility conditions,

$$\begin{cases} \frac{\partial u_0}{\partial \eta} = e^{pv_0} u_0^{\alpha}, \\ \frac{\partial v_0}{\partial \eta} = u_0^q e^{\beta v_0}, \end{cases} \quad x \in \Omega \tag{1.2}$$

and $\Delta u_0, \Delta v_0 \geq 0$, for $x \in \Omega$.

The system (1.1) can be used to describe heat propagation of a two-component combustible mixture in a bounded region. In this case, u and v represent the temperatures of the interacting components, thermal conductivity is supposed constant and equal for both substances, and a volume energy release given by powers of u and v is assumed; see

[1, 6]. The nonlinear Neumann boundary conditions can be physically interpreted as the cross-boundary fluxes, which obey different laws; some may obey power laws [4, 7, 10, 14], some may follow exponential laws [18]. It is interesting when the two types of boundary fluxes meet. In system (1.1), the coupled boundary flux obey a mixed type of power terms and exponential terms.

As we shall see, under certain conditions the solutions of this problem can become unbounded in a finite time. This phenomenon is known as blow-up, and has been observed for several scalar equation since the pioneering work of Fujita. Blow-up may also happen for systems, X. F. Song considered the blow-up conditions and blow-up rates of system (1.1), when $p, q > 0$, $0 \le \alpha < 1$ and $0 \le \beta < p$, in [16].

However, it can only show

$$\lim_{t\to T} \sup\big\{ \|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \big\} = \infty,$$

whether the blow-up is simultaneous or non-simultaneous is not known yet.

Recently, the simultaneous and non-simultaneous blow-up problems of parabolic systems have been widely considered by many authors [2, 3, 8, 9, 11–13, 15, 19, 20]. For example, B. C. Liu and F. J. Li [8] considered the nonlinear parabolic system

$$\begin{cases} u_t = \Delta u + u^m e^{pv}, \qquad v_t = \Delta v + u^q e^{nv}, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega. \end{cases}$$

They got a complete and optimal classification on non-simultaneous and simultaneous blow-ups by four sufficient and necessary conditions.

Motivated by the above works, we will focus on the simultaneous and non-simultaneous blow-up problems to system (1.1), and get our main results as follows.

Theorem 1.1 *There exist initial data such that the solutions of (1.1) blow up, if*

$$\alpha > 1, \quad or \quad \beta > 0, \quad or \quad pq > \beta(\alpha - 1).$$

In the sequel, we assume the blow-up indeed occurs. Then we get the conditions, under which simultaneous or non-simultaneous blow-up occurs.

Theorem 1.2 *There exist initial data such that non-simultaneous blow-up occurs if and only if*

$$\alpha > q + 1, \quad or \quad \beta > p.$$

Corollary 1.1 *Any blow-up is simultaneous if and only if*

$$\begin{cases} \alpha \le q + 1, \\ \beta \le p. \end{cases}$$

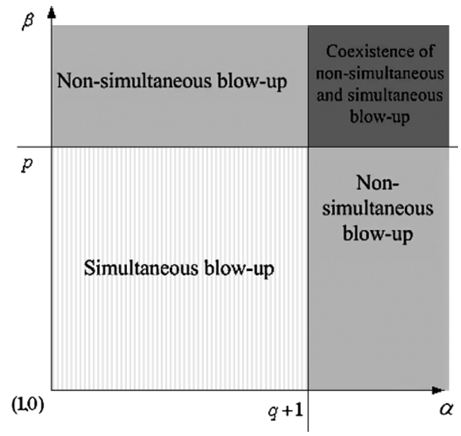

Figure 1 Non-simultaneous and simultaneous blow-ups.

Theorem 1.3 *If*

$$\begin{cases} \alpha > q+1, \\ \beta > p \end{cases}$$

both non-simultaneous and simultaneous blow-ups may occur.

In order to show the conditions more clearly, we graph Figure 1 with the region of non-simultaneous and simultaneous blow-ups occur in the parameter space.

The rest of this paper is organized as follows: In next section, we consider the blow-up conditions of system (1.1), give the proof of Theorem 1.1. In Section 3, we will study the sufficient and necessary conditions of non-simultaneous blow-up, in order to prove Theorem 1.2. In Section 4, we consider the coexistence of both simultaneous and non-simultaneous blow-ups; Theorem 1.3 is proved.

2 Blow-up

In this section, we prove the blow-up criterion of system (1.1). First, we check the monotonicity of the solution.

Lemma 2.1 *Let (u, v) be a solution of system (1.1), then $u_t, v_t \geq 0$, for all $(x,t) \in B_R \times (0,T)$.*

Proof Set

$$M = u_t, \qquad N = v_t, \quad (x,t) \in B_R \times (0,T).$$

From the hypothesis of initial data, we can get

$$\begin{cases} M_t = \Delta M, \qquad N_t = \Delta N, \quad (x,t) \in B_R \times (0,T), \\ \frac{\partial M}{\partial \eta} = pu^{\alpha}e^{pv}N + \alpha u^{\alpha-1}e^{pv}M, \\ \frac{\partial N}{\partial \eta} = qu^{q-1}e^{\beta v}M + \beta u^{q}e^{\beta v}N, \quad (x,t) \in \partial B_R \times (0,T), \\ M(x,0) = \Delta u_0 \geq 0, \qquad N(x,0) = \Delta v_0 \geq 0, \quad x \in B_R. \end{cases}$$

By the comparison principle, $M(x,t), N(x,t) \geq 0$, for $(x,t) \in B_R \times (0,T)$. □

Proof of Theorem 1.1 It is easy to check that

$$\begin{cases} \frac{\partial u}{\partial \eta} = e^{pv}u^{\alpha} \geq v^{p}u^{\alpha}, \\ \frac{\partial v}{\partial \eta} = u^{q}e^{\beta v} \geq u^{q} \cdot (\frac{\beta}{\beta+1})^{\beta+1} \cdot v^{\beta+1}. \end{cases}$$

Let $(\underline{u}, \underline{v})$ be a solution of the following system:

$$\begin{cases} \underline{u}_t = \Delta \underline{u}, & \underline{v}_t = \Delta \underline{v}, \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial \underline{u}}{\partial \eta} = \underline{u}^{\alpha}\underline{v}^{\beta}, & \frac{\partial \underline{v}}{\partial \eta} = (\frac{\beta}{\beta+1})^{\beta+1}\underline{u}^{q}\underline{v}^{\beta+1}, \quad (x,t) \in \partial\Omega \times (0,T), \\ \underline{u}(x,0) = u_0(x), & \underline{v}(x,0) = v_0(x), \quad x \in \Omega. \end{cases} \tag{2.1}$$

By the results of [17], the solutions of (2.1) blow up with large initial data if $\alpha > 1$, or $\beta > 0$, or $pq > \beta(\alpha - 1)$. By the comparison principle, $(\underline{u}, \underline{v})$ is a sub-solution of (1.1), thus the solutions of (1.1) also blow up. □

3 Non-simultaneous blow-up

In this section, we prove Theorem 1.2 with four lemmas. Firstly, we define the set of initial data with a fixed constant $\varepsilon \in (0,1)$,

$$\mathbb{V}_0 = \{(u_0, v_0) | \Delta u_0 - \varepsilon u_0^{\alpha}e^{pv_0} \geq 0, \Delta v_0 - \varepsilon u_0^{q}e^{\beta v_0} \geq 0, x \in B_R\}.$$

Lemma 3.1 *For any $(u_0, v_0) \in \mathbb{V}_0$, there must be*

$$\begin{aligned} u_t(x,t) &\geq \varepsilon(u^{\alpha}e^{pv})(x,t) \\ v_t(x,t) &\geq \varepsilon(u^{q}e^{\beta v})(x,t) \end{aligned} \quad (x,t) \in B_R \times [0,T). \tag{3.1}$$

Proof Set

$$J = u_t - \varepsilon u^{\alpha}e^{pv}, \qquad K = v_t - \varepsilon u^{q}e^{\beta v}, \quad (x,t) \in B_R \times [0,T).$$

By computations, we can check that

$$\begin{aligned} J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon\alpha u^{\alpha-1}e^{pv}(u_t - \Delta u) - \varepsilon p u^{q}e^{pv}(v_t - \Delta v) \\ &\quad + \varepsilon u^{\alpha-2}e^{pv}\big(\alpha(\alpha-1)u_r^2 + 2\alpha p u u_r v_r + p^2u^2v_r^2\big) \geq 0, \\ K_t - \Delta K &= (v_t - \Delta v)_t - \varepsilon q u^{q-1}e^{\beta v}(u_t - \Delta u) - \varepsilon\beta u^{q}e^{\beta v}(v_t - \Delta v) \\ &\quad + \varepsilon u^{q-2}e^{\beta v}\big(q(q-1)u_r^2 + 2q\beta u u_r v_r + \beta^2u^2v_r^2\big) \geq 0, \\ &\quad (x,t) \in B_R \times (0,T). \\ \frac{\partial J}{\partial \eta} &= \big(e^{pv}u^{\alpha}\big)_t - \varepsilon\alpha u^{\alpha-1}\frac{\partial u}{\partial \eta}e^{\beta v} - \varepsilon u^{\alpha}e^{pv}p\frac{\partial v}{\partial \eta} \\ &= e^{pv}pu^{\alpha}\big(v_t - \varepsilon u^{q}e^{\beta v}\big) + e^{pv}\alpha u^{\alpha-1}\big(u_t - \varepsilon u^{\alpha}e^{pv}\big) \\ &= pu^{\alpha}e^{pv}K + \alpha u^{\alpha-1}e^{pv}J, \\ \frac{\partial K}{\partial \eta} &= qu^{q-1}e^{\beta v}J + \beta u^{q}e^{\beta v}K, \quad (x,t) \in \partial B_R \times (0,T). \end{aligned}$$

$$J(x,0) = \Delta u_0 - \varepsilon u_0^\alpha e^{pv_0} \geq 0, \quad x \in B_R,$$
$$K(x,0) = \Delta v_0 - \varepsilon u_0^q e^{\beta v_0} \geq 0, \quad x \in B_R.$$

By the comparison principle, $J(x,t), K(x,t) \geq 0$, for $(x,t) \in B_R \times [0,T)$. □

Lemma 3.2 *For any* $t \in [0,T)$

$$u(0,t) \leq \left[(\alpha-1)\varepsilon e^{pv_0(0)}\right]^{-\frac{1}{\alpha-1}}(T-t)^{-\frac{1}{\alpha-1}} \quad (\alpha \neq 1), \tag{3.2}$$
$$v(0,t) \leq \ln\left\{\left[\beta\varepsilon u_0^q(0)\right]^{-\frac{1}{\beta}}(T-t)^{-\frac{1}{\beta}}\right\} \quad (\beta \neq 0). \tag{3.3}$$

Proof First, we prove (3.2). From (3.1), we get

$$u_t(0,t) \geq \varepsilon u^\alpha(0,t)e^{pv(0,t)},$$

then

$$u_t(0,t) \geq \varepsilon u^\alpha(0,t)e^{pv_0(0,t)}. \tag{3.4}$$

Integrating (3.4) from t to T,

$$\int_t^T \frac{u_t(0,t)\,dt}{u^\alpha(0,t)} \geq \varepsilon e^{pv_0(0)}(T-t),$$

thus

$$-u^{-\alpha+1}(0,t) \geq \varepsilon e^{pv_0(0)}(T-t)(-\alpha+1),$$

then

$$u(0,t) \leq \left[(\alpha-1)\varepsilon e^{pv_0(0)}\right]^{-\frac{1}{\alpha-1}}(T-t)^{-\frac{1}{\alpha-1}}.$$

Similarly, we can also prove (3.3) from (3.1),

$$v_t(0,t) \geq \varepsilon u^q(0,t)e^{\beta v(0,t)} \geq \varepsilon u_0^q(0)e^{\beta v(0,t)}.$$

Integrating the above inequality from t to T, then

$$\int_t^T e^{-\beta v(0,t)}v_t(0,t)\,dt \geq \varepsilon u_0^q(0)(T-t),$$
$$\frac{1}{\beta}e^{-\beta v(0,t)} \geq \varepsilon u_0^q(0)(T-t),$$
$$v(0,t) \leq -\frac{1}{\beta}\ln\left[\beta\varepsilon u_0^q(0)(T-t)\right].$$
□

The following lemma proves the sufficient and necessary condition on the existence of u blowing up alone.

Lemma 3.3 *There exist suitable initial data such that* u *blows up while* v *remains bounded if and only if* $\alpha > q+1$.

Proof Firstly, we prove the sufficiency.

Let

$$\Gamma(x,y,t,\tau)=\frac{1}{[4\pi(t-\tau)]^{N/2}}\cdot\exp\left\{\frac{-|x-y|^2}{4(t-\tau)}\right\}$$

be the fundamental solution of the heat equation. Assume $(\tilde{u}_0,\tilde{v}_0)$ is a pair of initial data such that the solution of (1.1) blows up. Fix radially symmetric v_0 $(\geq \tilde{v}_0)$ in B_R and take $M_0 > v_0(0)$. Let the minimum of u_0 $(\geq \tilde{u}_0)$ be large such that T is small and satisfies

$$M_0 \geq v_0(0)+\frac{\alpha-1}{\alpha-q-1}\bigl[(\alpha-1)\varepsilon e^{pv_0(0)}\bigr]^{-\frac{q}{\alpha-1}}e^{\beta M_0}\cdot T^{\frac{\alpha-q-1}{\alpha-1}}.$$

Consider the auxiliary problem

$$\begin{cases}\bar{v}_t=\Delta\bar{v}, & (x,t)\in B_R\times(0,T),\\ \frac{\partial\bar{v}}{\partial\eta}=[(\alpha-1)\varepsilon e^{pv_0(0)}]^{-\frac{q}{\alpha-1}}e^{\beta M_0}(T-t)^{-\frac{q}{\alpha-1}}, & (x,t)\in\partial B_R\times(0,T),\\ \bar{v}(x,0)=v_0(x), & x\in B_R.\end{cases}$$

For $\alpha > q+1$ and by Green's identity [5], we have

$$\begin{aligned}\bar{v}(x,t) &= \int_{B_R}\Gamma(x,y,t,0)\cdot v_0(y)\,dy+\int_0^t\int_{\partial B_R}\Gamma(x,y,t,\tau)\cdot\frac{\partial\bar{v}}{\partial\eta}\cdot dS_y\cdot d\tau\\ &= \int_{B_R}\Gamma(x,y,t,0)\cdot v_0(y)\,dy\\ &\quad+\int_0^t\int_{\partial B_R}\Gamma(x,y,t,\tau)\cdot\bigl[(\alpha-1)\varepsilon e^{pv_0(0)}\bigr]^{-\frac{q}{\alpha-1}}e^{\beta M_0}(T-\tau)^{-\frac{q}{\alpha-1}}\cdot dS_y\cdot d\tau\\ &\leq v_0(0)+\frac{\alpha-1}{\alpha-q-1}\bigl[(\alpha-1)\varepsilon e^{pv_0(0)}\bigr]^{-\frac{q}{\alpha-1}}e^{\beta M_0}T^{\frac{\alpha-q-1}{\alpha-1}}\\ &\leq M_0,\end{aligned}$$

thus, $M_0\geq\bar{v}(x,t)$, for any $(x,t)\in B_R\times(0,T)$. So $\bar{v}$ satisfies

$$\begin{cases}\bar{v}_t=\Delta\bar{v}, & (x,t)\in B_R\times(0,T),\\ \frac{\partial\bar{v}}{\partial\eta}\geq[(\alpha-1)\varepsilon e^{pv_0(0)}]^{-\frac{q}{\alpha-1}}e^{\beta\bar{v}}(T-t)^{-\frac{q}{\alpha-1}}, & (x,t)\in\partial B_R\times(0,T),\\ \bar{v}(x,0)=v_0(x), & x\in B_R.\end{cases}$$

Combining the radial symmetry and the monotonicity of the initial data with the estimate (3.2), we have

$$u^q\bigl(|x|,t\bigr)\leq u^q(0,t)\leq\bigl[(\alpha-1)\varepsilon e^{pv_0(0)}\bigr]^{-\frac{q}{\alpha-1}}(T-t)^{-\frac{q}{\alpha-1}}\quad (x,t)\in B_R\times(0,T).$$

So, v satisfies that

$$\begin{cases}v_t=\Delta v, & (x,t)\in B_R\times(0,T),\\ \frac{\partial v}{\partial\eta}\leq[(\alpha-1)\varepsilon e^{pv_0(0)}]^{-\frac{q}{\alpha-1}}e^{\beta v}(T-t)^{-\frac{q}{\alpha-1}}, & (x,t)\in\partial B_R\times(0,T),\\ v(x,0)=v_0(x), & x\in B_R.\end{cases}$$

By the comparison principle, $v \le \bar{v} \le M_0$, so v remains bounded up to time T. Since $(u_0, v_0) \ge (\tilde{u}_0, \tilde{v}_0)$, (u, v) blows up, hence only u blows up at time T.

Secondly, we prove the necessity. Assume u blows up while v remains bounded, say $v \le C$.

By Green's identity, we have

$$u(0,t) \le u(0,z) + Cu^{\alpha}(0,t)(T-z),$$

for any $z \in (0,T)$, take t such that $u(0,t) = 2u(0,z)$, then

$$u(0,z) \le Cu^{\alpha}(0,z)(T-z),$$

hence,

$$u(0,t) \ge C(T-t)^{-\frac{1}{\alpha-1}} \quad t \in (0,T).$$

For some $t_1 \in (0,T)$, we can find a suitable $\varepsilon_1 \in (0,1)$, such that

$$u_t(x,t_1) - \varepsilon_1\big(u_0^{\alpha} e^{pv_0}\big)(x,t_1) \ge 0.$$

Similarly to Lemma 3.1, we can prove there must be

$$\begin{aligned} u_t(x,t) &\ge \varepsilon_1\big(u^{\alpha} e^{pv}\big)(x,t) \\ v_t(x,t) &\ge \varepsilon_1\big(u^{q} e^{\beta v}\big)(x,t) \end{aligned} \quad (x,t) \in B_R \times [t_1,T). \tag{3.5}$$

Then

$$v_t(0,t) \ge \varepsilon_1 e^{\beta v_0(0)} C^q (T-t)^{-\frac{q}{\alpha-1}}, \quad t \in [t_1,T).$$

Integrating the above inequality from t_1 to t, we have

$$v(0,t) \ge \varepsilon_1 e^{\beta v_0(0)} C^q \int_{t_1}^{t} (T-\tau)^{-\frac{q}{\alpha-1}}\, d\tau + v(0,t_1).$$

The boundedness of v requires that $\alpha > q+1$. □

The following lemma proves the sufficient and necessary condition on the existence of v blowing up alone.

Lemma 3.4 *There exist suitable initial data such that v blows up while u remains bounded if and only if $\beta > p$.*

Proof Firstly, we prove the sufficiency. Assume $(\tilde{u}_0, \tilde{v}_0)$ is a pair of initial data such that the solution of (1.1) blows up. Fix radially symmetric u_0 ($\ge \tilde{u}_0$) in B_R and take $M_1 > u_0(0)$. Let the minimum of v_0 ($\ge \tilde{v}_0$) be large such that T is small and satisfies

$$M_1 \ge u_0(0) + \frac{\beta}{\beta-p}\big[\beta\varepsilon u_0^q(0)\big]^{-\frac{p}{\beta}} M_1^{\alpha} T^{\frac{\beta-p}{\beta}}.$$

Consider the auxiliary problem

$$\begin{cases} \bar{u}_t = \Delta \bar{u}, \quad (x,t) \in B_R \times (0,T), \\ \frac{\partial \bar{u}}{\partial \eta} = [\beta \varepsilon u_0^q(0)]^{-\frac{p}{\beta}} M_1^\alpha (T-t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0,T), \\ \bar{u}(x,0) = u_0(x), \quad x \in B_R. \end{cases}$$

For $\beta > p$, and by Green's identity, we have

$$\bar{u}(x,t) \le u_0(0) + \frac{\beta}{\beta - p}\big[\beta \varepsilon u_0^q(0)\big]^{-\frac{p}{\beta}} M_1^\alpha T^{\frac{\beta-p}{\beta}} \le M_1.$$

So $\bar{u}$ satisfies

$$\frac{\partial \bar{u}}{\partial \eta} \ge \big[\beta \varepsilon u_0^q(0)\big]^{-\frac{p}{\beta}} \bar{u}^\alpha (T-t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0,T).$$

From (3.3), we have

$$\frac{\partial u}{\partial \eta} \le \big[\beta \varepsilon u_0^q(0)\big]^{-\frac{p}{\beta}} u^\alpha (T-t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0,T).$$

By the comparison principle, $u \le \bar{u} \le M_1$. Since $(u_0, v_0) \ge (\tilde{u}_0, \tilde{v}_0)$, (u,v) blows up, hence only v blows up at time T.

Secondly, we prove the necessity. Assume v blows up while u remains bounded, say $u \le C$.

By Green's identity, we have

$$v(0,t) \le v(0,z) + Ce^{\beta v(0,t)}(T-z).$$

For any $z \in (0,T)$, take t such that $v(0,t) = v(0,z) + 1$, then

$$Ce^{\beta v(0,z)}(T-z) \ge 1,$$

thus

$$v(0,t) \ge \ln\big[C(T-t)\big]^{-\frac{1}{\beta}}, \quad t \in (0,T). \tag{3.6}$$

From (3.5) and (3.6), we have

$$u_t(0,t) \ge \varepsilon_1 u_0^\alpha(0) C^{-\frac{p}{\beta}} (T-t)^{-\frac{p}{\beta}}, \quad t \in (t_1, T). \tag{3.7}$$

Integrating (3.7) from t_1 to t, we obtain that

$$u(0,t) \ge u(0,t_1) + \varepsilon_1 u_0^\alpha(0) C^{-\frac{p}{\beta}} \int_{t_1}^t (T-\tau)^{-\frac{p}{\beta}}\, d\tau.$$

The boundedness of u requires that $\beta > p$. □

4 Coexistence of simultaneous and non-simultaneous blow-up

In this section, we consider the coexistence of both simultaneous and non-simultaneous blow-ups. In order to prove Theorem 1.3, we introduce following lemma.

Lemma 4.1 *The set of* (u_0, v_0) *in* $\mathbb{V}_0$ *such that v blows up while u remains bounded is open in* L^∞*-topology.*

Proof Let (u, v) be a solution of (1.1) with initial data $(u_0, v_0) \in \mathbb{V}_0$ such that v blows up at T while u remains bounded, that is $0 < u(0, t) \le M$. We only need to find a L^∞-neighborhood of (u_0, v_0) in $\mathbb{V}_0$, such that any solution $(\hat{u}, \hat{v})$ of (1.1) coming from this neighborhood maintains the property that $\hat{v}$ blows up while $\hat{u}$ remains bounded.

By Lemma 3.4, we know $\beta > p$. Take $M_2 > M + \frac{u_0(0)}{2}$, let $(\tilde{u}, \tilde{v})$ be the solution of the following problem:

$$\begin{cases} \tilde{u}_t = \Delta\tilde{u}, \quad \tilde{v}_t = \Delta\tilde{v}, \quad (x,t) \in B_R \times (0, T_0), \\ \frac{\partial\tilde{u}}{\partial\eta} = e^{p\tilde{v}}\tilde{u}^\alpha, \quad \frac{\partial\tilde{v}}{\partial\eta} = \tilde{u}^q e^{\beta\tilde{v}}, \quad (x,t) \in \partial B_R \times (0, T_0), \\ \tilde{u}(x, 0) = \tilde{u}_0(x), \quad \tilde{v}(x, 0) = \tilde{v}_0(x), \quad x \in B_R, \end{cases}$$

where radially symmetric $(\tilde{u}_0, \tilde{v}_0)$ is to be determined and T_0 is the maximal existence time.

Denote

$$\mathbb{N}(u_0, v_0) = \left\{ (\tilde{u}_0, \tilde{v}_0) \in \mathbb{V}_0 \middle| \left\| \tilde{u}_0(0) - u(0, T - \varepsilon_0) \right\|_\infty, \left\| \tilde{v}_0(0) - v(0, T - \varepsilon_0) \right\|_\infty < \frac{u_0(0)}{2} \right\}.$$

Since v blows up at time T, there exists small $\varepsilon_0 > 0$, such that $(\tilde{u}, \tilde{v})$ blows up and T_0 is small, satisfying

$$M_2 > M + \frac{u_0(0)}{2} + \frac{\beta}{\beta - p}\left[\beta\varepsilon\left(\frac{u_0(0)}{2}\right)^q\right]^{-\frac{p}{\beta}} T_0^{\frac{\beta - p}{\beta}} M_2^\alpha,$$

provided $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{N}(u_0, v_0)$.

Consider the auxiliary system,

$$\begin{cases} \bar{u}_t = \Delta\bar{u}, \quad (x,t) \in B_R \times (0, T_0), \\ \frac{\partial\bar{u}}{\partial\eta} = [\beta\varepsilon\tilde{u}_0^q(0)]^{-\frac{p}{\beta}} M_2^\alpha (T_0 - t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0, T_0), \\ \bar{u}(x, 0) = \tilde{u}_0(x), \quad x \in B_R. \end{cases}$$

By Green's identity, $\bar{u} \le M_2$. Hence,

$$\frac{\partial\bar{u}}{\partial\eta} \ge \left[\beta\varepsilon\tilde{u}_0^q(0)\right]^{-\frac{p}{\beta}} \bar{u}^\alpha (T_0 - t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0, T_0).$$

Meanwhile, from (3.3), we get

$$\tilde{v}(0, t) \le \ln\left\{\left[\beta\varepsilon\tilde{u}_0^q(0)\right]^{-\frac{1}{\beta}} (T_0 - t)^{-\frac{1}{\beta}}\right\}.$$

So, we have

$$\frac{\partial\tilde{u}}{\partial\eta} \le \left[\beta\varepsilon\tilde{u}_0^q(0)\right]^{-\frac{p}{\beta}} \tilde{u}^\alpha (T_0 - t)^{-\frac{p}{\beta}}, \quad (x,t) \in \partial B_R \times (0, T_0).$$

By the comparison principle, $\tilde{u} \le \bar{u} \le M_2$, then $\tilde{v}$ must blow up.

According to the continuity with respect to initial data for bounded solutions, there must exist a neighborhood of (u_0, v_0) in $\mathbb{V}_0$ such that every solution $(\hat{u}, \hat{v})$ starting from the neighborhood, will enter $\mathbb{N}(u_0, v_0)$ at time $T - \varepsilon_0$, and keeps the property that $\hat{v}$ blows up while $\hat{u}$ remains bounded. □

Similarly, we can prove the set of (u_0, v_0) in $\mathbb{V}_0$ such that u blows up while v remains bounded is open in L^∞-topology, we omit the proof here.

Now, we give the proof of Theorem 1.3.

Proof of Theorem 1.3 Under our assumptions, from Lemma 3.3, we know that the set of (u_0, v_0) in $\mathbb{V}_0$ such that u blows up and v remains bounded is nonempty. And from Lemma 3.4, we also know the set of (u_0, v_0) in $\mathbb{V}_0$ such that v blows up and u is bounded is nonempty.

Moreover, Lemma 4.1 shows that such sets are open. Clearly, the two open sets are disjoint. That is to say, there exists (u_0, v_0) such that u and v blow up simultaneously at a finite time T. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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