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# A program for predicting the intervals of oscillations in the solutions of ordinary second-order linear homogeneous differential equations

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**Abstract**

We derive a new criterion for deducing the intervals of oscillatory behavior in the solutions of ordinary second-order linear homogeneous differential equations from their coefficients. The validity of the method depends on one's ability to transform a given differential equation to its simplest possible form, so a program must be executed that involves transformations of both variables before the criterion can be applied. The payoff of the program is the detection of oscillations precisely where they may occur in finite or infinite intervals of the independent variable. We demonstrate how the oscillation-detection program can be carried out for a variety of well-known differential equations from applied mathematics and mathematical physics.

**MSC:** 34A25; 34A30

**Keywords:** oscillations; second-order linear differential equations; analytical theory; transformations

## 1 Introduction

In this work, we set out to investigate an apparently simple question concerning the theory of oscillatory solutions of ordinary second-order linear homogeneous differential equations: It is well known that such equations in the form

$$y'' + By' + Cy = 0, \tag{1}$$

where primes denote derivatives with respect to the independent variable $x$ and $B$ and $C$ are constant coefficients, admit oscillatory solutions $y(x)$ only if the discriminant $D \equiv B^2 - 4C < 0$, but the same discriminant is not a predictor of oscillatory behavior in cases where the coefficients are functions of $x$. The question then is why this criterion for oscillatory behavior fails in the general case of equations of the form

$$y'' + b(x)y' + c(x)y = 0, \tag{2}$$

and whether the corresponding discriminant function $d(x) \equiv b^2(x) - 4c(x)$ carries any information at all about the nature of the solutions of equation (2).

Our analysis, like many of the classical investigations of the past [1–12] (see also the reviews of Wong [12] and Agarwal *et al.* [13]), begins by transforming equations (1) and (2) to their canonical forms in which the first derivative terms are eliminated. Equation (1) then becomes

$$u'' + Qu = 0, \tag{3}$$

where $y(x) = u(x)\exp(-\frac{1}{2}Bx)$, and the constant coefficient

$$Q = -\frac{D}{4} = C - \frac{B^2}{4}. \tag{4}$$

Equation (2) becomes

$$u'' + q(x)u = 0, \tag{5}$$

where $y(x) = u(x)\exp(-\frac{1}{2}\int b(x)\,dx)$, and the coefficient

$$q(x) = -\frac{1}{4}\bigl[d(x) + 2b'(x)\bigr] = c(x) - \frac{b^2(x)}{4} - \frac{b'(x)}{2}. \tag{6}$$

A comparison between the two canonical forms reveals why the properties of the well-known damped and simple harmonic oscillators (equations (1) and (3)) do not carry over to the case of nonconstant coefficients (equations (2) and (5)): The former transformation to the canonical form effectively 'folds' the damping coefficient $B$ into the new constant term $Q$, where the negative term $-B^2/4$ clearly opposes the natural internal oscillatory term $C > 0$ of equation (1). On the other hand, the latter transformation to the canonical form folds into $q(x)$ the derivative term $-b'(x)/2$ in addition to the pure-damping negative term $-b^2(x)/4$. The term $-b'(x)/2$ sometimes acts as damping (when $b' > 0$), whereas other times it enhances the internal oscillatory term $c(x) > 0$ (when $b' < 0$). Therefore, the function $b(x)$ does not represent pure damping as in the case of the constant coefficient $B$ in the first derivative term of equation (1). This is also the reason that in the past the canonical form has been proven inadequate in predicting the oscillatory behavior of the solutions of equation (2) since the transformation to this form does not fold a pure-damping term into the coefficient $q(x)$ of equation (5) (for examples, see Section 2.3 in [13] and Section XI.1 in [14]).

It is rather obvious that the above difficulties with the term $-b'(x)/2$ do not materialize in cases where $b(x)$ is constant since then $b' = 0$ and the canonical coefficient $q(x)$ is not 'contaminated' by the term $-b'(x)/2$. This observation shows us how to circumvent the difficulties associated with $b'(x)$: Equations of the form (2) should be cast to their canonical form, and then this form should be cast into a new form in which the coefficient of the first derivative is constant. Then, another transformation to the canonical form will fold a constant-damping term into the $q(x)$ term of equation (5) that will clearly oppose any oscillatory tendency without introducing contamination from $b'(x)$. Once the competition between oscillation and damping has been established in that new $q(x)$ term, a criterion for oscillatory solutions could presumably be found by using that final canonical form of equation (2) and Sturm's [8] comparison theorem. We analyze this procedure in Section 2 below, where we establish a program for deriving a criterion for oscillatory solutions for equation (5). Then, in Sections 3-5, we apply this methodology to some well-known and commonly used equations in applied mathematics and mathematical physics [15–17], and we predict the precise intervals of oscillations of their solutions. Finally, in Section 6, we summarize and discuss our results.

## 2 A program for detecting oscillatory solutions

### 2.1 Cauchy-Euler equation

The Cauchy-Euler equation

$$y'' + \frac{B}{x}y' + \frac{C}{x^2}y = 0, \tag{7}$$

where $B$ and $C$ are constants, shows that a criterion for oscillatory solutions can be easily established when a given differential equation can be transformed to a form that contains constant coefficients. By applying the Euler transformation

$$x = \exp(t), \tag{8}$$

this equation takes the form of a damped harmonic oscillator with constant coefficients

$$\ddot{y} + (B-1)\dot{y} + Cy = 0, \tag{9}$$

where dots denote derivatives with respect to $t$, and oscillatory solutions appear when the discriminant $D = (B-1)^2 - 4C < 0$. This conclusion is well known from the theory of second-order differential equations with constant coefficients. It can also be obtained by casting equation (9) to its canonical form and then by applying Sturm's [8] comparison theorem.

The case of the Cauchy-Euler equation indicates that the first step in establishing a criterion for oscillatory solutions must be an attempt to transform an equation of the form (5) to another form in which all the coefficients are constant. We show in Section 2.2 that such a task cannot be accomplished by a transformation of the dependent variable $u(x)$ because then the constant damping of the first derivative term cannot be folded into the coefficient of the final canonical form. Then we show in Section 2.3 that the task can be carried out successfully by a transformation of the independent variable $x$ but only for some specific equations that are generalized forms of the Cauchy-Euler equation. Therefore, equation (7) is not merely a simple case that can be handled with ease; on the contrary, it is a representative of the one and only one type of differential equation that can be transformed to a damped harmonic oscillator with constant coefficients in all of its terms.

When the above step fails (for equations that do not have the symmetries of equation (7)), it is still possible to transform a given equation to a form in which only the first derivative term has a constant coefficient. Then this constant damping can be folded into the coefficient of the final canonical form where it will oppose oscillatory tendencies. We show in Section 2.4 how this step is carried out and how the criterion for oscillatory solutions then emerges. In what follows, we always begin with the canonical form (5) since all ordinary second-order linear homogeneous differential equations can be initially cast into this form. We note however that if a given equation is already in canonical form, then one may not assume that the damping has already been folded into $q(x)$; as equation (6) shows, the given $q(x)$ may already be contaminated by the $-b'(x)/2$ term, which may not be acting as pure damping. It is for this reason that the above-discussed oscillation-detection program must still be carried out in its entirety for a given equation of the form (5) so that a constant-damping term will be created, and then it will be explicitly folded into the original $q(x)$ term. This procedure will ensure that every effort has been made for pure damping to oppose the natural tendency for oscillatory behavior that the given $q(x)$ term may possess.

### 2.2 The transformation of the dependent variable fails

The substitution $u(x) = v(x)z(x)$ into equation (5) gives

$$z'' + \frac{2v'}{v}z' + \left[\frac{v''}{v} + q(x)\right]z = 0. \tag{10}$$

In this equation, we can choose the function $v(x)$ freely. The requirement that the coefficient of $z'$ should be a constant $k$ leads to $v(x) = \exp(\frac{1}{2}kx)$, and then equation (10) becomes

$$z'' + kz' + \left[\frac{k^2}{4} + q(x)\right]z = 0. \tag{11}$$

We see now that the constant damping $k$ of the $z'$ term cannot be folded into the coefficient $\hat{q}(x)$ of the final canonical form. The constant $k$ drops out of the discriminant $d(x)$ of equation (11):

$$d(x) = k^2 - 4\left[\frac{k^2}{4} + q(x)\right] = -4q(x), \tag{12}$$

and $\hat{q}(x)$ reverts back to the original $q(x)$:

$$\hat{q}(x) = -\frac{d(x)}{4} = q(x). \tag{13}$$

Therefore, our program cannot be carried out by a transformation of the original function $u(x)$ in equation (5), and we turn next to transformations of the independent variable $x$.

### 2.3 Transformations of the independent variable

The substitution $x = f(t)$ into equation (5) gives

$$\ddot{u} - \frac{\ddot{f}}{\dot{f}}\dot{u} + \dot{f}^2 q(x)u = 0, \tag{14}$$

where dots denote derivatives with respect to $t$. In this equation, we can choose the function $f(t)$ freely. The requirement that the coefficient of $\dot{u}$ should be a constant $-k$ leads to a solution of the equation $\ddot{f} - k\dot{f} = 0$ of the form $f(t) = c_1 + c_2 \exp(kt)$, where $c_1$ and $c_2$ are arbitrary constants, and then equation (14) can be written as

$$\ddot{u} - k\dot{u} + k^2(x - c_1)^2 q(x)u = 0. \tag{15}$$

Its canonical form is

$$\ddot{w} + \hat{q}(x)w = 0, \tag{16}$$

where $u(t) = w(t)\exp(\frac{1}{2}kt)$ and

$$\hat{q}(x) = k^2\left[(x - c_1)^2 q(x) - \frac{1}{4}\right]. \tag{17}$$

The above solution for $x = f(t)$ is a generalization of the classical Euler transformation $x = \exp(t)$ and inserts into equations (15) and (17) the two arbitrary constants $k$ and $c_1$. The constant $k^2$ acts as a scale factor in the coefficient $\hat{q}(x)$ of the canonical form. As the transformation $x = f(t)$ changes the $x$-scale of the original equation, $k$ rescales accordingly the 'oscillation' frequency $\sqrt{\hat{q}(x)}$. The constant $c_1$ is more interesting: It represents a 'horizontal' shift of the independent variable $x$. By an appropriate choice of $c_1$, the shifted term $(x - c_1)^2$ in equation (17) is capable of eliminating any one regular singular point that the original given term $q(x)$ may contain. Furthermore, the coefficient $\hat{q}(x)$ ends up being a constant, and equations (15) and (16) end up having constant coefficients only in cases where $q(x)$ contains precisely one regular singular point at $x = c_1$, that is, when

$$q(x) \propto \frac{1}{(x - c_1)^2}, \tag{18}$$

and the original equation (5) is of the Cauchy-Euler type.[a] In any other case, the coefficient of $u(t)$ in equation (15) cannot be a constant. Such cases are analyzed in Section 2.4.

The results described indicate that differential equations of the Cauchy-Euler type should always be transformed to a form with constant coefficients before an investigation of oscillatory behavior in their solutions is carried out. At the same time, there exist differential equations that can be transformed to the Cauchy-Euler type, and the entire procedure that leads to constant coefficients must then be applied to them as well. We provide a related example in Section 3.3, where we study the Riemann-Weber [18] equations, a long-standing counterexample to the discovery of a robust criterion for oscillatory solutions by considering the $q(x)$ term alone of a differential equation given in the canonical form (5).

### 2.4 Constant damping and the criterion for oscillatory solutions

The differential equations that we study can all be cast in the canonical form (5), and this form can always be recast in the form (15) with constant damping, but in the most commonly occurring cases, the coefficient of $u(t)$ will not be constant. For such equations, we can still fold the constant damping $k$ into the coefficient (17) of the canonical form (16), where it will be allowed to oppose internal oscillatory tendencies. The resulting equation

$$\ddot{w} + k^2\left[(x - c_1)^2 q(x) - \frac{1}{4}\right]w = 0, \tag{19}$$

can then be used to establish a criterion for oscillatory solutions despite the dependence of the coefficient of $w(t)$ on $x(t)$. Applying Sturm's [8] comparison theorem, a comparison of equation (19) to the simple harmonic oscillator

$$\ddot{y} + \epsilon^2 y = 0, \tag{20}$$

where the constant $\epsilon \to 0$, shows that oscillatory solutions occur in equation (19) and therefore also in equation (5) in intervals of $x$ where

$$q(x) > \frac{1}{4(x - c_1)^2}. \tag{21}$$

This criterion provides 'to within $\epsilon$' a necessary and sufficient condition for oscillations in the solutions of equation (5) (see also [12] and [19]). A surprising element is the presence of the arbitrary constant $c_1$. As we have mentioned, this constant is useful in eliminating regular singularities from the coefficient of equation (19), in which case the search for oscillatory solutions circumvents any pole that may be embedded in the $q(x)$ term of an equation given in the canonical form (5). For many equations of applied mathematics, singularities in their coefficients occur at $x = 0$, in which case we set $c_1 = 0$, and the criterion takes the form

$$q(x) > \frac{1}{4x^2}. \tag{22}$$

The validity of this criterion for oscillatory solutions is confirmed in Section 3 for several complicated cases of differential equations with known oscillatory characteristics.

Our investigation is not however limited only to the conventional definition of oscillation as a sequence of infinitely many zeros in the solution of a differential equation. In what follows, we define oscillatory behavior as the appearance of successive critical points of the same kind (maxima, or minima, or inflection points) in the graph of a solution. This definition allows us to study oscillations in solutions with a finite number of zeros or no zeros (Section 3.4) and in solutions defined in finite domains (Section 5), as well as some characteristic high-frequency oscillations that tend to occur in the vicinity of $x = 0$ (Section 3.1 and Section 4.3).

## 3 Confirmations of the criterion

### 3.1 Bessel equation and equations transformed to the Bessel type

The canonical form of Bessel's differential equation [16] is

$$u'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right)u = 0. \tag{23}$$

A nontrivial application of Sturm's comparison and separation theorems shows that the solutions (the Bessel functions) oscillate for all $|x| > |n|$ (Theorem 9.3 in [20]). The same result is also obtained by the criterion (22) with $q(x) = 1 + (1 - 4n^2)/(4x^2)$ that leads to the inequality $x^2 > n^2$. More importantly, having established the criterion for oscillations of the Bessel functions, we can now confirm the validity of inequality (22) for a large family of differential equations that can be cast to the Bessel type. These are the equations of the form

$$u'' + x^n u = 0, \tag{24}$$

where $n \neq 0, -2$ is a constant, and they can be transformed to a Bessel equation of order $m = 1/(n+2)$ with independent variable $\xi = 2mx^{1/(2m)}$. The condition $|\xi| > |m|$ for the Bessel equation then predicts oscillatory solutions in equation (24) for

$$|x|^{n+2} > \frac{1}{4}. \tag{25}$$

The same result can also be obtained, quite easily, by using $q(x) = x^n$ into the criterion (22). The agreement between the two derivations of the criterion confirms the validity of inequality (22).

Equation (25) indicates that there exist two distinct regions of the parameter $n$ (separated by the $n = -2$ case) in which the solutions exhibit different oscillatory characters. The above analysis is not applicable for $n = -2$, but in this case the original equation (24) is a Cauchy-Euler equation (equation (7) with $B = 0$ and $C = 1$) that was discussed in Section 2.1. For $n > -2$, then $m = 1/(n+2) > 0$, and equation (25) can be written as

$$|x| > \left(\frac{1}{4}\right)^m, \tag{26}$$

so oscillations occur in two semi-infinite intervals on the $x$-axis. On the other hand, for $n < -2$, then $m < 0$, and equation (25) can be written as

$$|x| < \left(\frac{1}{4}\right)^m, \tag{27}$$

so oscillations occur only in a finite interval around $x = 0$. This distinct behavior of the two types of solutions has been verified by numerical integrations of equation (24). The same dual behavior has also been observed in the solutions of a more complicated differential equation that is discussed in Section 4.3.

### 3.2 Modified Bessel equation

The canonical form of the modified Bessel differential equation [16] is

$$u'' + \left(\frac{1-4n^2}{4x^2} - 1\right)u = 0. \tag{28}$$

It is well known that its particular solutions, the modified Bessel functions, are nonoscillatory. The same result is also obtained by the criterion (22) with $q(x) = (1-4n^2)/(4x^2) - 1$, which leads to $x^2 + n^2 < 0$, an inequality that is not satisfied for any values of $x$ and $n$.

### 3.3 Riemann-Weber equations

An interesting family of extensions of the Cauchy-Euler equation, known as the Riemann-Weber [18] equations, have long served as a counterexample to finding a robust criterion for oscillatory solutions by examining the coefficient $q(x)$ alone (see [12–14, 19]). The first few members of this family are:

$$u'' + \frac{\mu}{x^2} u = 0, \tag{29}$$

$$u'' + \frac{1}{x^2}\left(\frac{1}{4} + \frac{\mu}{\ln^2 x}\right) u = 0, \tag{30}$$

and

$$u'' + \frac{1}{x^2}\left[\frac{1}{4} + \frac{1}{\ln^2 x}\left(\frac{1}{4} + \frac{\mu}{\ln^2(\ln x)}\right)\right] u = 0, \tag{31}$$

where $\mu$ is a constant. Subsequent members of the family can be written down by using the recursion formula

$$q_n(x) = \frac{1}{x^2}\left[\frac{1}{4} + q_{n-1}(\ln x)\right], \tag{32}$$

for the coefficient of $u(x)$. By applying a sequence of successive Euler transformations of the form (8) and by returning to the canonical form in each iteration, every equation in this family can be cast into a simple harmonic oscillator of the form

$$w_{ss} + \left(\mu - \frac{1}{4}\right) w = 0, \tag{33}$$

where the subscript *ss* denotes the second derivative with respect to the final independent variable $s$. This final canonical form implies that the criterion for oscillatory solutions is $\mu > 1/4$ for all members of the family, a result that has also been produced from a variety of more complicated calculations [13, 14]. With the exception of the canonical Cauchy-Euler equation (29) that does not contain logarithms, this criterion cannot be obtained by considering the given Riemann-Weber [18] equations or any of their intermediate forms during the iterative procedure that reduces them to the form (33). This underlines the need for transforming a given equation to the simplest possible form before a criterion for oscillatory solutions can be established.

### 3.4 Wong-Willett equations with oscillatory coefficients

Wong [12] and Willett [9] have studied the oscillatory properties of the equation

$$u'' + \frac{\alpha \sin(\gamma x)}{x} u = 0 \quad (x \geq 0), \tag{34}$$

where $\alpha$ and $\gamma$ are nonzero constants. The criterion for oscillatory solutions is $|\alpha/\gamma| > 1/\sqrt{2}$ (see also Section 2.2 in [13]) and relies on the oscillatory nature of $\sin(\gamma x)$ that is included in the coefficient of $u(x)$. When the inequality is not satisfied, the solutions do not oscillate in the sense that they do not possess an infinite number of zeros along the $x$-axis.

On the other hand, the final canonical form of equation (34) is

$$\ddot{w} + \left[\alpha x \sin(\gamma x) - \frac{1}{4}\right] w = 0, \tag{35}$$

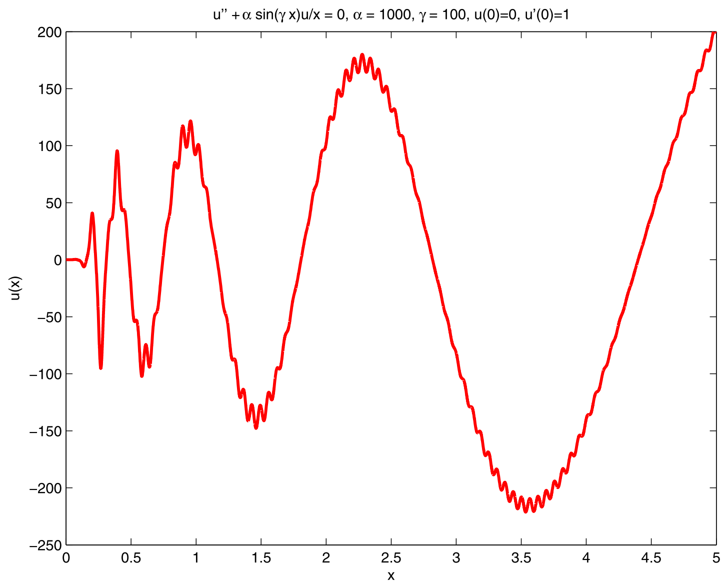


**Figure 1 The numerical solution of equation (34) with $\alpha = 1{,}000$, $\gamma = 100$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$.**

where $x = \exp(t)$ and $u(t) = w(t)\exp(t/2)$; if we take $\alpha > 0$, then inequality (22) predicts that oscillations will appear in the solutions for $\sin(\gamma x) > 1/(4\alpha x)$. This condition is satisfied periodically over repeated intervals in $x$, and it implies an additional type of oscillation that rides on top of the large-scale oscillation predicted about the $x$-axis by Wong [12] and Willett [9].

Figure 1 shows the numerical solution of equation (34) with $\alpha = 1{,}000$, $\gamma = 100$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$. Both types of oscillations are clearly visible. Figure 2 shows another numerical solution with $\alpha = 10$, $\gamma = 20$, and the same set of boundary conditions. For these choices of the constants, the Wong-Willett criterion is not satisfied, and the solution does not oscillate about the $x$-axis. The oscillations predicted by inequality (22) are however still visible, confirming the validity of criterion (22). We note that, because of the steep monotonic rise of the solution, the oscillations are manifested as a sequence of inflection points rather than as a sequence of extrema. This interesting behavior is discussed further in Section 5.1.

Wong [12] and Willett [9] have also studied several other differential equations with oscillatory coefficients. We note two equations that are extensions of equation (34):

(a) Wong [12] studied the equation

$$u'' + \frac{\alpha \sin(\gamma x)}{x^k} u = 0 \quad (x \geq 0), \tag{36}$$

where $\alpha$, $\gamma$, and $k$ are nonzero constants. He found that, for $k > 1$, the solutions are nonoscillatory in the sense that they do not possess infinitely many zeros. In contrast, if we consider $\alpha > 0$, then the criterion (22) predicts oscillations for $\sin(\gamma x) > x^{k-2}/(4\alpha)$. For $k = 2$, this condition reduces to $\sin(\gamma x) > 1/(4\alpha)$, and then, for $0 < \alpha \leq 1/4$, the solutions are predicted to be nonoscillatory by our definition as well. But for $\alpha > 1/4$, our criterion predicts oscillations over repeated intervals in $x$. Such oscillations can be seen in

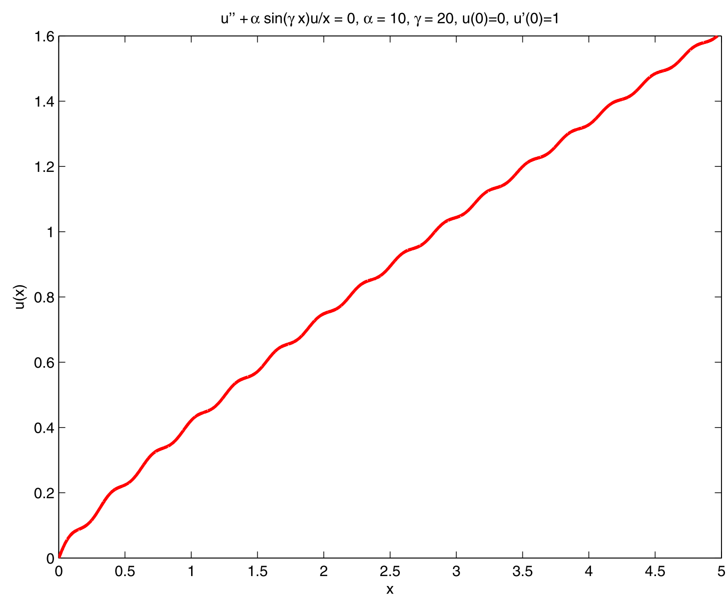


**Figure 2** The numerical solution of equation (34) with $\alpha = 10$, $\gamma = 20$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$.

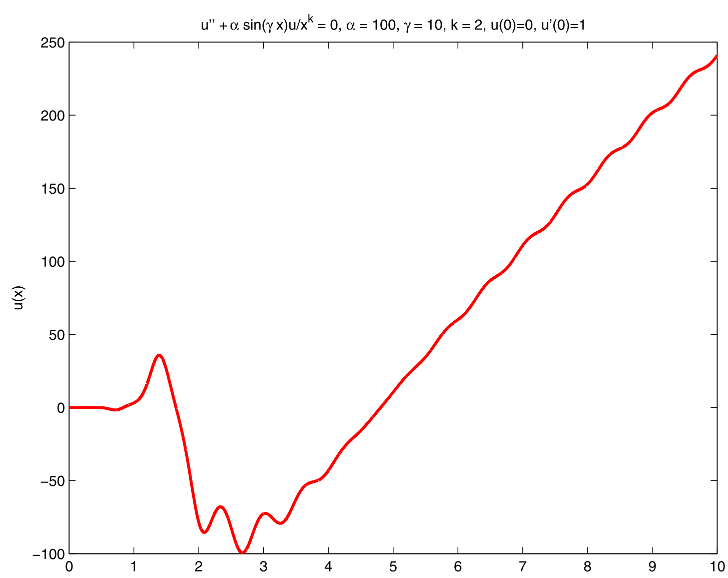


**Figure 3** The numerical solution of equation (36) with $\alpha = 100$, $\gamma = 10$, $k = 2$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$.

Figure 3, which shows the numerical solution of equation (36) with $\alpha = 100$, $\gamma = 10$, $k = 2$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$. The solution clearly oscillates for $x < 4$, and then, as it runs away to infinity, it exhibits a sequence of inflection points that give it a distinct oscillatory character.

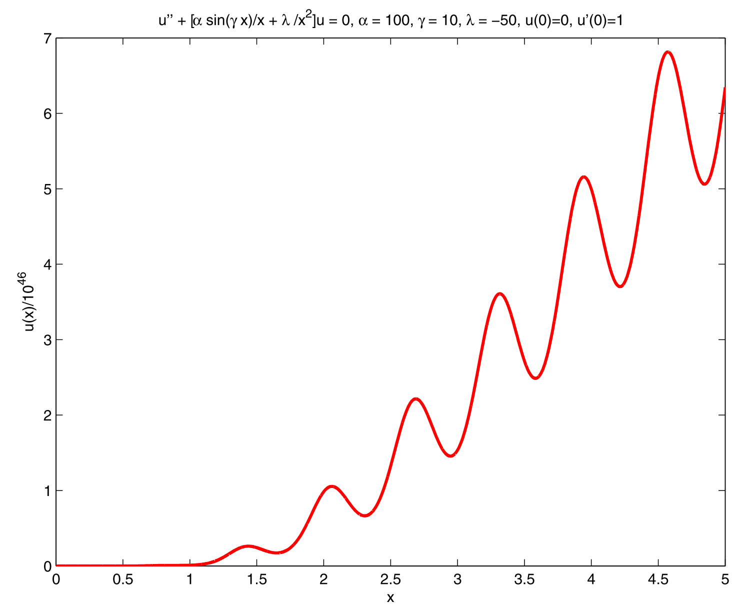


**Figure 4 The numerical solution of equation (37) with $\alpha = 100$, $\gamma = 10$, $\lambda = -50$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$.**

(b) Willett [9] studied the equation

$$u'' + \left[\frac{\alpha \sin(\gamma x)}{x} + \frac{\lambda}{x^2}\right]u = 0 \quad (x \geq 0), \tag{37}$$

where $\alpha$, $\gamma$, and $\lambda$ are nonzero constants. In conjunction with a result obtained by Wong [12] for a special case, Willett found that, for $\lambda \leq 1/4 - (\alpha/\gamma)^2/2$, the solutions are nonoscillatory in the sense that they do not possess infinitely many zeros. On the other hand, if we consider $\alpha > 0$, then the criterion (22) predicts oscillations for $\sin(\gamma x) > (1 - 4\lambda)/(4\alpha x)$, a condition that can be easily satisfied in successive intervals for large values of $x$, irrespective of the value of $\lambda$. An example of such oscillations can be seen in Figure 4, which shows the numerical solution of equation (37) with $\alpha = 100$, $\gamma = 10$, $\lambda = -50$, and boundary conditions $u(0) = 0$ and $u'(0) = 1$. For these values of the constants, the solution does not oscillate about the $x$-axis, but the oscillations predicted by the criterion (22) are clearly visible.

## 4 Applications of the criterion in semiinfinite domains

In this section, we apply inequality (22) to equations that are known empirically to exhibit oscillatory solutions over semi-infinite intervals in the variable $x$.

### 4.1 First form of the parabolic cylinder equation

We call the equation

$$u'' + \left(\frac{x^2}{4} - n\right)u = 0, \tag{38}$$

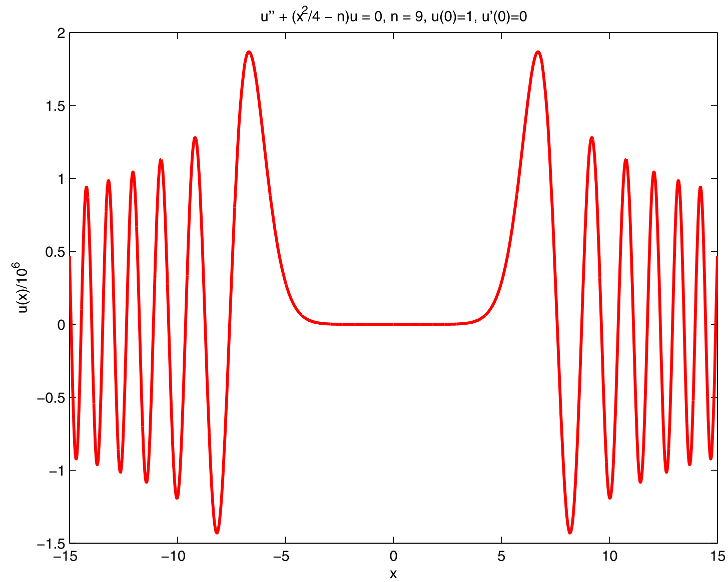


**Figure 5 The numerical solution of equation (38) with $n = 9$ and boundary conditions $u(0) = 1$ and $u'(0) = 0$.**

where $n$ is a constant [15], the first form of the parabolic cylinder differential equation. Its final canonical form is

$$\ddot{w} + \left[x^2\left(\frac{x^2}{4} - n\right) - \frac{1}{4}\right]w = 0, \tag{39}$$

where $x = \exp(t)$ and $u(t) = w(t)\exp(t/2)$. According to equation (22), the intervals over which the solutions may oscillate are given by the solutions of the algebraic inequality

$$x^4 - 4nx^2 - 1 > 0. \tag{40}$$

For $x \gg 1$, this condition can be approximated by $x^2 - 4n > 0$ and, for $n > 0$, the criterion then is $|x| > 2\sqrt{n}$. Equation (40) has only two real roots, so oscillations occur only in the two semiinfinite intervals specified by this approximate criterion. Figure 5 shows the numerical solution of equation (38) with $n = 9$ and boundary conditions $u(0) = 1$ and $u'(0) = 0$. Clearly, the oscillations set in for $|x| > 6$, as predicted by the above approximation.

### 4.2 Airy equation

The Airy differential equation [15]

$$u'' - xu = 0, \tag{41}$$

has $q(x) = -x$. According to equation (22), its solutions oscillate for

$$x < -\left(\frac{1}{4}\right)^{1/3} = -0.63. \tag{42}$$

This result is significant in that it shows that oscillations set in over a semi-infinite interval in $x$ that does not include a finite region on the left of $x = 0$. This corrects the empirical notion that the Airy functions oscillate for all negative values of $x$.

### 4.3 A complicated canonical form

Although complicated, the canonical form

$$u'' + \left[(n+1)^2 k^2 x^{2n} - \frac{n(n+2)}{4x^2}\right] u = 0, \tag{43}$$

where $k \neq 0$ and $n$ are constants, admits a simple analytic solution

$$u(x) = x^{-n/2} \sin\bigl(kx^{n+1}\bigr). \tag{44}$$

For $n = -1$, equation (43) reduces to a Cauchy-Euler equation with $q(x) = 1/(4x^2)$ and nonoscillatory solutions. For $n > -1$, the sine term in equation (44) is clearly oscillatory all the way to infinity, whereas for $n < -1$, some high-frequency oscillations are restricted to a small but finite region around $x = 0$ owing to the particular dependence of the argument of the sine on a positive power of $(1/x)$.

This varied behavior of the solutions (44) can be easily deduced from criterion (22) as well. After some algebra, we find that the solutions are oscillatory for

$$|x|^{n+1} > \frac{1}{2|k|}. \tag{45}$$

For $n > -1$, this inequality can be written as

$$|x| > \left(\frac{1}{2|k|}\right)^{\frac{1}{n+1}}, \tag{46}$$

which predicts oscillations in two distinct semiinfinite intervals on the $x$-axis.[b] On the other hand, for $n < -1$, equation (45) can be written as

$$|x| < \left(\frac{1}{2|k|}\right)^{\frac{1}{n+1}}, \tag{47}$$

which predicts oscillations over a finite interval around $x = 0$.

## 5 Oscillatory solutions in finite domains

In this section, we apply inequality (22) to equations that are known empirically to exhibit oscillatory solutions over finite intervals in the variable $x$.

### 5.1 Second form of the parabolic cylinder equation

We call the equation

$$u'' - \left(\frac{x^2}{4} + n\right) u = 0, \tag{48}$$

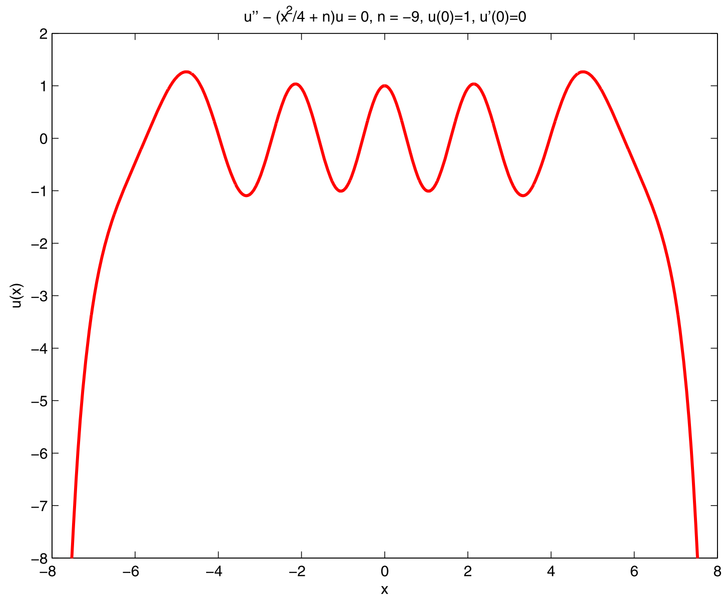


**Figure 6 The numerical solution of equation (48) with $n = -9$ and boundary conditions $u(0) = 1$ and $u'(0) = 0$.**

where $n$ is a constant [15], the second form of the parabolic cylinder differential equation. Its final canonical form is

$$\ddot{w} - \left[x^2\left(\frac{x^2}{4} + n\right) + \frac{1}{4}\right]w = 0, \tag{49}$$

where $x = \exp(t)$ and $u(t) = w(t)\exp(t/2)$. According to equation (22), the intervals over which the solutions may oscillate are given by the solutions of the algebraic inequality

$$x^4 + 4nx^2 + 1 < 0. \tag{50}$$

For $x \gg 1$, this condition can be approximated by $x^2 + 4n < 0$, and, for $n < 0$, the criterion then is $|x| < 2\sqrt{-n}$. Therefore, oscillations are restricted only in a finite interval in $x$. Figure 6 shows the numerical solution of equation (48) with $n = -9$ and boundary conditions $u(0) = 1$ and $u'(0) = 0$. Clearly, the oscillations set in for $|x| < 6$, as predicted by the above approximation.

Equation (50) has four real roots, so it predicts also a small region of no oscillation around $x = 0$. This region is too narrow (its size is $\approx 1/\sqrt{-n}$ for $n \leq -1$) for a break in oscillation to be observed in Figure 6. The nonoscillatory part of the solution around $x = 0$ ($|x| \leq 1/6$ for $n = -9$) is effectively squeezed by the two larger oscillatory regions on either side of $x = 0$. Nevertheless, the finite extent of these regions raises the question of how one can define oscillatory behavior in finite domains.

We have defined in Section 2.4 oscillatory behavior as the appearance of successive critical points of the same kind (maxima, or minima, or inflection points as in Figure 2) in the graph of a solution of a differential equation. In addition, for oscillation in a finite interval $[x_1, x_2]$, we expect to see at least one full 'cycle' in the open interval $(x_1, x_2)$, that is, at least two critical points of the same kind. As a result of this definition, low-order polynomial

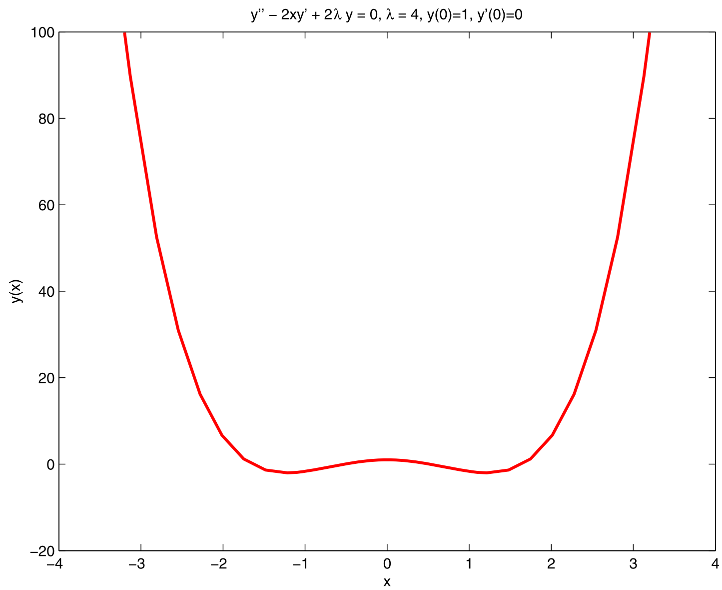


**Figure 7 The numerical solution of equation (51) with $\lambda = 4$ and boundary conditions $y(0) = 1$ and $y'(0) = 0$.**

solutions of degree $n \le 2$ are not oscillatory in any interval, but higher-order polynomial solutions will be called oscillatory in $[x_1, x_2]$ if two or more critical points of the same kind do appear in $(x_1, x_2)$. Some borderline cases with just two such critical points are described in Sections 5.2, 5.4, and 5.5.

### 5.2 Hermite equation

The Hermite differential equation [21]

$$y'' - 2xy' + 2\lambda y = 0, \tag{51}$$

where $\lambda > 0$ is a constant, is first transformed to its canonical form

$$u'' - \left[x^2 - (2\lambda + 1)\right]u = 0, \tag{52}$$

where $y(x) = u(x)\exp(x^2/2)$. Equation (52) is similar to the second form of the parabolic cylinder differential equation (48), which was analyzed in Section 5.1; therefore, the oscillatory characteristics of the Hermite solutions are expected to be similar to those described in Section 5.1. Applying the criterion (22) to $q(x) = (2\lambda + 1) - x^2$ of equation (52), we find that the intervals over which the solutions of the Hermite equation may oscillate are given by the solutions of the algebraic inequality

$$x^4 - (2\lambda + 1)x^2 + \frac{1}{4} < 0. \tag{53}$$

For $x \gg 1$, this condition can be approximated by $x^2 < 2\lambda + 1$, and the criterion then is $|x| < \sqrt{2\lambda + 1}$. Therefore, oscillations occur in a finite interval in $x$. Figure 7 shows the numerical solution of equation (51) with $\lambda = 4$ and boundary conditions $y(0) = 1$ and $y'(0) = 0$. Clearly, an oscillation occurs for $|x| < 3$, as predicted by the above approximation.

Equation (53) also predicts a region of no oscillation around $x = 0$, but this region is too narrow for a break in oscillation to be observed in Figure 7. We call this solution (a polynomial of degree $\lambda = 4$) oscillatory because, according to the definition given at the end of Section 5.1, a full 'cycle' (two minima) can be seen in the open interval $(-3, 3)$. In fact, numerical integrations using the same boundary conditions show that the lowest-order Hermite solution that exhibits such an oscillation has $\lambda = 3$; for $\lambda \leq 2$, the predicted region for oscillation, $|x| < \sqrt{2\lambda + 1}$, does not host two critical points of the same kind, and we call such solutions nonoscillatory. We note however that the $\lambda = 3$ oscillatory case is a borderline case; for different choices of boundary conditions (*e.g.*, for $y(0) = 0$ and $y'(0) = 1$), numerical integrations produce solutions that are nonoscillatory. This demonstrates the heavy influence of the adopted boundary conditions to the low-order polynomial solutions.

### 5.3 CDOS equation

An equation studied by Chuaqui *et al.* [22] (hereafter CDOS) provides an example of oscillatory solutions defined over a small finite interval. The CDOS equation is

$$u'' + \frac{C}{(1-x^2)^2} u = 0 \quad \bigl(|x| < 1\bigr), \tag{54}$$

where $C > 0$ is a constant. Its solutions are known to be oscillatory for $C > 1$, a critical value that is not singled out in the form of equation (54).

Equation (54) can be transformed to a simple harmonic oscillator as follows: The transformation

$$x = \tanh(t), \tag{55}$$

leads to the equation

$$\ddot{u} + 2\tanh(t)\dot{u} + Cu = 0, \tag{56}$$

where dots denote derivatives with respect to $t$, and the canonical form of this equation is

$$\ddot{w} + (C-1)w = 0, \tag{57}$$

where $u(t) = w(t)\,\mathrm{sech}(t)$. Alternatively, the substitution of

$$x = \tanh\left(\frac{t}{\sqrt{C}}\right), \tag{58}$$

into equation (54) leads to the equation

$$\ddot{u} + \frac{2}{\sqrt{C}} \tanh\left(\frac{t}{\sqrt{C}}\right)\dot{u} + u = 0, \tag{59}$$

where dots denote derivatives with respect to the new $t$, and the canonical form of this equation is

$$\ddot{w} + \frac{C-1}{C} w = 0, \tag{60}$$

where $u(t) = w(t)\,\mathrm{sech}(t/\sqrt{C})$. It is now obvious that the solutions are oscillatory for $C > 1$. This example shows that the oscillation-detection program must be carried out in its entirety. The criterion for oscillatory solutions cannot be obtained by considering equation (54) or the intermediate forms that lead to equations (57) or (60) (see also Section 3.3).

### 5.4 Chebyshev equation

The Chebyshev equation [21], written in the form of equation (2), is

$$y'' - \frac{x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0 \quad \bigl(|x| < 1\bigr), \tag{61}$$

where $n$ is a constant. It can be transformed to a simple harmonic oscillator by the transformation

$$x = \sin(t), \tag{62}$$

which leads directly to the equation

$$\ddot{y} + n^2 y = 0, \tag{63}$$

where dots denote derivatives with respect to $t$. Alternatively, the transformation

$$x = \sin\left(\frac{t}{n}\right), \tag{64}$$

casts equation (61) directly to the equation

$$\ddot{y} + y = 0, \tag{65}$$

where dots denote derivatives with respect to the new $t$. The particular solutions of this equation, $\sin(t)$ and $\cos(t)$, are not necessarily oscillatory because $t \propto \sin^{-1}(x)$. This raises the question of how we could define oscillatory behavior in the solutions of equation (61). Our answer makes use of the definition of oscillation given at the end of Section 5.1: The solutions of equation (61), the Chebyshev polynomials, are defined in the closed interval $[-1, 1]$; therefore, the lowest-order solution that can be called oscillatory according to our definition is the $n = 3$ polynomial. This case is shown in Figure 8, which depicts the numerical solution of equation (61) with $n = 3$ and boundary conditions $y(0) = 1$ and $y'(0) = 0$. Two minima are observed in the open interval $(-1, 1)$. This case is however a borderline case; for different choices of boundary conditions (*e.g.*, for $y(0) = 0$ and $y'(0) = 1$), numerical integrations produce solutions that are nonoscillatory. This example is another demonstration of the heavy influence of the boundary conditions to the low-order polynomial solutions (see also Section 5.2).

### 5.5 Laguerre equation

The Laguerre equation [21], written in the form of equation (2) is

$$y'' + \frac{1-x}{x}y' + \frac{\lambda}{x}y = 0 \quad (x \ge 0), \tag{66}$$

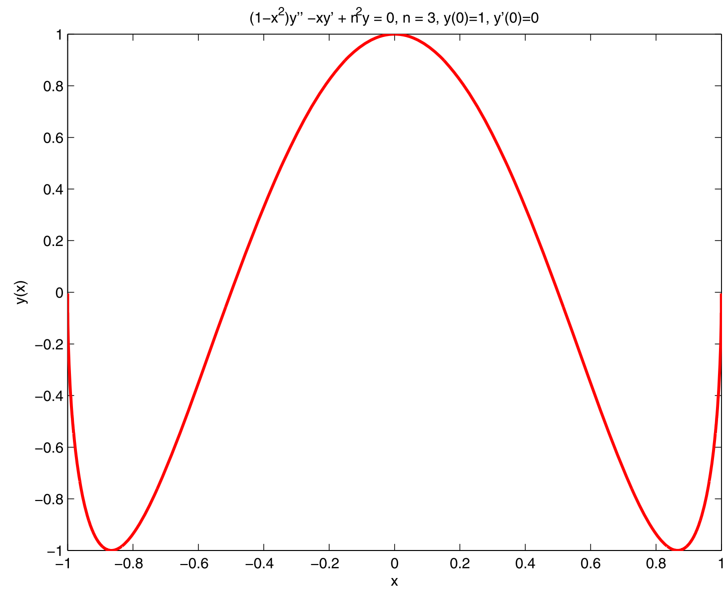


**Figure 8 The numerical solution of equation (61) with $n = 3$ and boundary conditions $y(0) = 1$ and $y'(0) = 0$.**

where $\lambda > 0$ is a constant. Its canonical form is

$$u'' + \left(\frac{1}{4x^2} + \frac{2\lambda + 1}{2x} - \frac{1}{4}\right)u = 0, \tag{67}$$

where again $x \geq 0$ and $y(x) = u(x)\exp(x/2)/\sqrt{x}$. Then criterion (22) predicts oscillatory solutions for

$$x < 2(2\lambda + 1). \tag{68}$$

This condition can be tested only approximately by numerical integrations of equation (66) because the solutions $y(x)$ break off oscillating and run away too steeply right before the critical value $2(2\lambda + 1)$ is reached. This difficulty is not present in the canonical form, so we present as examples two solutions of the canonical form. Figure 9 shows the numerical solution of equation (67) with $\lambda = 7$ and boundary conditions $u(0) = 0$ and $u'(0) = 1$. Clearly, the oscillations break off precisely at $x = 30$ (which is a point of inflection), just as predicted by inequality (68) for $\lambda = 7$. Similarly, Figure 10 shows the lowest-order numerical solution, which is oscillatory according to our definition; this solution has $\lambda = 3$ and obeys the same boundary conditions. Again, the oscillation breaks off precisely at the inflection point $x = 14$, as predicted by condition (68) with $\lambda = 3$. In the same $\lambda = 3$ case, another numerical integration of equation (67) with different boundary conditions ($u(0) = 1$ and $u'(0) = 0$) also produces a low-order oscillatory solution. This is in contrast to the lowest-order borderline cases for the Hermite and Chebyshev equations (Section 5.2 and Section 5.4, respectively), where we found some nonoscillatory solutions.

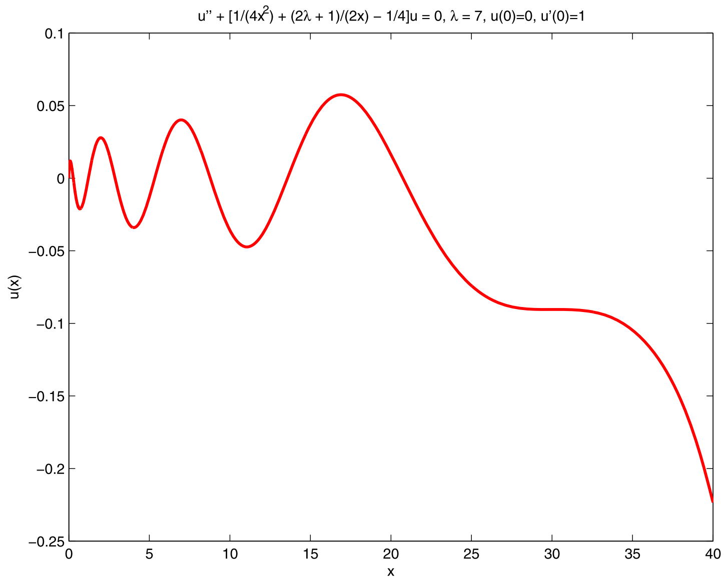


**Figure 9 The numerical solution of equation (67) with $\lambda = 7$ and boundary conditions $u(0) = 0$ and $u'(0) = 1$.**

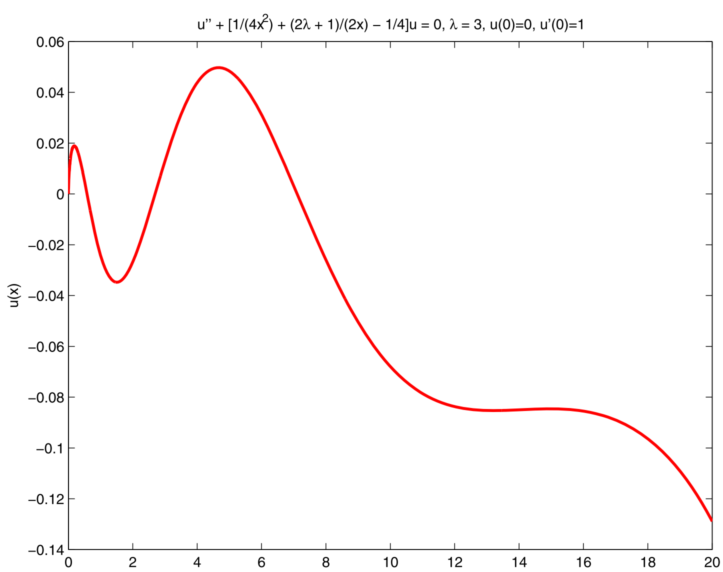


**Figure 10 The numerical solution of equation (67) with $\lambda = 3$ and boundary conditions $u(0) = 0$ and $u'(0) = 1$.**

## 6 Summary and discussion

### 6.1 Summary

In this paper, we have presented a new methodology for predicting the intervals of oscillations in the solutions of ordinary second-order linear homogeneous differential equations by examining the behavior of their coefficients. We have defined oscillatory behavior as the appearance of successive critical points of the same kind (maxima, minima, or inflection points) in the graph of a solution (Section 2.4). According to this definition, low-order polynomial solutions can be oscillatory even in finite intervals (see Section 5.1). Because of the subtleties involved in investigating oscillations in finite and infinite intervals, an entire program must be carried out with the goal of transforming a given equation to its simplest possible form. The starting point of the program is the canonical form (5) of any given equation in the form (2).

In the first step, an attempt must be made to transform the canonical form into a damped harmonic oscillator with constant coefficients in the form (1). Then the criterion for oscillatory solutions can be established, quite easily, from the discriminant of the characteristic quadratic equation, as is well known. It turns out that the Cauchy-Euler equation (Section 2.1), the Riemann-Weber equations (Section 3.3), the CDOS equation (Section 5.3), and the Chebyshev equation (Section 5.4) can all be transformed to equations with constant coefficients during this step of the program.

In the event that the above step is not viable, a different type of transformation may still be applied to the independent variable (Section 2.3; see also Section 2.2 for a failure to transform the dependent variable). In this second step, the canonical form (5) must be cast to a form in which the coefficient of the first derivative is constant. This constant represents pure damping that is capable of opposing any natural oscillatory tendency that the solutions may possess (Section 2.4). Finally, when this form is transformed to its canonical form, the constant damping is folded into the coefficient of this final form, where it will certainly oppose oscillation. Then, an application of Sturm's [8] comparison theorem produces a criterion (equations (21) and (22)) that can detect the intervals of oscillatory behavior in this step of the program. In Sections 3-5, we have presented several examples of equations from applied mathematics and mathematical physics in which our oscillation-detection program can be carried out successfully.

### 6.2 Discussion

For the general second-order differential equation of the form (2), we can use the criterion (22) and equation (6) to answer the question posed in the beginning of this paper (Section 1) about any useful information that may be obtained directly from the discriminant of equation (2), $d(x) \equiv b^2(x) - 4c(x)$. By combining these equations, the criterion for oscillatory solutions can be written as

$$d(x) < -\frac{1}{x^2} - 2b'(x). \tag{69}$$

This inequality shows that it is not sufficient for $d(x)$ to be negative, as in the case of the harmonic oscillator with constant coefficients (equation (1)). Solving equation (69) for $c(x)$, we find that

$$c(x) > \frac{b^2(x)}{4} + \left[\frac{b'(x)}{2} + \frac{1}{4x^2}\right]. \tag{70}$$

The term $1/(4x^2)$ represents the lowest-level resistance to oscillation that is present even in cases with $b(x) = 0$. For $b' > 0$, the terms in square brackets represent additional resistance to oscillation on top of the term $b^2/4$, which is analogous to the damping $B^2/4$ in the constant-coefficients case (see equation (4)). The term $c(x)$ must overcome these terms as well for oscillation to appear in the solutions. Only in the special case of $b(x) = 1/(2x) + c_0$, where $c_0$ is an arbitrary constant, do the bracketed terms cancel one another, and the criterion ($c > b^2/4$ or, equivalently, $d < 0$) then resembles the constant-coefficients case. In this case, as well as in all the other cases where $b' < 0$, the term $b'/2$ works to diminish the effect of damping in equations of the form (2). Cases in which $b' < 0$ are quite common in the equations of mathematical physics (Bessel, Hermite, Chebyshev, and Laguerre equations all belong to this category).

Two more interesting cases that make use of special forms of $b(x)$ can be delineated from inequality (70):

(a) For

$$b(x) = \frac{2}{x}, \tag{71}$$

the $b$-dependent terms in equation (70) cancel out, and the criterion reduces to $c(x) > 1/(4x^2)$. Therefore, in this case, there is no need to transform equation (2) to the canonical form (5) since $q(x) \equiv c(x)$, unless of course equation (2) can be transformed to a harmonic oscillator with constant coefficients. An example of this exception is the equation

$$y'' + \frac{2}{x}y' + \frac{1}{x^2}\left(\frac{1}{4} + \frac{\mu}{\ln^2 x}\right)u = 0, \tag{72}$$

where $\mu$ is a constant. This extension of the Riemann-Weber [18] equations (see Section 3.3) can be transformed to an equation with constant coefficients of the form (33).

(b) For

$$b(x) = \frac{1}{x}, \tag{73}$$

all terms at the right-hand side of equation (70) cancel out, and the criterion reduces to $c(x) > 0$. This case is directly applicable to Bessel equations and shows that differential equations of the form

$$y'' + \frac{1}{x}y' + c(x)y = 0, \tag{74}$$

do not contain any damping of the term $c(x)$. The category includes also the 'fuel cell' equation [23]

$$-\left(xy'\right)' - x^3 y = \lambda xy \quad (x > 0, \lambda = \text{const.}), \tag{75}$$

which can be written in the form (74) with $c(x) = x^2 + \lambda$. The absence of damping from equation (74) is consistent with what is known about the Bessel differential equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0 \quad (n = \text{const.}), \tag{76}$$

in which the $y'/x$ term is an inertial term that appears when a cylindrical coordinate system is used to solve Laplace's equation (the origin of this term is the curvature of the coordinate system). As such, this term should neither oppose nor assist oscillations in the solutions.[c]

The Riccati equations of these two cases ($2b' + b^2 = 0$ and $2b' + b^2 + 1/x^2 = 0$, respectively) can both be solved to yield the general solutions for $b(x)$ that lead to the above special criteria, but the particular solutions (71) and (73) given here seem to be the most relevant $b(x)$-forms for the equations of mathematical physics. The general solution of the latter case may however be of some theoretical interest. The most general form of $b(x)$ that represents the complete absence of damping in equation (2) is

$$b(x) = \frac{1}{x}\left[1 + \frac{2}{\ln|c_0 x|}\right], \tag{77}$$

where $c_0$ is an arbitrary constant. The particular solution (73) of physical interest is recovered from this equation in the limit as $c_0 \to 0$.

Finally, we return to the Cauchy-Euler equation (7), the CDOS equation (54), and the Chebyshev equation (61), which can be transformed to harmonic oscillators with constant coefficients in the first step of the program, thereby simplifying considerably the study of their oscillatory properties. An examination of the transformations $x = g(t)$ that we have used in each case (equation (8), equations (55), (58), and equations (62), (64), respectively) reveals that the inverse transformation $t = g^{-1}(x)$ can be obtained by the integration

$$t = \int \sqrt{c(x)}\, dx, \tag{78}$$

where $c(x)$ is the coefficient of the $y$ term that appears in the general form (2). The multiplicative constant that appears in the $c(x)$ in each case may be retained or dropped (see Section 5.3 and Section 5.4), and still the transformation $x = g(t)$ is successful in casting each of these equations to a final form with constant coefficients. This suggests that such a transformation should always be attempted in the first step of the program.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors drafted the manuscript, and they read and approved the submitted version.

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**Endnotes**

[a] If $q(x)$ contains two or more regular singular points, then the oscillatory properties of the solutions can still be investigated across each singularity by applying a different $x = f(t)$ transformation in each interval that contains one such singularity.

[b] This is also the case for elementary periodic functions such as $\sin(kx)$ and $\cos(kx)$. For $n = 0$, the solution (44) of equation (43) reduces to $u(x) = \sin(kx)$, and the criterion (46) for oscillations reduces to $|x| > 1/(2|k|)$. Thus, the oscillations set in outside of a finite interval of width $\pm P/(4\pi)$, where $P = 2\pi/|k|$ is the fundamental period. This interval is however too narrow for a break in oscillation to be observed in the graphs of $\sin(kx)$ and $\cos(kx)$ (*cf.* Section 5.1).

[c] When Laplace's equation is separated in Cartesian or spherical coordinates, the resulting inertial terms $by'$ are 0 and of the form $(2/x)y'$, respectively. It is interesting that the criterion (70) reduces to $c(x) > 1/(4x^2)$ in both of these cases. This inequality indicates that the lowest-level resistance to oscillation is present in these coordinate systems unlike in the case of cylindrical coordinates.

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