Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 767620, 14 pages
doi:10.1155/2010/767620

Research Article

Solvability of a Higher-Order Nonlinear Neutral Delay Difference Equation

Min Liu and Zhenyu Guo

School of Sciences, Liaoning Shihua University, Fushun, Liaoning 113001, China

Correspondence should be addressed to Zhenyu Guo, guozy@163.com

Received 19 March 2010; Revised 10 July 2010; Accepted 5 September 2010

Academic Editor: S. Grace

The existence of bounded nonoscillatory solutions of a higher-order nonlinear neutral delay difference equation $\Delta(a_{kn}\cdots\Delta(a_{2n}\Delta(a_{1n}\Delta(x_n+b_nx_{n-d}))))+f(n,x_{n-r_{1n}},x_{n-r_{2n}},\ldots,x_{n-r_{sn}})=0$, $n\geq n_0$, where $n_0\geq 0$, $d>0$, $k>0$, and $s>0$ are integers, $\{a_{in}\}_{n\geq n_0}$ $(i=1,2,\ldots,k)$ and $\{b_n\}_{n\geq n_0}$ are real sequences, $\bigcup_{j=1}^{s}\{r_{jn}\}_{n\geq n_0}\subseteq\mathbb{Z}$, and $f:\{n:n\geq n_0\}\times\mathbb{R}^s\to\mathbb{R}$ is a mapping, is studied. Some sufficient conditions for the existence of bounded nonoscillatory solutions of this equation are established by using Schauder fixed point theorem and Krasnoselskii fixed point theorem and expatiated through seven theorems according to the range of value of the sequence $\{b_n\}_{n\geq n_0}$. Moreover, these sufficient conditions guarantee that this equation has not only one bounded nonoscillatory solution but also uncountably many bounded nonoscillatory solutions.

1. Introduction and Preliminaries

Recently, the interest in the study of the solvability of difference equations has been increasing (see [1–17] and references cited therein). Some authors have paied their attention to various difference equations. For example,

$$\Delta(a_n\Delta x_n)+p_nx_{g(n)}=0,\quad n\geq 0 \tag{1.1}$$

(see [14]),

$$\Delta(a_n\Delta x_n)=q_nx_{n+1},\qquad \Delta(a_n\Delta x_n)=q_nf(x_{n+1}),\quad n\geq 0 \tag{1.2}$$

(see [11]),

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0 \tag{1.3}$$

(see [6]),

$$\Delta^2(x_n + px_{n-k}) + f(n, x_n) = 0, \quad n \geq 1 \tag{1.4}$$

(see [10]),

$$\Delta^2(x_n - px_{n-\tau}) = \sum_{i=1}^{m} q_i f_i(x_{n-\sigma_i}), \quad n \geq n_0 \tag{1.5}$$

(see [9]),

$$\Delta(a_n \Delta(x_n + bx_{n-\tau})) + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \dots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0 \tag{1.6}$$

(see [8]),

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} = 0, \quad n \geq n_0 \tag{1.7}$$

(see [15]),

$$\Delta^m(x_n + c_n x_{n-k}) + p_n f(x_{n-r}) = 0, \quad n \geq n_0 \tag{1.8}$$

(see [3, 4, 12, 13]),

$$\Delta^m(x_n + cx_{n-k}) + \sum_{s=1}^{u} p_n^s f_s(x_{n-r_s}) = q_n, \quad n \geq n_0 \tag{1.9}$$

(see [16]),

$$\Delta^m(x_n + cx_{n-k}) + p_n x_{n-r} - q_n x_{n-l} = 0, \quad n \geq n_0 \tag{1.10}$$

(see [17]).

Motivated and inspired by the papers mentioned above, in this paper, we investigate the following higher-order nonlinear neutral delay difference equation:

$$\Delta(a_{kn} \cdots \Delta(a_{2n} \Delta(a_{1n} \Delta(x_n + b_n x_{n-d})))) + f(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \dots, x_{n-r_{sn}}) = 0, \quad n \geq n_0, \tag{1.11}$$

where $n_0 \geq 0$, $d > 0$, $k > 0$, and $s > 0$ are integers, $\{a_{in}\}_{n \geq n_0}$ $(i = 1, 2, \dots, k)$ and $\{b_n\}_{n \geq n_0}$ are real sequences, $\bigcup_{j=1}^{s} \{r_{jn}\}_{n \geq n_0} \subseteq \mathbb{Z}$, and $f : \{n : n \geq n_0\} \times \mathbb{R}^s \to \mathbb{R}$ is a mapping. Clearly, difference equations (1.1)–(1.10) are special cases of (1.11). By using Schauder fixed point theorem and Krasnoselskii fixed point theorem, the existence of bounded nonoscillatory solutions of (1.11) is established.

Lemma 1.1 (Schauder fixed point theorem). *Let Ω be a nonempty closed convex subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω.*

Lemma 1.2 (Krasnoselskii fixed point theorem). *Let Ω be a bounded closed convex subset of a Banach space X, and let $T_1, T_2 : \Omega \to X$ satisfy $T_1x + T_2y \in \Omega$ for each $x, y \in \Omega$. If T_1 is a contraction mapping and T_2 is a completely continuous mapping, then the equation $T_1x + T_2x = x$ has at least one solution in Ω.*

The forward difference Δ is defined as usual, that is, $\Delta x_n = x_{n+1} - x_n$. The higher-order difference for a positive integer m is defined as $\Delta^m x_n = \Delta(\Delta^{m-1}x_n)$, $\Delta^0 x_n = x_n$. Throughout this paper, assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ and $\mathbb{Z}$ stand for the sets of all positive integers and integers, respectively, $\alpha = \inf\{n - r_{jn} : 1 \le j \le s, n \ge n_0\}$, $\beta = \min\{n_0 - d, \alpha\}$, $\lim_{n\to\infty}(n - r_{jn}) = +\infty$, $1 \le j \le s$, and l_β^∞ denotes the set of real sequences defined on the set of positive integers lager than β where any individual sequence is bounded with respect to the usual supremum norm $\|x\| = \sup_{n\ge\beta}|x_n|$ for $x = \{x_n\}_{n\ge\beta} \in l_\beta^\infty$. It is well known that l_β^∞ is a Banach space under the supremum norm. A subset Ω of a Banach space X is relatively compact if every sequence in Ω has a subsequence converging to an element of X.

Definition 1.3 (see [5]). A set Ω of sequences in l_β^∞ is uniformly Cauchy (or equi-Cauchy) if, for every $\varepsilon > 0$, there exists an integer N_0 such that

$$\left|x_i - x_j\right| < \varepsilon, \tag{1.12}$$

whenever $i, j > N_0$ for any $x = \{x_k\}_{k\ge\beta}$ in Ω.

Lemma 1.4 (discrete Arzela-Ascoli's theorem [5]). *A bounded, uniformly Cauchy subset Ω of l_β^∞ is relatively compact.*

Let

$$A(M,N) = \left\{x = \{x_n\}_{n\ge\beta} \in l_\beta^\infty : M \le x_n \le N, \forall n \ge \beta\right\} \quad \text{for } N > M > 0. \tag{1.13}$$

Obviously, $A(M,N)$ is a bounded closed and convex subset of l_β^∞. Put

$$\overline{b} = \limsup_{n\to\infty} b_n, \qquad \underline{b} = \liminf_{n\to\infty} b_n. \tag{1.14}$$

By a solution of (1.11), we mean a sequence $\{x_n\}_{n\ge\beta}$ with a positive integer $N_0 \ge n_0 + d + |\alpha|$ such that (1.11) is satisfied for all $n \ge N_0$. As is customary, a solution of (1.11) is said to be oscillatory about zero, or simply oscillatory, if the terms x_n of the sequence $\{x_n\}_{n\ge\beta}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. Existence of Nonoscillatory Solutions

In this section, a few sufficient conditions of the existence of bounded nonoscillatory solutions of (1.11) are given.

Theorem 2.1. *Assume that there exist constants* M *and* N *with* $N > M > 0$ *and sequences* $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, *and* $\{q_n\}_{n\geq n_0}$ *such that, for* $n \geq n_0$,

$$b_n \equiv -1, \quad \textit{eventually}, \tag{2.1}$$

$$\left|f(n,u_1,u_2,\dots,u_s) - f(n,v_1,v_2,\dots,v_s)\right| \leq h_n \max\{|u_i - v_i| : u_i, v_i \in [M,N],\ 1 \leq i \leq s\}, \tag{2.2}$$

$$\left|f(n,u_1,u_2,\dots,u_s)\right| \leq q_n, \quad u_i \in [M,N],\ 1 \leq i \leq s, \tag{2.3}$$

$$\sum_{t=n_0}^{\infty} \max\left\{\frac{1}{|a_{it}|}, h_t, q_t : 1 \leq i \leq k\right\} < +\infty. \tag{2.4}$$

Then (1.11) *has a bounded nonoscillatory solution in* $A(M,N)$.

Proof. Choose $L \in (M,N)$. By (2.1), (2.4), and the definition of convergence of series, an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$b_n \equiv -1, \quad \forall n \geq N_0, \tag{2.5}$$

$$\sum_{j=1}^{\infty} \sum_{t_1=N_0+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \leq \min\{L - M, N - L\}. \tag{2.6}$$

Define a mapping $T_L : A(M,N) \rightarrow X$ by

$$(T_L x)_n = \begin{cases} L - (-1)^k \displaystyle\sum_{j=1}^{\infty} \sum_{t_1=n+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, & n \geq N_0, \\ (T_L x)_{N_0}, & \beta \leq n < N_0 \end{cases} \tag{2.7}$$

for all $x \in A(M,N)$.

(i) It is claimed that $T_L x \in A(M,N)$, for all $x \in A(M,N)$.

In fact, for every $x \in A(M,N)$ and $n \geq N_0$, it follows from (2.3) and (2.6) that

$$
\begin{aligned}
(T_L x)_n &\geq L - \sum_{j=1}^{\infty} \sum_{t_1=n+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left| f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}}) \right|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\
&\geq L - \sum_{j=1}^{\infty} \sum_{t_1=N_0+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\
&\geq M, \\
(T_L x)_n &\leq L + \sum_{j=1}^{\infty} \sum_{t_1=N_0+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\
&\leq N.
\end{aligned}
\tag{2.8}
$$

That is, $(T_L x)(A(M,N)) \subseteq A(M,N)$.

(ii) It is declared that T_L is continuous.

Let $x = \{x_n\} \in A(M,N)$ and $x^{(u)} = \{x_n^{(u)}\} \in A(M,N)$ be any sequence such that $x_n^{(u)} \to x_n$ as $u \to \infty$. For $n \geq N_0$, (2.2) guarantees that

$$
\begin{aligned}
&\left| T_L x_n^{(u)} - T_L x_n \right| \\
&\quad \leq \sum_{j=1}^{\infty} \sum_{t_1=n+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left| f\left(t, x_{t-r_{1t}}^{(u)}, x_{t-r_{2t}}^{(u)}, \dots, x_{t-r_{st}}^{(u)}\right) - f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}}) \right|}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\
&\quad \leq \sum_{j=1}^{\infty} \sum_{t_1=n+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t \max\left\{ \left| x_{t-r_{jt}}^{(u)} - x_{t-r_{jt}} \right| : 1 \leq j \leq s \right\}}{\left| \prod_{i=1}^{k} a_{it_i} \right|} \\
&\quad \leq \left\| x^{(u)} - x \right\| \sum_{j=1}^{\infty} \sum_{t_1=N_0+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{h_t}{\left| \prod_{i=1}^{k} a_{it_i} \right|}.
\end{aligned}
\tag{2.9}
$$

This inequality and (2.4) imply that T_L is continuous.

(iii) It can be asserted that $T_L A(M,N)$ is relatively compact.

By (2.4), for any $\varepsilon > 0$, take $N_3 \geq N_0$ large enough so that

$$\sum_{j=1}^{\infty} \sum_{t_1=N_3+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} < \frac{\varepsilon}{2}. \tag{2.10}$$

Then, for any $x = \{x_n\} \in A(M,N)$ and $n_1, n_2 \geq N_3$, (2.10) ensures that

$$\begin{aligned}
|T_L x_{n_1} - T_L x_{n_2}| &\leq \sum_{j=1}^{\infty} \sum_{t_1=n_1+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\quad + \sum_{j=1}^{\infty} \sum_{t_1=n_2+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{\left|f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\leq \sum_{j=1}^{\infty} \sum_{t_1=N_3+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&\quad + \sum_{j=1}^{\infty} \sum_{t_1=N_3+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned} \tag{2.11}$$

which means that $T_L A(M,N)$ is uniformly Cauchy. Therefore, by Lemma 1.4, $T_L A(M,N)$ is relatively compact.

By Lemma 1.1, there exists $x = \{x_n\} \in A(M,N)$ such that $T_L x = x$, which is a bounded nonoscillatory solution of (1.11). In fact, for $n \geq N_0 + d$,

$$\begin{aligned}
x_n &= L - (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+jd}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, \\
x_{n-d} &= L - (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+(j-1)d}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}},
\end{aligned} \tag{2.12}$$

which derives that

$$
\begin{aligned}
x_n - x_{n-d} &= (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+(j-1)d}^{n+jd-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, \\
\Delta(x_n - x_{n-d}) &= (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+1+(j-1)d}^{n+jd} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}} \\
&\quad - (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+(j-1)d}^{n+jd-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}} \\
&= -(-1)^k \sum_{j=1}^{\infty} \sum_{t_2=n+(j-1)d}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{a_{1(n+(j-1)d)} \prod_{i=2}^{k} a_{it_i}} \\
&\quad + (-1)^k \sum_{j=1}^{\infty} \sum_{t_2=n+jd}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{a_{1(n+jd)} \prod_{i=2}^{k} a_{it_i}} \\
&= (-1)^{k-1} \sum_{t_2=n}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{a_{1n} \prod_{i=2}^{k} a_{it_i}}.
\end{aligned}
\tag{2.13}
$$

That is,

$$
a_{1n} \Delta(x_n - x_{n-d}) = (-1)^{k-1} \sum_{t_2=n}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=2}^{k} a_{it_i}}, \tag{2.14}
$$

by which it follows that

$$
\begin{aligned}
\Delta(a_{1n} \Delta(x_n - x_{n-d})) &= (-1)^{k-1} \sum_{t_2=n+1}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=2}^{k} a_{it_i}} \\
&\quad - (-1)^{k-1} \sum_{t_2=n}^{\infty} \sum_{t_3=t_2}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{\prod_{i=2}^{k} a_{it_i}} \\
&= (-1)^{k-2} \sum_{t_3=n}^{\infty} \sum_{t_4=t_3}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \dots, x_{t-r_{st}})}{a_{2n} \prod_{i=3}^{k} a_{it_i}}, \\
&\ \vdots \\
\Delta(a_{kn} \cdots \Delta(a_{2n} \Delta(a_{1n} \Delta(x_n + b_n x_{n-d})))) &= (-1)^{k-(k+1)} f(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \dots, x_{n-r_{sn}}) \\
&= -f(n, x_{n-r_{1n}}, x_{n-r_{2n}}, \dots, x_{n-r_{sn}}).
\end{aligned}
\tag{2.15}
$$

Therefore, x is a bounded nonoscillatory solution of (1.11). This completes the proof. □

Remark 2.2. The conditions of Theorem 2.1 ensure the (1.11) has not only one bounded nonoscillatory solution but also uncountably many bounded nonoscillatory solutions. In fact, let $L_1, L_2 \in (M, N)$ with $L_1 \neq L_2$. For L_1 and L_2, as the preceding proof in Theorem 2.1, there exist integers $N_1, N_2 \geq n_0 + d + |\alpha|$ and mappings T_{L_1}, T_{L_2} satisfying (2.5)–(2.7), where L, N_0 are replaced by L_1, N_1 and L_2, N_2, respectively, and $\sum_{j=1}^{\infty}\sum_{t_1=N_4+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}(h_t/|\prod_{i=1}^{k}a_{it_i}|) < |L_1 - L_2|/2N$ for some $N_4 \geq \max\{N_1, N_2\}$. Then the mappings T_{L_1} and T_{L_2} have fixed points $x, y \in A(M, N)$, respectively, which are bounded nonoscillatory solutions of (1.11) in $A(M, N)$. For the sake of proving that (1.11) possesses uncountably many bounded nonoscillatory solutions in $A(M, N)$, it is only needed to show that $x \neq y$. In fact, by (2.7), we know that, for $n \geq N_4$,

$$\begin{aligned} x_n &= L_1 - (-1)^k \sum_{j=1}^{\infty}\sum_{t_1=n+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k}a_{it_i}}, \\ y_n &= L_2 - (-1)^k \sum_{j=1}^{\infty}\sum_{t_1=n+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{f(t, y_{t-r_{1t}}, y_{t-r_{2t}}, \ldots, y_{t-r_{st}})}{\prod_{i=1}^{k}a_{it_i}}. \end{aligned} \tag{2.16}$$

Then,

$$\begin{aligned} |x_n - y_n| &\geq |L_1 - L_2| \\ &\quad - \sum_{j=1}^{\infty}\sum_{t_1=n+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{\left|f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}}) - f(t, y_{t-r_{1t}}, y_{t-r_{2t}}, \ldots, y_{t-r_{st}})\right|}{\left|\prod_{i=1}^{k}a_{it_i}\right|} \\ &\geq |L_1 - L_2| - \|x - y\|\sum_{j=1}^{\infty}\sum_{t_1=N_4+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|} \\ &\geq |L_1 - L_2| - 2N\sum_{j=1}^{\infty}\sum_{t_1=N_4+jd}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{h_t}{\left|\prod_{i=1}^{k}a_{it_i}\right|} \\ &> 0, \quad n \geq N_4, \end{aligned} \tag{2.17}$$

that is, $x \neq y$.

Theorem 2.3. *Assume that there exist constants M and N with $N > M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, satisfying* (2.2)–(2.4) *and*

$$b_n \equiv 1, \quad \textit{eventually}. \tag{2.18}$$

Then (1.11) *has a bounded nonoscillatory solution in $A(M, N)$.*

Proof. Choose $L \in (M, N)$. By (2.18) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$b_n \equiv 1, \quad \forall n \geq N_0,$$
$$\sum_{j=1}^{\infty} \sum_{t_1=N_0+(2j-1)d}^{N_0+2jd-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \leq \min\{L - M, N - L\}. \tag{2.19}$$

Define a mapping $T_L : A(M, N) \to X$ by

$$(T_L x)_n = \begin{cases} L + (-1)^k \displaystyle\sum_{j=1}^{\infty} \sum_{t_1=n+(2j-1)d}^{n+2jd-1} \sum_{t_2=t_1}^{\infty} \\ \quad \cdots \displaystyle\sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, & n \geq N_0, \\ (T_L x)_{N_0}, & \beta \leq n < N_0 \end{cases} \tag{2.20}$$

for all $x \in A(M, N)$.

The proof that T_L has a fixed point $x = \{x_n\} \in A(M, N)$ is analogous to that in Theorem 2.1. It is claimed that the fixed point x is a bounded nonoscillatory solution of (1.11). In fact, for $n \geq N_0 + d$,

$$x_n = L + (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+(2j-1)d}^{n+2jd-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}},$$
$$x_{n-d} = L + (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+2(j-1)d}^{n+(2j-1)d-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, \tag{2.21}$$

by which it follows that

$$x_n + x_{n-d} = 2L + (-1)^k \sum_{j=1}^{\infty} \sum_{t_1=n+(j-1)d}^{n+jd-1} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}. \tag{2.22}$$

The rest of the proof is similar to that in Theorem 2.1. This completes the proof. □

Theorem 2.4. *Assume that there exist constants b, M, and N with $N > M > 0$ and sequences $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, satisfying* (2.2)–(2.4) *and*

$$|b_n| \leq b < \frac{N - M}{2N}, \quad \textit{eventually}. \tag{2.23}$$

Then (1.11) *has a bounded nonoscillatory solution in $A(M, N)$.*

Proof. Choose $L \in (M + bN, N - bN)$. By (2.23) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$
\begin{gathered}
|b_n| \le b < \frac{N - M}{2N}, \quad \forall n \ge N_0, \\
\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \le \min\{L - bN - M, N - bN - L\}.
\end{gathered}
\tag{2.24}
$$

Define two mappings $T_{1L}, T_{2L} : A(M,N) \to X$ by

$$
\begin{gathered}
(T_{1L}x)_n = \begin{cases} L - b_n x_{n-d}, & n \ge N_0, \\ (T_{1L}x)_{N_0}, & \beta \le n < N_0, \end{cases} \\
(T_{2L}x)_n = \begin{cases} (-1)^k \displaystyle\sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, & n \ge N_0, \\ (T_{2L}x)_{N_0}, & \beta \le n < N_0 \end{cases}
\end{gathered}
\tag{2.25}
$$

for all $x \in A(M,N)$.

(i) It is claimed that $T_{1L}x + T_{2L}y \in A(M,N)$, for all $x, y \in A(M,N)$.

In fact, for every $x, y \in A(M,N)$ and $n \ge N_0$, it follows from (2.3), (2.24) that

$$
\begin{gathered}
\left(T_{1L}x + T_{2L}y\right)_n \ge L - bN - \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \ge M, \\
\left(T_{1L}x + T_{2L}y\right)_n \le L + bN + \sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \le N.
\end{gathered}
\tag{2.26}
$$

That is, $(T_{1L}x + T_{2L}y)(A(M,N)) \subseteq A(M,N)$.

(ii) It is declared that T_{1L} is a contraction mapping on $A(M,N)$.

In reality, for any $x, y \in A(M,\mathrm{N})$ and $n \ge N_0$, it is easy to derive that

$$
\left|(T_{1L}x)_n - \left(T_{1L}y\right)_n\right| \le |b_n| \left|x_{n-d} - y_{n-d}\right| \le b\|x - y\|, \tag{2.27}
$$

which implies that

$$
\|T_{1L}x - T_{1L}y\| \le b\|x - y\|. \tag{2.28}
$$

Then, $b < (N - M)/2N < 1$ ensures that T_{1L} is a contraction mapping on $A(M,N)$.

(iii) Similar to (ii) and (iii) in the proof of Theorem 2.1, it can be showed that T_{2L} is completely continuous.

By Lemma 1.2, there exists $x = \{x_n\} \in A(M,N)$ such that $T_{1L}x + T_{2L}x = x$, which is a bounded nonoscillatory solution of (1.11). This completes the proof. □

Theorem 2.5. *Assume that there exist constants* M *and* N *with* $N > ((2-\underline{b})/(1-\overline{b}))M > 0$ *and sequences* $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, *satisfying* (2.2)–(2.4) *and*

$$b_n \geq 0, \ \textit{eventually}, \ \textit{and} \ 0 \leq \underline{b} \leq \overline{b} < 1. \tag{2.29}$$

Then (1.11) *has a bounded nonoscillatory solution in* $A(M,N)$.

Proof. Choose $L \in (M + ((1+\overline{b})/2)N, N + (\underline{b}/2)M)$. By (2.29) and (2.4), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\begin{gathered}\frac{\underline{b}}{2} \leq b_n \leq \frac{1+\overline{b}}{2}, \quad \forall n \geq N_0, \\ \sum_{t_1=N_0}^{\infty}\sum_{t_2=t_1}^{\infty}\cdots\sum_{t_k=t_{k-1}}^{\infty}\sum_{t=t_k}^{\infty}\frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \leq \min\left\{L - M - \frac{1+\overline{b}}{2}N, N - L + \frac{\underline{b}}{2}M\right\}.\end{gathered} \tag{2.30}$$

Define two mappings $T_{1L}, T_{2L} : A(M,N) \to X$ as (2.25). The rest of the proof is analogous to that in Theorem 2.4. This completes the proof. □

Similar to the proof of Theorem 2.5, we have the following theorem.

Theorem 2.6. *Assume that there exist constants* M *and* N *with* $N > ((2+\overline{b})/(1+\underline{b}))M > 0$ *and sequences* $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, *satisfying* (2.2)–(2.4) *and*

$$b_n \leq 0, \ \textit{eventually}, \ \textit{and} \ -1 < \underline{b} \leq \overline{b} \leq 0. \tag{2.31}$$

Then (1.11) *has a bounded nonoscillatory solution in* $A(M,N)$.

Theorem 2.7. *Assume that there exist constants* M *and* N *with* $N > (\underline{b}(\overline{b}^2 - \underline{b})/\overline{b}(\underline{b}^2 - \overline{b}))M > 0$ *and sequences* $\{a_{in}\}_{n\geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n\geq n_0}$, $\{h_n\}_{n\geq n_0}$, $\{q_n\}_{n\geq n_0}$, *satisfying* (2.2)–(2.4) *and*

$$b_n > 1, \ \textit{eventually}, \ 1 < \underline{b} \ \textit{and} \ \overline{b} < \underline{b}^2 < +\infty. \tag{2.32}$$

Then (1.11) *has a bounded nonoscillatory solution in* $A(M,N)$.

Proof. Take $\varepsilon \in (0, \underline{b}-1)$ sufficiently small satisfying

$$\begin{gathered}1 < \underline{b} - \varepsilon < \overline{b} + \varepsilon < \left(\underline{b} - \varepsilon\right)^2, \\ \left(\left(\overline{b}+\varepsilon\right)\left(\underline{b}-\varepsilon\right)^2 - \left(\overline{b}+\varepsilon\right)^2\right)N > \left(\left(\overline{b}+\varepsilon\right)^2\left(\underline{b}-\varepsilon\right) - \left(\underline{b}-\varepsilon\right)^2\right)M.\end{gathered} \tag{2.33}$$

Choose $L \in ((\overline{b}+\varepsilon)M + ((\overline{b}+\varepsilon)/(\underline{b}-\varepsilon))N, (\underline{b}-\varepsilon)N + ((\underline{b}-\varepsilon)/(\overline{b}+\varepsilon))M)$. By (2.33), an integer $N_0 > n_0 + d + |\alpha|$ can be chosen such that

$$\underline{b} - \varepsilon < b_n < \overline{b} + \varepsilon, \quad \forall b \geq N_0,$$
$$\sum_{t_1=N_0}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{q_t}{\left|\prod_{i=1}^{k} a_{it_i}\right|} \leq \min\left\{ \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon} L - (\underline{b}-\varepsilon)M - N, \frac{\underline{b}-\varepsilon}{\overline{b}+\varepsilon} M + (\underline{b}-\varepsilon)N - L \right\}. \tag{2.34}$$

Define two mappings $T_{1L}, T_{2L} : A(M,N) \to X$ by

$$(T_{1L}x)_n = \begin{cases} \dfrac{L}{b_{n+d}} - \dfrac{x_{n+d}}{b_{n+d}}, & n \geq N_0, \\ (T_{1L}x)_{N_0}, & \beta \leq n < N_0, \end{cases}$$
$$(T_{2L}x)_n = \begin{cases} \dfrac{(-1)^k}{b_{n+d}} \displaystyle\sum_{t_1=n}^{\infty} \sum_{t_2=t_1}^{\infty} \cdots \sum_{t_k=t_{k-1}}^{\infty} \sum_{t=t_k}^{\infty} \frac{f(t, x_{t-r_{1t}}, x_{t-r_{2t}}, \ldots, x_{t-r_{st}})}{\prod_{i=1}^{k} a_{it_i}}, & n \geq N_0, \\ (T_{2L}x)_{N_0}, & \beta \leq n < N_0 \end{cases} \tag{2.35}$$

for all $x \in A(M,N)$. The rest of the proof is analogous to that in Theorem 2.4. This completes the proof. □

Similar to the proof of Theorem 2.7, we have

Theorem 2.8. *Assume that there exist constants M and N with $N > ((1+\underline{b})/(1+\overline{b}))M > 0$ and sequences $\{a_{in}\}_{n \geq n_0}$ $(1 \leq i \leq k)$, $\{b_n\}_{n \geq n_0}$, $\{h_n\}_{n \geq n_0}$, $\{q_n\}_{n \geq n_0}$, satisfying* (2.2)–(2.4) *and*

$$b_n < -1, \text{ eventually}, \ -\infty < \underline{b} \text{ and } \overline{b} < -1. \tag{2.36}$$

Then (1.11) *has a bounded nonoscillatory solution in $A(M,N)$.*

Remark 2.9. Similar to Remark 2.2, we can also prove that the conditions of Theorems 2.3–2.8 ensure that (1.11) has not only one bounded nonoscillatory solution but also uncountably many bounded nonoscillatory solutions.

Remark 2.10. Theorems 2.1–2.8 extend and improve Theorem 1 of Cheng [6], Theorems 2.1–2.7 of Liu et al. [8], and corresponding theorems in [3, 4, 9–17].

3. Examples

In this section, two examples are presented to illustrate the advantage of the above results.

Example 3.1. Consider the following fourth-order nonlinear neutral delay difference equation:

$$\Delta(4^n \Delta(3^n \Delta(2^n \Delta(x_n - x_{n-1})))) = 0, \quad n \geq 1. \tag{3.1}$$

Choose $M = 1$ and $N = 2$. It is easy to verify that the conditions of Theorem 2.1 are satisfied. Therefore Theorem 2.1 ensures that (3.1) has a nonoscillatory solution in $A(1,2)$. However, the results in [3, 4, 6, 8–17] are not applicable for (3.1).

Example 3.2. Consider the following third-order nonlinear neutral delay difference equation:

$$\Delta\left((2^n - n)\Delta\left(\left(n^2 - n + 1\right)\Delta\left(x_n + \frac{2^n - 1}{3^n} x_{n-4}\right)\right)\right) + \frac{\sin(2x_{n-2})}{n^2} - \frac{\cos(3x_{n-3})}{n^3} = 0, \quad n \geq 5, \tag{3.2}$$

where

$$\begin{aligned} a_{1n} &= n^2 - n + 1, \qquad a_{2n} = 2^n - n, \qquad b_n = \frac{2^n - 1}{3^n}, \\ f(n, u_1, u_2) &= \frac{\sin(2u_1)}{n^2} - \frac{\cos(3u_2)}{n^3}, \qquad h_n = q_n = \frac{2}{n^2}. \end{aligned} \tag{3.3}$$

Choose $M = 1$ and $N = 5$. It can be verified that the assumptions of Theorem 2.5 are fulfilled. It follows from Theorem 2.5 that (3.2) has a nonoscillatory solution in $A(1,5)$. However, the results in [3, 4, 6, 8–17] are unapplicable for (3.2).

Acknowledgment

The authors are grateful to the editor and the referee for their kind help, careful reading and editing, valuable comments and suggestions.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Application*, vol. 228 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[2] R. P. Agarwal, S. R. Grace, and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[3] R. P. Agarwal, E. Thandapani, and P. J. Y. Wong, "Oscillations of higher-order neutral difference equations," *Applied Mathematics Letters*, vol. 10, no. 1, pp. 71–78, 1997.

[4] R. P. Agarwal and S. R. Grace, "Oscillation of higher-order nonlinear difference equations of neutral type," *Applied Mathematics Letters*, vol. 12, no. 8, pp. 77–83, 1999.

[5] S. S. Cheng and W. T. Patula, "An existence theorem for a nonlinear difference equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 20, no. 3, pp. 193–203, 1993.

[6] J. Cheng, "Existence of a nonoscillatory solution of a second-order linear neutral difference equation," *Applied Mathematics Letters*, vol. 20, no. 8, pp. 892–899, 2007.

[7] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, NY, USA, 1991.

[8] Z. Liu, Y. Xu, and S. M. Kang, "Global solvability for a second order nonlinear neutral delay difference equation," *Computers & Mathematics with Applications*, vol. 57, no. 4, pp. 587–595, 2009.

[9] Q. Meng and J. Yan, "Bounded oscillation for second-order nonlinear neutral difference equations in critical and non-critical states," *Journal of Computational and Applied Mathematics*, vol. 211, no. 2, pp. 156–172, 2008.

[10] M. Migda and J. Migda, "Asymptotic properties of solutions of second-order neutral difference equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e789–e799, 2005.

[11] E. Thandapani, M. M. S. Manuel, J. R. Graef, and P. W. Spikes, "Monotone properties of certain classes of solutions of second-order difference equations," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 291–297, 1998.
[12] F. J. Yang and J. C. Liu, "Positive solution of even-order nonlinear neutral difference equations with variable delay," *Journal of Systems Science and Mathematical Sciences*, vol. 22, no. 1, pp. 85–89, 2002.
[13] B. G. Zhang and B. Yang, "Oscillation in higher-order nonlinear difference equations," *Chinese Annals of Mathematics*, vol. 20, no. 1, pp. 71–80, 1999.
[14] Z. Zhang and Q. Li, "Oscillation theorems for second-order advanced functional difference equations," *Computers & Mathematics with Applications*, vol. 36, no. 6, pp. 11–18, 1998.
[15] Y. Zhou, "Existence of nonoscillatory solutions of higher-order neutral difference equations with general coefficients," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 785–791, 2002.
[16] Y. Zhou and Y. Q. Huang, "Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations," *Journal of Mathematical Analysis and Applications*, vol. 280, no. 1, pp. 63–76, 2003.
[17] Y. Zhou and B. G. Zhang, "Existence of nonoscillatory solutions of higher-order neutral delay difference equations with variable coefficients," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 991–1000, 2003.