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Impulse lethal control and contraception control of seasonal breeding rodent

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Abstract

The rodent is one important component of an ecosystem; however, abundance of rodents may have negative effects to humans. Based on a seasonally breeding model and the logistic model, the effect of impulse lethal control and impulse contraception control on rodent population dynamics is investigated. The existence and stability of the periodic solution were analyzed. The condition of rodent population dying out is the same for lethal and contraception control. However, the process of rodent population tending to a stable periodic solution is different obviously. Under lethal control, the rodent population tends to a stable periodic solution more quickly, whereas under contraception control, the rodent population develops slowly.

Keywords: seasonally breeding; impulse equation; periodic solution; contraception control; lethal control

1 Introduction

The rodent is one important component of an ecosystem, it is indispensable and irreplaceable. Due to the seasonal variation of food availability and physical environment, some species of rodents are seasonally breeding and their population sizes vary in a wavelike way. Examples of seasonally breeding rodents are listed in Table 1.

The amount of rodents may increase dramatically due to some reason and bring negative effects to humans. In practice, lethal control and contraception control are usually used to control overabundance of rodents. Lethal control is killing rodents by means of rodenticide or mousetrap and its aim is increasing the death rate of rodent population. Contraception control is preventing all or part of the rodent to produce normally by means of sterilant or special virus so as to reduce the birth rate of rodent population.

How the controlled rodent population develops is one important issue. Despite the fact that it is difficult to engage in this issue through experiments, mathematical modeling may be helpful. With the prerequisite of seasonally breeding, some discrete models have been formulated and analyzed to investigate the effect of lethal control or contraception control on population dynamics [1, 2].

In the present paper, based on seasonally breeding model and the logistic model, the effect of lethal control and contraception control on rodent population dynamics is investigated. In the seasonally breeding model, one year is divided into breeding season and non-breeding season and we describe the rodent population dynamics in these two time periods with different continuous mathematical models. When the long-term dynamics

Table 1 Seasonally breeding rodent

Species of rodent	Region	Breeding season	Reference
<i>Peromyscus</i>	North America	Seasonally breeding	[3, 4]
<i>Ochotona curzoniae</i>	Heimahe, China	Seasonally breeding	[5]
<i>Arvicanthis niloticus</i>	Oursi, Burkina Faso	Seasonally breeding	[6]
<i>Mastomys erythroleucus</i>			
<i>Taterillus gracilis</i>			
<i>Gerbillus nigeriae</i>			
<i>Taterillus petteri</i>			
<i>Acomys</i>			
<i>Cryptomys hottentotus hottentotus</i>	Western Cape, South Africa	September to November	[7]
<i>Cryptomys hottentotus pretoriae</i>	Gauteng, South Africa	July to November	[8]
<i>Bathyergus janetta</i>	Kamieskroon, South Africa	July to September	[9]
<i>Lophuromys flavopunctatus</i>	Magamba, Tanzania	February to May	[10]
<i>Grammomys dolichurus</i>			
<i>Praomys delectorum</i>			
<i>Lasiopodomys brandtii</i>		March to September	[11]
<i>Meriones unguiculatus</i>	Taipusi, China	Spring and summer	[12]
<i>Acomys dimidiatus</i>	Taif, Saudi Arabia	Spring to Autumn	[13]
<i>Calomys tener</i>	Federal District, Brazil	Rainy season	[14]

of the rodent population is concerned, one year is only a time point, the seasonality of breeding is erased, and the population dynamics is described with the logistic model. Generically, lethal control is actualized through rodenticide and contraception control is actualized through sterilant. After ingesting bait that includes rodenticide or sterilant, the rodent would die or be sterile during a period not longer than several days. This period is very short compared to the whole lifetime of rodent, so we suppose that the control is accomplished at a time point instantaneously. Usually, there is a long time interval between two control practices, so we assume that the controls are implemented at discrete times with equal intervals.

2 Logistic model and seasonally breeding model

2.1 Logistic model

The logistic model is

$$x' = rx \left(1 - \frac{x}{K} \right),$$

where $x(t)$ is the population size at time t , $r > 0$ is the intrinsic increasing rate and K is the carrying capacity. The solution of the logistic model with initial condition $x(t_0) = x_0$ is $x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-r(t-t_0)}}$. For arbitrary $x_0 \neq 0$, $\lim_{t \rightarrow +\infty} x(t) = K$, that is, the positive equilibrium $x = K$ is globally asymptotically stable (eliminate the case $x_0 = 0$).

2.2 Seasonally breeding model

For seasonally breeding rodents, assume that the breeding season and the non-breeding season are present in turn, the length of the breeding season is T_1 , the length of the non-breeding season is T_2 , and the sum of T_1 and T_2 is one year, denoted by T . Let $x(t)$ be the size of the rodent population at time t . For convenience, let $t = 0$ be the beginning of a breeding season. During the breeding season, $x(t)$ increases gradually, and beyond the breeding season $x(t)$ decreases gradually due to death. Assume that $x(t)$ satisfies the logistic model during the breeding season and $x(t)$ satisfies the Malthusian model during

non-breeding season. Then one has

$$\begin{cases} x' = r_1x(1 - \frac{x}{K}), & t \in [nT, nT + T_1], \\ x' = -r_2x, & t \in [nT + T_1, (n + 1)T], \end{cases} \quad n = 0, 1, 2, \dots, \tag{1}$$

where $r_1 > 0$ is the intrinsic increasing rate during breeding season, K is the carrying capacity, and $r_2 > 0$ is the death rate during non-breeding season. This model has been studied [15].

Lemma 1 is useful in studying the dynamical behavior of model (1).

Lemma 1 *The sequence $\{y_n\}$ satisfies*

$$y_{n+1} = G(y_n) = \frac{ay_n}{by_n + c}, \tag{2}$$

where a, b, c are all positive constants. Then 0 is always an equilibrium of (2), when $a > c$, (2) has another unique positive equilibrium $y^* = \frac{a-c}{b}$. There is no other $h (>1)$ periodic equilibrium at all. If $c \geq a$, then 0 is globally asymptotically stable and if $c < a$, then y^* asymptotically attracts all positive solutions.

Proof Simple calculation gives the existence of equilibria. For positive integer $h > 1$, $y_{n+h} = G^h(y_n) = \frac{a^h(a-c)y_n}{b(a^h-c^h)y_n + c^h(a-c)}$ and it has no other equilibrium than 0 and y^* . That is, (2) has no other $h (>1)$ periodic equilibrium at all.

Obviously, if $y_0 = 0$, then $y_n \equiv 0$ and if $y_0 \neq 0$, then, for any positive integer n , $y_n \neq 0$. When $y_0 \neq 0$, rewrite equation (2) as

$$\frac{1}{y_{n+1}} = \frac{b}{a} + \frac{c}{a} \cdot \frac{1}{y_n}. \tag{3}$$

When $a = c$, (3) is $\frac{1}{y_{n+1}} = \frac{b}{a} + \frac{1}{y_n}$, its solution is $y_n = \frac{1}{\frac{1}{y_0} + \frac{nb}{a}}$, so $\lim_{n \rightarrow +\infty} y_n = 0$.

When $a \neq c$, (3) implies $\frac{1}{y_{n+1}} - \frac{b}{a-c} = \frac{c}{a}(\frac{1}{y_n} - \frac{b}{a-c})$, its solution is $y_n = \frac{1}{(\frac{c}{a})^n(\frac{1}{y_0} - \frac{b}{a-c}) + \frac{b}{a-c}}$. Thus, if $c > a$, then $\lim_{n \rightarrow +\infty} y_n = 0$ and if $c < a$, then $\lim_{n \rightarrow +\infty} y_n = y^*$. The proof is complete. \square

Solving model (1) on time interval $[nT, (n + 1)T]$, one has the mapping (4) from $x(nT)$ to $x((n + 1)T)$,

$$x((n + 1)T) = G_1(x(nT)) = \frac{Kx(nT)e^{-r_2T_2}}{(1 - e^{-r_1T_1})x(nT) + Ke^{-r_1T_1}}. \tag{4}$$

Then Theorem 1 can be obtained from Lemma 1 immediately.

Theorem 1 *0 is always an equilibrium of (4), if $r_2T_2 < r_1T_1$, then (4) has another unique positive equilibrium $x_1^* = \frac{K(e^{-r_2T_2} - e^{-r_1T_1})}{1 - e^{-r_1T_1}}$. There is no other $h (>1)$ periodic equilibrium at all. If $r_2T_2 \geq r_1T_1$, then 0 is globally asymptotically stable and model (1) has a globally asymptotically stable zero solution. If $r_2T_2 < r_1T_1$, then x_1^* asymptotically attracts all posi-*

tive solutions of (4) and model (1) has globally asymptotically stable periodic solution

$$\begin{cases} x = \frac{Kx_1^*}{x_1^* + (K - x_1^*)e^{-r_1(t-nT)}}, & t \in [nT, nT + T_1], \\ x = \frac{Kx_1^*e^{-r_2(t-nT-T_1)}}{x_1^* + (K - x_1^*)e^{-r_1T_1}}, & t \in [nT + T_1, (n + 1)T], \end{cases} \quad n = 0, 1, 2, \dots$$

Obviously, when $r_2T_2 \geq r_1T_1$, the rodent pest dies out, there is no damage. Thus rodent control is considered under the condition $r_2T_2 < r_1T_1$ hereafter.

3 Impulse lethal control

3.1 Logistic model with lethal control

The logistic model with impulse lethal control is

$$\begin{cases} x' = rx(1 - \frac{x}{K}), & t \neq nU, \\ x(nU^+) = (1 - p)x(nU), & t = nU, \end{cases} \quad (5)$$

where p is the removal rate and lethal control is implemented once at regular time intervals U .

Solving model (5) on time interval $[nU^+, (n + 1)U^+]$, one has the mapping (6) from $x(nU^+)$ to $x((n + 1)U^+)$,

$$x((n + 1)U^+) = G_2(x(nU^+)) = \frac{(1 - p)Kx(nU^+)}{(1 - e^{-rU})x(nU^+) + Ke^{-rU}}. \quad (6)$$

Then mapping (6) and Lemma 1 imply Theorem 2.

Theorem 2 *0 is always an equilibrium of (6), when $p < 1 - e^{-rU}$, then (6) has another unique positive equilibrium $x_2^* = \frac{(1 - e^{-rU} - p)K}{1 - e^{-rU}}$. There is no other $h (>1)$ periodic equilibrium at all. If $p \geq 1 - e^{-rU}$, then 0 is globally asymptotically stable and model (5) has globally asymptotically stable zero solution. If $p < 1 - e^{-rU}$, then x_2^* asymptotically attracts all positive solutions of (6) and model (5) has the globally asymptotically stable periodic solution $x(t) = \frac{Kx_2^*}{x_2^* + (K - x_2^*)e^{-r(t-nU)}}$, $t \in [nU^+, (n + 1)U]$, $n = 0, 1, 2, \dots$*

3.2 Seasonally breeding model with lethal control

This section deals with the seasonally breeding model with impulse lethal control. For convenience, suppose that lethal control is implemented at nLT ($n = 0, 1, 2, \dots$) with removal rate p , that is, put the control in practice once every L years and at the beginning of some breeding seasons. Thus, one can formulate model (7),

$$\begin{cases} x' = r_1x(1 - \frac{x}{K}), & t \in [nT, nT + T_1], t \neq kLT, \\ x' = -r_2x, & t \in [nT + T_1, (n + 1)T], t \neq kLT, \\ x(kLT^+) = (1 - p)x(kLT), & t = kLT, \end{cases} \quad \begin{matrix} n = 0, 1, 2, \dots, \\ k = 0, 1, 2, \dots \end{matrix} \quad (7)$$

Solving model (7) and denoting $x(nLT^+)$ by X_n gives

$$X_{n+1} = G_3(X_n) = \frac{(1 - p)KX_n e^{-Lr_2T_2}(e^{-r_2T_2} - e^{-r_1T_1})}{X_n(1 - e^{-r_1T_1})(e^{-Lr_2T_2} - e^{-Lr_1T_1}) + Ke^{-Lr_1T_1}(e^{-r_2T_2} - e^{-r_1T_1})}. \quad (8)$$

Thus Theorem 3 is obtained from Lemma 1 and (8).

Theorem 3 *0 is always an equilibrium of (8), when $p < 1 - e^{L(r_2 T_2 - r_1 T_1)}$, then (8) has another unique positive equilibrium $x_3^* = \frac{K(e^{-r_2 T_2} - e^{-r_1 T_1})[(1-p)e^{-Lr_2 T_2} - e^{-Lr_1 T_1}]}{(1 - e^{-r_1 T_1})(e^{-Lr_2 T_2} - e^{-Lr_1 T_1})}$. There is no other $h (>1)$ periodic equilibrium at all. If $p \geq 1 - e^{L(r_2 T_2 - r_1 T_1)}$, then 0 is globally asymptotically stable and model (7) has globally asymptotically stable zero solution. If $p < 1 - e^{L(r_2 T_2 - r_1 T_1)}$, then x_3^* asymptotically attracts all positive solutions of (8) and model (7) has a globally asymptotically stable LT periodic solution.*

4 Impulse contraception control

4.1 Logistic model with contraception control

Under contraception control, the rodent population is divided into a fertile subpopulation and a sterile subpopulation. Let $f(t)$ and $s(t)$ denote the sizes of the fertile subpopulation and the sterile subpopulation at time t , respectively. Assume that control is implemented once at regular time intervals U and the contraception rate is q . The sterile individuals cannot produce offspring, but occupy resources normally. Denote the death rate of sterile subpopulation by d . Then one can formulate the logistic model with contraception control (9),

$$\begin{cases} f' = rf(1 - \frac{f+s}{K}), & t \neq nU, \\ s' = -ds, & t \neq nU, \\ f(nU^+) = (1 - q)f(nU), & t = nU, \\ s(nU^+) = s(nU) + qf(nU), & t = nU. \end{cases} \tag{9}$$

Solving model (9) on the interval $[nU^+, (n + 1)U^+]$, one can obtain the mapping (10) from $f(nU^+), s(nU^+)$ to $f((n + 1)U^+), s((n + 1)U^+)$,

$$\begin{cases} f((n + 1)U^+) = \frac{(1-q)Kf(nU^+)\pi(U, s(nU^+))}{K\pi(0, s(nU^+)) + rf(nU^+) \int_0^U \pi(v, s(nU^+)) dv}, \\ s((n + 1)U^+) = s(nU^+)e^{-dU} + \frac{qKf(nU^+)\pi(U, s(nU^+))}{K\pi(0, s(nU^+)) + rf(nU^+) \int_0^U \pi(v, s(nU^+)) dv}, \end{cases} \tag{10}$$

where $\pi(t, u) = \exp(rt + \frac{ru}{dK}e^{-dt})$.

Theorem 4 *(0, 0) is always an equilibrium of (10), if $q < 1 - e^{-rU}$, then (10) has another unique positive equilibrium (f^*, s^*) such that $f^* = \frac{(1-q)(1 - e^{-dU})s^*}{q}$, $s^* < s_0$ and $s_0 = \frac{q(r+d)K}{r[1 - (1-q)e^{-dU}]}$.*

Proof It is easy to see that (0, 0) is an equilibrium of (10) and another equilibrium (f, s) of (10) satisfies (11),

$$\begin{cases} 1 = \frac{(1-q)K\pi(U, s)}{K\pi(0, s) + rf \int_0^U \pi(v, s) dv}, \\ s = se^{-dU} + \frac{qKf\pi(U, s)}{K\pi(0, s) + rf \int_0^U \pi(v, s) dv}. \end{cases} \tag{11}$$

Taking account the first equation in (11), one can rewrite the second equation in (11) as $f = \frac{(1-q)(1 - e^{-dU})s}{q}$. Substituting this new relation into the first equation in (11) gives

$$\varphi(s) \triangleq (1 - q)K\pi(U, s) - K\pi(0, s) - \frac{r(1 - q)(1 - e^{-dU})s}{q} \int_0^U \pi(v, s) dv = 0.$$

Because $\varphi(0) = Ke^{rU}(1 - q - e^{-rU})$, so if $q < 1 - e^{-rU}$, then $\varphi(0) > 0$, if $q \geq 1 - e^{-rU}$, then $\varphi(0) \leq 0$.

When $q \geq 1 - e^{-rU}$, $(1 - q)e^{rU} \leq 1$ holds. Note that $\exp(\frac{rs}{dK}e^{-dU}) < \exp(\frac{rs}{dK})$, hence $(1 - q)Ke^{rU} \exp(\frac{rs}{dK}e^{-dU}) < K \exp(\frac{rs}{dK})$, that is, $(1 - q)K\pi(U, s) - K\pi(0, s) < 0$. Thus $\varphi(s) < 0$ for all $s > 0$, and then $\varphi(s) = 0$ has no positive solution.

Next, consider the existence of a positive solution of $\varphi(s) = 0$ under the condition $q < 1 - e^{-rU}$. The fact that $\lim_{s \rightarrow +\infty} [rU + \frac{rs}{dK}(e^{-dU} - 1)] = -\infty$ implies

$$\lim_{s \rightarrow +\infty} \left\{ (1 - q) \exp \left[rU + \frac{rs}{dK} (e^{-dU} - 1) \right] - 1 \right\} = -1.$$

This result, together with the fact that $\lim_{s \rightarrow +\infty} \exp(\frac{rs}{dK}) = +\infty$ gives

$$\lim_{s \rightarrow +\infty} \exp \left(\frac{rs}{dK} \right) \left\{ (1 - q) \exp \left[rU + \frac{rs}{dK} (e^{-dU} - 1) \right] - 1 \right\} = -\infty,$$

that is, when $s \rightarrow +\infty$, the sum of the first two terms of $\varphi(s)$ approaches $-\infty$. Obviously, when $s \rightarrow +\infty$, the third term of $\varphi(s)$ approaches $-\infty$. So, $\lim_{s \rightarrow +\infty} \varphi(s) = -\infty$ and then when $q < 1 - e^{-rU}$, $\varphi(s) = 0$ has positive solution.

When $q < 1 - e^{-rU}$, let s^* be a positive solution of $\varphi(s) = 0$, then $\varphi(s^*) = 0$ holds, that is,

$$(1 - q)K\pi(U, s^*) - K\pi(0, s^*) = \frac{r(1 - q)(1 - e^{-dU})s^*}{q} \int_0^U \pi(v, s^*) dv > 0. \tag{12}$$

Because

$$\begin{aligned} r \int_0^U \pi(v, s^*) dv &= \int_0^U \exp \left(\frac{rs^*}{dK} e^{-dv} \right) de^{rv} \\ &= \pi(T, s^*) - \pi(0, s^*) + \frac{rs^*}{K} \int_0^U e^{-dv} \pi(v, s^*) dv \end{aligned}$$

we have

$$\frac{rs^*}{K} \int_0^U e^{-dv} \pi(v, s^*) dv = r \int_0^U \pi(v, s^*) dv - \pi(U, s^*) + \pi(0, s^*).$$

Thus the derivative of φ with respect to s at s^* is

$$\varphi'(s^*) = \frac{r[1 - (1 - q)e^{-dU}]}{qs^*} [(1 - q)\pi(U, s^*) - \pi(0, s^*)](s^* - s_0).$$

Therefore, when $s^* > s_0$, $\varphi'(s^*) > 0$ holds; when $s^* < s_0$, $\varphi'(s^*) < 0$ holds; when $s^* = s_0$, $\varphi'(s^*) = 0$ holds.

If there exists s^* that is larger than s_0 , and s_1 is the maximum one of all s^* , then $\varphi'(s_1) > 0$. Thus there is a point \bar{s}_1 in a certain right neighborhood of s_1 such that $\bar{s}_1 > s_1$ and $\varphi(\bar{s}_1) > 0$. From the result of $\lim_{s \rightarrow +\infty} \varphi(s) = -\infty$, one knows that $\varphi(s) = 0$ has a solution which is

larger than \bar{s}_1 . This is contrary to s_1 being maximum. Accordingly, $\varphi(s)$ does not have a zero point that is larger than s_0 .

If there are at least two zero points of $\varphi(s)$ that are less than s_0 then one can select two points among all these zero points and denote them by s_2, s_3 . So $\varphi'(s_2) < 0, \varphi'(s_3) < 0$. Without loss of generality, suppose $s_2 < s_3$. Thus there is a point \bar{s}_2 in a certain right neighborhood of s_2 and a point \bar{s}_3 in a certain left neighborhood of s_3 such that $\bar{s}_2 < \bar{s}_3, \varphi(\bar{s}_2) < 0$ and $\varphi(\bar{s}_3) > 0$. So there exists a zero point of $\varphi(s)$ between \bar{s}_2 and \bar{s}_3 , thereby between s_2 and s_3 . This result shows that there is at least one zero point of $\varphi(s)$ between any two zero points of it. So, the zero point of $\varphi(s)$ between s_2 and s_3 is dense. Then $\varphi'(s_2)$ (or $\varphi'(s_3)$) is 0 or does not exist, this is contrary to the fact that $\varphi'(s_2) < 0$ (or $\varphi'(s_3) < 0$). Therefore $\varphi(s)$ has no more than one zero point that is less than s_0 .

If $s^* = s_0$, then $\varphi(s_0) = \varphi'(s_0) = 0$. Due to

$$\begin{aligned} (r-d) \int_0^U \pi(v, s^*) e^{-dv} dv &= \int_0^U \exp\left(\frac{rs^*}{dK} e^{-dv}\right) d e^{(r-d)v} \\ &= \pi(U, s^*) e^{-dU} - \pi(0, s^*) + \frac{rs^*}{K} \int_0^U e^{-2dv} \pi(v, s^*) dv, \end{aligned}$$

we have $\frac{rs^*}{K} \int_0^U e^{-2dv} \pi(v, s^*) dv = (r-d) \int_0^U \pi(v, s^*) e^{-dv} dv - \pi(U, s^*) e^{-dU} + \pi(0, s^*)$, and then

$$\varphi''(s_0) = \frac{r^2 [1 - (1-q)e^{-dU}]^2}{dKq^2(r+d)} [(1-q)\pi(U, s_0) - \pi(0, s_0)] > 0.$$

Hence, $\varphi(s)$ takes a local minimum value at point s_0 . Then there is a point \bar{s}_0 in a certain right neighborhood of s_0 such that $\bar{s}_0 > s_0, \varphi(\bar{s}_0) > 0$. This result, together with $\lim_{s \rightarrow +\infty} \varphi(s) = -\infty$, implies that $\varphi(s)$ has zero point that is larger than \bar{s}_0 , thereby larger than s_0 . This is contrary to the fact that $\varphi(s)$ does not have zero point that is larger than s_0 . Consequently, s_0 is not zero point of $\varphi(s)$.

Summarily, $\varphi(s)$ has an unique positive zero point s^* such that $s^* < s_0$. The proof is complete. □

Theorem 5 *If $q \geq 1 - e^{-rU}$, the equilibrium $(0, 0)$ of (10) is globally asymptotically stable and model (9) has a globally asymptotically stable zero solution.*

Proof It is easy to see that $f(nU^+) > 0, s(nU^+) > 0$ for $n \in N = \{0, 1, 2, 3, \dots\}$ when $f(0^+) > 0, s(0^+) > 0$. The condition $(1 - q)e^{rU} \leq 1$ and the facts $\exp(\frac{rs(nU^+)}{dK} e^{-dU}) < \exp(\frac{rs(nU^+)}{dK})$, $rf(nU^+) \int_0^U \pi(v, S(nU^+)) dv > 0$ imply $0 < (1 - q)K\pi(U, S(nU^+)) < K\pi(0, S(nU^+)) + rf(nU^+) \int_0^U \pi(v, S(nU^+)) dv$. This indicates that $f((n+1)U^+) < f(nU^+)$, that is, the sequence $\{f(nU^+)\}$ is descending. Since 0 is the lower bound of $\{f(nU^+)\}$, sequence $\{f(nU^+)\}$ converges and its limit is larger than or equals 0.

If there exists k such that $s((k+1)U^+) < s(kU^+)$, then $s((n+1)U^+) = s(nU^+)e^{-dU} + \frac{qf((n+1)U^+)}{1-q}$ and $\{f(nU^+)\}$ being descending imply $s((n+1)U^+) < s(nU^+)$ for all $n \geq k$, that is, the sequence $\{s(nU^+)\}$ is descending for $n \geq k$. Since 0 is the lower bound of $\{s(nU^+)\}$, the sequence $\{s(nU^+)\}$ converges.

If there is no k such that $s((k+1)U^+) < s(kU^+)$, namely, the sequence $\{s(nU^+)\}$ is ascending, then from the relation $s(nU^+) \leq s((n+1)U^+) = s(nU^+)e^{-dU} + \frac{qf((n+1)U^+)}{1-q}$, one can

conclude $s(nU^+) \leq \frac{qf((n+1)U^+)}{(1-q)(1-e^{-dU})}$, that is, $\{s(nU^+)\}$ is bounded. Thus, the sequence $\{s(nU^+)\}$ converges.

The sequence $\{s(nU^+)\}$ always converges.

Let $\lim_{n \rightarrow +\infty} f(nU^+) = A$ and $\lim_{n \rightarrow +\infty} s(nU^+) = B$. Taking limits at both sides of the two equations in (10) gives

$$\begin{cases} A = \frac{(1-q)KA\pi(U,B)}{K\pi(0,B)+rA \int_0^U \pi(v,B) dv}, \\ B = Be^{-dU} + \frac{qKA\pi(U,B)}{K\pi(0,B)+rA \int_0^U \pi(v,B) dv}. \end{cases} \tag{13}$$

Form the first formula in (13), one obtains $A = 0$ or $A = \frac{(1-q)KA\pi(U,B)-K\pi(0,B)}{r \int_0^U \pi(v,B) dv}$. Because the latter result is less than 0 and $A \geq 0$, we have $A = 0$. Afterwards, it is easy to see that $B = 0$ from the second formula in (13).

Therefore, the equilibrium $(0, 0)$ of (10) is globally asymptotically stable. □

Theorem 6 *Let*

$$\begin{aligned} \Lambda = & \frac{1}{(1-q)\pi(U,s^*)} \left\{ (1 + e^{-dU})[(1-q)\pi(U,s^*) + \pi(0,s^*)] \right. \\ & + \frac{r}{dK(1-q)} \{ [1 - (1-q)e^{-dU}]f^* - K(1-q)(1 - e^{-dU}) \} \\ & \left. \times [(1-q)\pi(U,s^*) - \pi(0,s^*)] \right\}. \end{aligned}$$

If $q < 1 - e^{-rU}$ and $\Lambda > 0$, the positive equilibrium (f^*, s^*) of (10) is locally asymptotically stable.

Proof The linearized matrix of (10) at (f^*, s^*) is $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$, here $J_{11} = \frac{\pi(0,s^*)}{(1-q)\pi(T,s^*)}$, $J_{12} = \frac{r}{dKq(1-q)} \{ [1 - (1-q)e^{-dU}]f^* - (1-q)(1 - e^{-dU})K \} [(1-q)\pi(U,s^*) - \pi(0,s^*)]$, $J_{21} = \frac{qJ_{11}}{(1-q)}$, $J_{22} = e^{-dU} + \frac{qJ_{12}}{(1-q)}$.

The characteristic equation of J is $\lambda^2 + a_1\lambda + a_2 = 0$, here $a_1 = -J_{11} - J_{22}$, $a_2 = |J| = e^{-dU} \frac{\pi(0,s^*)}{(1-q)\pi(U,s^*)}$. Equation (12) tells us $(1-q)\pi(T,s^*) > \pi(0,s^*) > 0$, so $0 < a_2 < 1$. Since

$$1 + a_1 + a_2 = \frac{1 - e^{-dU}}{d(1-q)\pi(U,s^*)} [(1-q)\pi(U,s^*) - \pi(0,s^*)] \left[r + d - \frac{r[1 - (1-q)e^{-dU}]s^*}{qK} \right]$$

and $s^* < s_0$, we have $1 + a_1 + a_2 > 0$. Calculation shows $1 - a_1 + a_2 = \Lambda > 0$. Therefore, on the basis of the Jury criterion, the positive equilibrium (f^*, s^*) of (10) is locally asymptotically stable. □

4.2 Seasonally breeding model with contraception control

Next, consider impulse contraception control based on the seasonally breeding model (1). For convenience, suppose that contraception control is implemented at nLT ($n = 0, 1, 2, \dots$) with contraception rate q , that is, put the control in practice once every L years and at the

beginning of some breeding seasons. Thus, one can formulate model (14),

$$\begin{cases} f' = r_1 f \left(1 - \frac{f+s}{K}\right), & t \in [nT, nT + T_1], t \neq kLT, \\ s' = -d_1 s, & t \in [nT, nT + T_1], t \neq kLT, \\ f' = -r_2 f, & t \in [nT + T_1, (n+1)T], t \neq kLT, \quad n = 0, 1, 2, \dots, \\ s' = -r_2 s, & t \in [nT + T_1, (n+1)T], t \neq kLT, \quad k = 0, 1, 2, \dots \\ f(kLT^+) = (1 - q)f(kLT), & t = kLT, \\ s(kLT^+) = s(kLT) + qf(kLT), & t = kLT, \end{cases} \quad (14)$$

Solving model (14) and denoting $f(nLT^+)$, $s(nLT^+)$ as F_n , S_n , respectively, gives

$$\begin{cases} F_{n+1} = \frac{(1-q)KF_n g_1(S_n)}{K+r_1 F_n g_2(S_n)}, \\ S_{n+1} = S_n e^{-L(d_1 T_1+r_2 T_2)} + \frac{qKF_n g_1(S_n)}{K+r_1 F_n g_2(S_n)}, \end{cases} \quad (15)$$

where

$$\begin{aligned} g_1(S_n) &= \exp\left\{L(r_1 T_1 - r_2 T_2) + \frac{r_1 S_n [1 - e^{-L(d_1 T_1+r_2 T_2)}]}{d_1 K (1 - e^{-d_1 T_1-r_2 T_2})} (e^{-d_1 T_1} - 1)\right\}, \\ g_2(S_n) &= \int_0^{T_1} \exp\left[r_1 v + \frac{r_1 S_n}{d_1 K} (e^{-d_1 v} - 1)\right] dv + \exp\left[r_1 T_1 - r_2 T_2 + \frac{r_1 S_n}{d_1 K} (e^{-d_1 T_1} - 1)\right] \\ &\quad \times \int_0^{T_1} \exp\left[r_1 v + \frac{r_1 S_n}{d_1 K} (e^{-d_1 v} - 1) e^{-d_1 T_1-r_2 T_2}\right] dv \\ &\quad + \exp\left[2r_1 T_1 - 2r_2 T_2 + \frac{r_1 S_n}{d_1 K} (e^{-d_1 T_1} - 1) (1 + e^{-d_1 T_1-r_2 T_2})\right] \\ &\quad \times \int_0^{T_1} \exp\left[r_1 v + \frac{r_1 S_n}{d_1 K} (e^{-d_1 v} - 1) e^{-2(d_1 T_1+r_2 T_2)}\right] dv \\ &\quad + \dots \\ &\quad + \exp\left[(L-1)(r_1 T_1 - 2r_2 T_2) \right. \\ &\quad \left. + \frac{r_1 S_n}{d_1 K} (e^{-d_1 T_1} - 1) (1 + e^{-d_1 T_1-r_2 T_2} + \dots + e^{-(L-2)(d_1 T_1+r_2 T_2)})\right] \\ &\quad \times \int_0^{T_1} \exp\left[r_1 v + \frac{r_1 S_n}{d_1 K} (e^{-d_1 v} - 1) e^{-(L-1)(d_1 T_1+r_2 T_2)}\right] dv. \end{aligned}$$

Theorem 7 $(0, 0)$ is always an equilibrium of (15), if $q < 1 - e^{-L(r_1 T_1-r_2 T_2)}$, then (15) has another positive equilibrium.

Proof Obviously, $(0, 0)$ is equilibrium of (15) and other equilibrium (F, S) of (15) satisfies (16)

$$\begin{cases} 1 = \frac{(1-q)KF(S)}{K+r_1 Fg_2(S)}, \\ S = S e^{-L(d_1 T_1+r_2 T_2)} + \frac{qF}{1-q}. \end{cases} \quad (16)$$

From the first equation in (16), one sees that $F = \frac{(1-q)Kg_1(S)-K}{r_1g_2(S)}$. Substituting this new relation into the second equation in (16) gives

$$0 = \phi(S) \triangleq (1-q)Kg_1(S) - K - \frac{(1-q)r_1KSg_1(S)}{q} [1 - e^{-L(d_1T_1+r_2T_2)}]. \tag{17}$$

Note that $\phi(0) = K[(1-q)e^{L(r_1T_1-r_2T_2)} - 1]$, so if $q < 1 - e^{-L(r_1T_1-r_2T_2)}$, then $\phi(0) > 0$ and if $q \geq 1 - e^{-L(r_1T_1-r_2T_2)}$, then $\phi(0) \leq 0$.

When $q \geq 1 - e^{-L(r_1T_1-r_2T_2)}$, $(1-q)e^{L(r_1T_1-r_2T_2)} \leq 1$ holds. Hence the first term of $\phi(S)$ is less than K and then $\phi(S) < 0$ for arbitrary $S \geq 0$. This means that $\phi(S) = 0$ does not have a positive solution.

When $q < 1 - e^{-L(r_1T_1-r_2T_2)}$, the first term of $\phi(S)$ tends to 0 as S approaching to $+\infty$, the second term is a negative constant, the third term is less than 0. So when S is large enough, $\phi(S) < 0$ holds. It implies that $\phi(S) = 0$ has a positive solution. □

Theorem 8 *If $q \geq 1 - e^{-L(r_1T_1-r_2T_2)}$, the equilibrium $(0, 0)$ of (15) is globally asymptotically stable and model (14) has a globally asymptotically stable zero solution.*

Proof Obviously, when $F_0 > 0, S_0 > 0, F_n > 0, S_n > 0$ hold for $n \in N = \{0, 1, 2, 3, \dots\}$. When the stated conditions hold, $0 < (1-q)e^{L(r_1T_1-r_2T_2)} \leq 1, \frac{r_1S_n[1-e^{-L(d_1T_1+r_2T_2)}]}{d_1K(1-e^{-L(r_1T_1-r_2T_2)})} (e^{-d_1T_1} - 1) < 0$, all these result in $0 < (1-q)Kg_1(S_n) < K$. In addition, $r_1F_n g_2(S_n) > 0$, so $0 < (1-q)KF_n g_1(S_n) < K + r_1F_n g_2(S_n)$. Thus $F_{n+1} < F_n$, that is, $\{F_n\}$ is descending. Since 0 is the lower bound of $\{F_n\}$, the sequence $\{F_n\}$ converges and its limit is larger than or equals 0.

If there exists k such that $S_{k+1} < S_k$, then the facts that $S_{n+1} = S_n e^{-L(d_1T_1+r_2T_2)} + \frac{qF_{n+1}}{1-q}$ and $\{F_n\}$ is descending imply $S_{n+1} < S_n$ for all $n \geq k$, that is, the sequence $\{S_n\}$ is descending for $n \geq k$. Since 0 is the lower bound of $\{S_n\}$, the sequence $\{S_n\}$ converges.

If there is no k such that $S_{k+1} < S_k$, namely, the sequence $\{S_n\}$ is ascending, then, from the relation $S_n \leq S_{n+1} = S_n e^{-L(d_1T_1+r_2T_2)} + \frac{qF_{n+1}}{1-q}$, one can conclude $S_n \leq \frac{qF_{n+1}}{(1-q)[1-e^{-L(d_1T_1+r_2T_2)}]}$, that is, $\{S_n\}$ is bounded. Thus, the sequence $\{S_n\}$ converges.

Thus, the sequence $\{S_n\}$ always converges.

Assume that $\lim_{n \rightarrow +\infty} F_n = A$ and $\lim_{n \rightarrow +\infty} S_n = B$. Taking limits at both sides of the two equations in (15) gives

$$\begin{cases} A = \frac{(1-q)KA g_1(B)}{K+r_1A g_2(B)}, \\ B = B e^{-L(d_1T_1+r_2T_2)} + \frac{qKA g_1(B)}{K+r_1A g_2(B)}. \end{cases} \tag{18}$$

From the first formula in (18), one obtains $A = 0$ or $A = \frac{(1-q)Kg_1(B)-K}{r_1g_2(B)}$. The latter result is less than 0 and $A \geq 0$, so $A = 0$. Afterwards, it is easy to see that $B = 0$ from the second formula in (18).

Therefore, the equilibrium $(0, 0)$ of (15) is globally asymptotically stable. □

Theorem 9 *Denote*

$$\begin{aligned} \Lambda_1 = & 1 - \frac{1}{(1-q)g_1(S^*)} - e^{-L(d_1T_1+r_2T_2)} \\ & - \frac{(1-q)qKF^*g_1'(S^*) - qr_1(F^*)^2g_2'(S^*)}{(1-q)^2Kg_1(S^*)} + \frac{e^{-L(d_1T_1+r_2T_2)}}{(1-q)g_1(S^*)}, \end{aligned}$$

$$\Lambda_2 = 1 + \frac{1}{(1-q)g_1(S^*)} + e^{-L(d_1T_1+r_2T_2)} + \frac{(1-q)qKF^*g'_1(S^*) - qr_1(F^*)^2g'_2(S^*)}{(1-q)^2Kg_1(S^*)} + \frac{e^{-L(d_1T_1+r_2T_2)}}{(1-q)g_1(S^*)}.$$

When $\Lambda_1 > 0$, $\Lambda_2 > 0$, $q < 1 - e^{-L(r_1T_1-r_2T_2)}$ hold, the positive equilibrium of (15) is locally asymptotically stable.

Proof The linearized matrix of the map (15) at positive equilibrium (F^*, S^*) is $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ with $M_{11} = \frac{1}{(1-q)g_1(S^*)}$, $M_{12} = \frac{(1-q)KF^*g'_1(S^*) - r_1(F^*)^2g'_2(S^*)}{(1-q)Kg_1(S^*)}$, $M_{21} = \frac{q}{1-q}M_{11}$, $M_{22} = e^{-L(d_1T_1+r_2T_2)} + \frac{q}{1-q}M_{12}$.

The characteristic equation of M is $\lambda^2 + b_1\lambda + b_2 = 0$, here $b_1 = -M_{11} - M_{22}$, $b_2 = |M| = \frac{e^{-L(d_1T_1+r_2T_2)}}{(1-q)g_1(S^*)}$. Equation (17) tells us $(1-q)g_1(S^*) > 1$, so $0 < b_2 < 1$. Simple calculation shows $1 - b_1 + b_2 = \Lambda_1 > 0$, $1 + b_1 + b_2 = \Lambda_2 > 0$. Therefore, on the basis of the Jury criterion, the positive equilibrium of (15) is locally asymptotically stable. \square

5 Numerical simulation

The foregoing analysis shows that the condition of the rodent population dying out is approximately the same for lethal control and contraception control. For the logistic model with lethal control (5) and the logistic model with contraception control (10), when the control rate (p or q) is larger than $1 - e^{-rU}$, the rodent dies out, and when the control rate (p or q) is smaller than $1 - e^{-rU}$, the rodent persists. For the seasonally breeding model with lethal control (7) and the seasonally breeding model with contraception control (14), when the control rate (p or q) is larger than $1 - e^{L(r_2T_2-r_1T_1)}$, the rodent dies out, and when the control rate (p or q) is smaller than $1 - e^{L(r_2T_2-r_1T_1)}$, the rodent persists. However, these two controls are different, the process of the rodent population tending to a stable solution obviously is different (Figures 1 and 2).

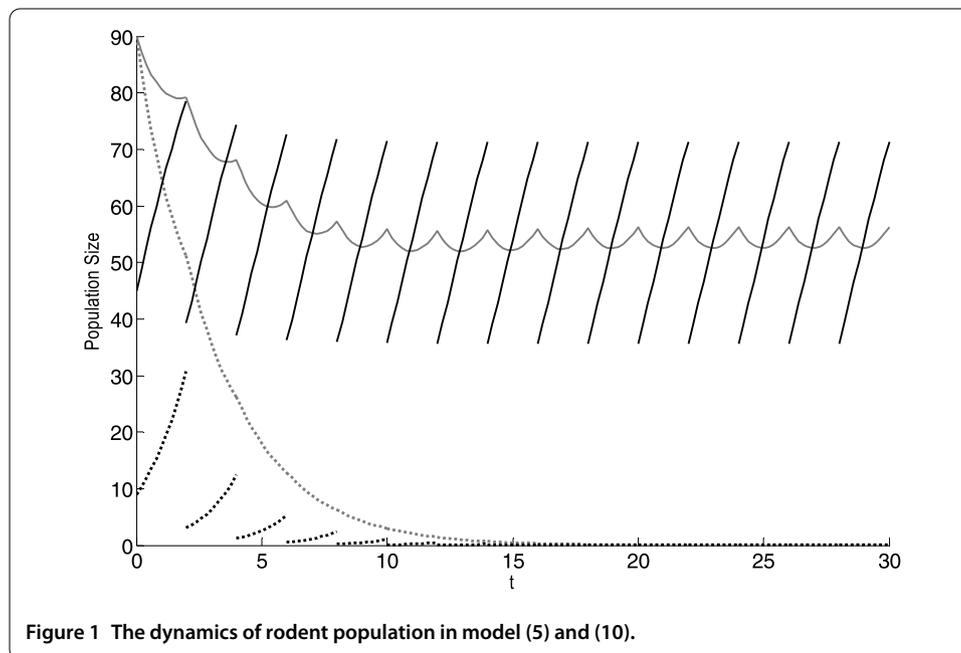


Figure 1 The dynamics of rodent population in model (5) and (10).

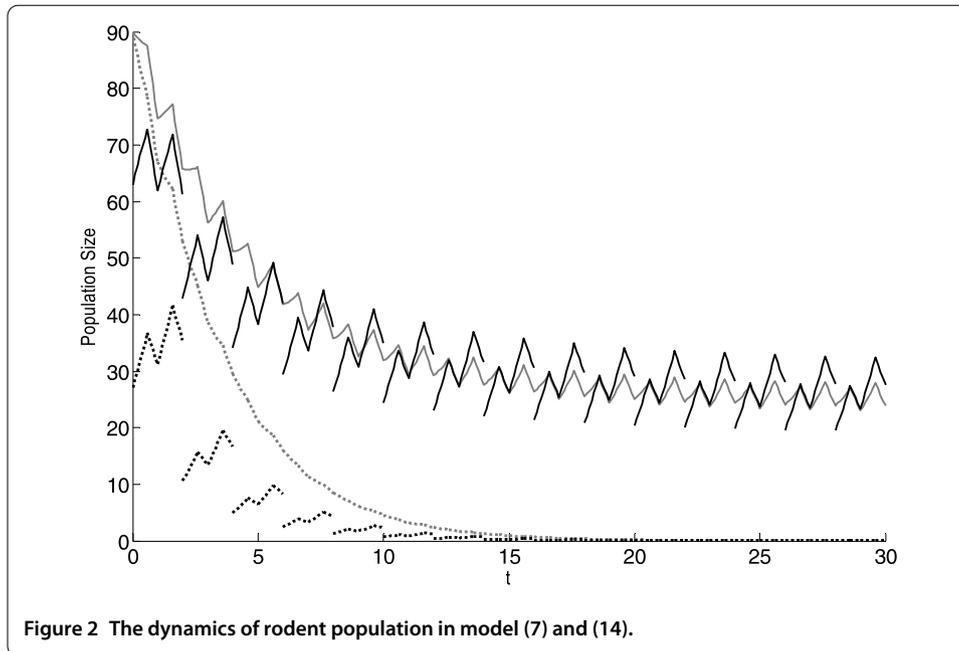


Figure 1 shows the dynamics of the rodent population in model (5) and (10), where $r = 0.75$, $K = 100$, $d = 0.4$, $U = 2$. For the solid line, the control rate (p or q) is 0.5, whereas, for the dotted line, it is 0.9. Figure 2 shows the dynamics of rodent population in model (7) and (14), where $r_1 = 0.75$, $K = 100$, $r_2 = 0.4$, $T_1 = 0.6$, $T_2 = 0.4$. For the solid line, the control rate (p or q) is 0.3, whereas, for the dotted line, it is 0.7. In both figures, the black line and gray line refer to the dynamics of the rodent population with lethal control and contraception control, respectively, and the initial value is $x(0) = 90$ or $f(0) = 90$, $s(0) = 0$.

Under lethal control, the rodent population approaches a stable periodic solution more quickly. So, if it was urgent to wipe off the rodent, the lethal control would be more effective. Under contraception control, the rodent population develops slowly. The variation of the rodent population is one kind of disturbance to the ecosystem, if this disturbance went beyond the tolerance of an ecosystem, the ecosystem would be disorganized, even it might collapse. Thus, for fragile ecosystems, contraception control is appropriate.

6 Discussion

In this paper, dynamics models are formulated based on the logistic model and the seasonally breeding model to investigate the effect of impulse lethal and contraception control on rodent population dynamics. The existence and stability of the periodic solution are analyzed. The condition of the rodent population dying out is the same for these two controls and each control has its own traits.

If the rodent control can wipe it off thoroughly then there would never exist damage in the future. However, this situation is awful. Among all biotic and abiotic components of an ecosystem, there are complicated relations. If the rodent dies out, the dynamics of the other components would vary correspondingly. When these variations are large enough, the ecosystem would be disorganized, even collapse and result in greater damage. Thus, the aim of rodent control is guiding rodent population to a suitable size, not exterminating it. For example, the plateau pika brings great damage to degraded alpine meadow, but it is a keystone species also [16, 17]. The extinction of plateau pika may reduce the loss of pasture

grasses that plateau pika feeds on, but, most importantly, this may induce extinction of some raptors that prey on plateau pika, and it may induce extinction of some birds that inhabit holes of plateau pika.

For an epidemic model, there is an important parameter, namely, the basic reproductive number R_0 . Usually, the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ and the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. For population dynamics model, frequently, when some conditions hold there is a stable positive equilibrium, otherwise there is a stable zero equilibrium. From these conditions, one can dig out a parameter that is similar to R_0 . Denote this parameter by R_0 . In this paper, R_0 is the maximum finite rate of increase during a specific time interval. Consider a population of which the size satisfies the Malthus model $x' = rx$, and suppose its size is x_0 at time t_0 . Then its solution is $x(t) = x_0 e^{r(t-t_0)}$, and after a time interval T , that is, at time $t_0 + T$, the population size is $x(t_0 + T) = x_0 e^{rT}$. So during this time interval of length T , the multiplication factor of the population size is e^{rT} , and this is the finite rate of increase during this time interval.

In the seasonally breeding model (1), the condition of a positive equilibrium existing and being stable is $r_2 T_2 < r_1 T_1$, namely, $e^{r_1 T_1} e^{-r_2 T_2} > 1$. Model (1) is a periodic system with 1-year period. For the breeding season, its length is T_1 and the maximum increasing rate is r_1 , so the maximum finite rate of increase is $e^{r_1 T_1}$. For the non-breeding season, its length is T_2 and the maximum increasing rate is $-r_2$, so the maximum finite rate of increase is $e^{-r_2 T_2}$. Therefore, during one period, the maximum finite rate of increase is $e^{r_1 T_1} e^{-r_2 T_2}$, and this is the quantity R_0 . In model (9), which is derived from the logistic model and in which the rodent are subject to impulse contraception control, the condition of positive equilibrium existing and being stable is $q < 1 - e^{-rU}$, namely, $(1 - q)e^{rU} > 1$. Consider one control period of length U and assume that control is implemented at the beginning of this period, so the ratio of fertile individuals to all individuals is $1 - q$. During one time interval of length U , the maximum finite rate of increase is e^{rU} , and then during one control period, the maximum finite rate of increase is $(1 - q)e^{rU}$, and this is R_0 . For other models, one can obtain the maximum finite rate of increase in a similar way and see that it is just R_0 .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HL designed and performed all the steps of the proofs in this research. FZ and QL participated in the design of the study and suggested many good ideas that made this paper possible. FZ performed the simulation. QL wrote the paper. All authors read and approved the final manuscript.

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