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Aspects of univalent holomorphic functions involving multiplier transformation and Ruscheweyh derivative

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Abstract

Making use multiplier transformation and Ruscheweyh derivative, we introduce a new class of analytic functions $\mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity and neighborhood property for functions belonging to the class $\mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$.

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1 Introduction

Let \mathcal{A} denote the class of functions of the form $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, which are analytic and univalent in the open unit disc $U = \{z : z \in \mathbb{C} : |z| < 1\}$. \mathcal{T} is a subclass of \mathcal{A} consisting of the functions of the form $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$. For functions $f, g \in \mathcal{A}$ given by $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, we define the Hadamard product (or convolution) of f and g by $(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$, $z \in U$.

Definition 1.1 (Ruscheweyh [1]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= z f'(z), \quad \dots, \\ (n+1)R^{n+1} f(z) &= z(R^n f(z))' + nR^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.1 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.
If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z - \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.2 [2, 3] For $f \in \mathcal{A}$, $n \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(n, \lambda, l)f(z)$ is defined by the following infinite series:

$$I(n, \lambda, l)f(z) := z + \sum_{j=2}^{\infty} \left(\frac{\lambda(j-1) + l + 1}{l + 1} \right)^n a_j z^j.$$

Remark 1.2 [4] It follows from the above definition that

$$I(0, \lambda, l)f(z) = f(z),$$

$$(l + 1)I(n + 1, \lambda, l)f(z) = (l + 1 - \lambda)I(n, \lambda, l)f(z) + \lambda z(I(n, \lambda, l)f(z))', \quad z \in U.$$

Remark 1.3 The operator $I(n, \lambda, 0) = D_\lambda^n$ is the generalized Sălăgean operator introduced by Al-Oboudi [5], and $I(n, 1, 0) = S^n$ is the Sălăgean differential operator [6].

Definition 1.3 [7, 8] Let $\gamma, \lambda, l \geq 0, n \in \mathbb{N}$. Denote by $RI_{n,\lambda,l}^\gamma$ the operator given by $RI_{n,\lambda,l}^\gamma : \mathcal{A} \rightarrow \mathcal{A}$,

$$RI_{n,\lambda,l}^\gamma f(z) = (1 - \gamma)R^n f(z) + \gamma I(n, \lambda, l)f(z), \quad z \in U.$$

Remark 1.4 If $f \in \mathcal{A}, f(z) = z + \sum_{j=2}^\infty a_j z^j$, then

$$RI_{n,\lambda,l}^\gamma f(z) = z + \sum_{j=2}^\infty \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j, \quad z \in U.$$

If $f \in \mathcal{T}, f(z) = z - \sum_{j=2}^\infty a_j z^j$, then

$$RI_{n,\lambda,l}^\gamma f(z) = z - \sum_{j=2}^\infty \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j, \quad z \in U.$$

Remark 1.5 The operator $RI_{n,\lambda,0}^\gamma f(z) = RD_{\lambda,\gamma}^n f(z)$ which was introduced in [9] and the operator $RI_{n,1,0}^\gamma f(z) = L_\gamma^n f(z)$ which was introduced in [10].

Following the work of Najafzadeh and Pezeshki [11] we can define the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ as follows.

Definition 1.4 For $\gamma, \lambda, l \geq 0, 0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ be the subclass of \mathcal{T} consisting of functions that satisfying the inequality

$$\left| \frac{RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1}{2v(RI_{n,\lambda,l}^{\mu,\gamma} f(z) - \alpha) - (RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1)} \right| < \beta, \tag{1.1}$$

where

$$RI_{n,\lambda,l}^{\mu,\gamma} f(z) = (1 - \mu) \frac{RI_{n,\lambda,l}^\gamma f(z)}{z} + \mu (RI_{n,\lambda,l}^\gamma f(z))', \tag{1.2}$$

$0 < v \leq 1$.

Remark 1.6 If $f \in \mathcal{T}, f(z) = z - \sum_{j=2}^\infty a_j z^j$, then

$$RI_{n,\lambda,l}^{\mu,\gamma} f(z) = 1 - \sum_{j=t+1}^\infty [1 + \mu(j-1)]$$

$$\times \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}, \quad z \in U.$$

Remark 1.7 The class $\mathcal{RI}(\gamma, \lambda, 0, \alpha, \beta) = \mathcal{RD}(\gamma, \lambda, \alpha, \beta)$ defined and studied in [12] and $\mathcal{RI}(\gamma, 1, 0, \alpha, \beta) = \mathcal{L}(\gamma, \alpha, \beta)$ defined and studied in [13].

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions in $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Theorem 2.1 *Let the function $f \in \mathcal{T}$. Then $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ if and only if*

$$\sum_{j=t+1}^{\infty} (1 + \mu(j-1)) [1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j < 2\beta\nu(1-\alpha). \tag{2.1}$$

The result is sharp for the function $F(z)$ defined by

$$F(z) = z - \frac{2\beta\nu(1-\alpha)}{(1 + \mu(j-1)) [1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j, \quad j \geq t+1.$$

Proof Suppose f satisfies (2.1). Then for $|z| < 1$, we have

$$\begin{aligned} & \left| RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1 \right| - \beta \left| 2\nu (RI_{n,\lambda,l}^{\mu,\gamma} f(z) - \alpha) - (RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1) \right| \\ &= \left| - \sum_{j=t+1}^{\infty} (1 + \mu(j-1)) \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1} \right| \\ & \quad - \beta \left| 2\nu(1-\alpha) - (2\nu-1) \sum_{j=t+1}^{\infty} \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} \right. \\ & \quad \left. \times [1 + \mu(j-1)] a_j z^{j-1} \right| \\ &\leq \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_k - 2\beta\nu(1-\alpha) \\ & \quad + \sum_{j=t+1}^{\infty} \beta(2\nu-1) (1 + \mu(j-1)) \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \\ &= \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] [1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \\ & \quad - 2\beta\nu(1-\alpha) < 0. \end{aligned}$$

Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$. Conversely, assume that

$$\begin{aligned} & \left| \frac{RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1}{2\nu (RI_{n,\lambda,l}^{\mu,\gamma} f(z) - \alpha) - (RI_{n,\lambda,l}^{\mu,\gamma} f(z) - 1)} \right| \\ &= \left| \frac{- \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}}{2\nu(1-\alpha) - \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] (2\nu-1) \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}} \right| \\ &< \beta, \quad z \in U. \end{aligned}$$

Since $\operatorname{Re}(z) \leq |z|$ for all $z \in U$, we have

$$\operatorname{Re} \left\{ \frac{\sum_{j=t+1}^{\infty} [1 + \mu(j-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}}{2\nu(1-\alpha) - \sum_{j=t+1}^{\infty} [1 + \mu(j-1)] (2\nu-1) \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^{j-1}} \right\} < \beta. \tag{2.2}$$

By choosing choose values of z on the real axis so that $RI_{n,\lambda,l}^{\mu,\gamma} f(z)$ is real and letting $z \rightarrow 1$ through real values, we obtain the desired inequality (2.1). \square

Corollary 2.2 *If $f \in \mathcal{T}$ is in $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, then*

$$a_j \leq \frac{2\beta\nu(1-\alpha)}{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}, \quad j \geq t+1, \tag{2.3}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z) = z$ and*

$$f_j(z) = z - \frac{2\beta\nu(1-\alpha)}{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j, \quad j \geq t+1, \tag{2.4}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma, \lambda, l \geq 0$ and $0 < \nu \leq 1$. Then $f(z)$ is in the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=t}^{\infty} \omega_j f_j(z), \tag{2.5}$$

where $\omega_j \geq 0$ and $\sum_{j=1}^{\infty} \omega_j = 1$.

Proof Suppose $f(z)$ can be written as in (2.5). Then

$$f(z) = z - \sum_{j=t+1}^{\infty} \omega_j \frac{2\beta\nu(1-\alpha)}{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j.$$

Now,

$$\begin{aligned} & \sum_{j=t+1}^{\infty} \frac{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{2\beta\nu(1-\alpha)} \omega_j \\ & \times \frac{2\beta\nu(1-\alpha)}{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} = \sum_{j=t+1}^{\infty} \omega_j = 1 - \omega_1 \leq 1. \end{aligned}$$

Thus $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Conversely, let $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$. Then by using (2.3), setting

$$\omega_j = \frac{[1 + \mu(j-1)][1 + \beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{2\beta\nu(1-\alpha)} a_j, \quad j \geq t+1,$$

and $\omega_1 = 1 - \sum_{j=2}^{\infty} \omega_j$, we have $f(z) = \sum_{j=t}^{\infty} \omega_j f_j(z)$. This completes the proof of Theorem 2.3. \square

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Theorem 3.1 *If $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, then*

$$\begin{aligned} r - \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \\ \leq |f(z)| \leq r + \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \end{aligned} \quad (3.1)$$

holds if the sequence $\{\sigma_j(\gamma, \lambda, l, \beta, v)\}_{j=t+1}^{\infty}$ is non-decreasing, and

$$\begin{aligned} 1 - \frac{2\beta v(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^t \\ \leq |f'(z)| \leq 1 + \frac{2\beta v(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^t \end{aligned} \quad (3.2)$$

holds if the sequence $\{\frac{\sigma_j(\gamma, \lambda, l, \beta, v)}{j}\}_{j=t+1}^{\infty}$ is non-decreasing, where

$$\sigma_j(\gamma, \beta, v) = [1 + \mu(j-1)][1 + \beta(2v-1)] \left\{ \gamma \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}.$$

The bounds in (3.1) and (3.2) are sharp, for $f(z)$ given by

$$f(z) = z - \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} z^{t+1}, \quad z = \pm r. \quad (3.3)$$

Proof In view of Theorem 2.1, we have

$$\sum_{j=t+1}^{\infty} a_j \leq \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}. \quad (3.4)$$

We obtain

$$|z| - |z|^{t+1} \sum_{j=t+1}^{\infty} a_j \leq |f(z)| \leq |z| + |z|^{t+1} \sum_{j=t+1}^{\infty} a_j.$$

Thus

$$\begin{aligned} r - \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \\ \leq |f(z)| \leq r + \frac{2\beta v(1-\alpha)}{(1+\mu t)[1+\beta(2v-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1}. \end{aligned} \quad (3.5)$$

Hence (3.1) follows from (3.5). Further,

$$\sum_{j=t+1}^{\infty} ja_j \leq \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!}\}}.$$

Hence (3.2) follows from

$$1 - r^t \sum_{j=t+1}^{\infty} ja_j \leq |f'(z)| \leq 1 + r^t \sum_{j=t+1}^{\infty} ja_j. \quad \square$$

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness, and convexity for the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ are given in this section.

Theorem 4.1 *Let the function $f \in \mathcal{T}$ belong to the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, Then $f(z)$ is close-to-convex of order δ , $0 \leq \delta < 1$, in the disc $|z| < r$, where*

$$r := \inf_{j \geq t+1} \left[\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(1-\alpha)} \right]^{\frac{1}{t}}. \quad (4.1)$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

Proof For given $f \in \mathcal{T}$ we must show that

$$|f'(z) - 1| < 1 - \delta. \quad (4.2)$$

By a simple calculation we have

$$|f'(z) - 1| \leq \sum_{j=t+1}^{\infty} ja_j |z|^t.$$

The last expression is less than $1 - \delta$ if

$$\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j |z|^t < 1.$$

We use the fact that $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)} a_j \leq 1.$$

Equation (4.2) holds true if

$$\frac{j}{1-\delta} |z|^t \leq \sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t \leq \sum_{j=t+1}^{\infty} \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(1-\alpha)},$$

which completes the proof. □

Theorem 4.2 *Let $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$. Then*

1. *f is starlike of order δ , $0 \leq \delta < 1$, in the disc $|z| < r_1$, where*

$$r_1 = \inf_{j \geq t+1} \left\{ \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)(j-\delta)} \right\}^{\frac{1}{t}}.$$

2. *f is convex of order δ , $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,*

$$r_2 = \inf_{j \geq t+1} \left\{ \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(j-1)(1-\alpha)} \right\}^{\frac{1}{t}}.$$

Each of these results is sharp for the extremal function $f(z)$ given by (2.5).

Proof 1. For $0 \leq \delta < 1$ we need to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \tag{4.3}$$

We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{\sum_{j=t+1}^{\infty} (j-1)a_j |z|^t}{1 - \sum_{j=t+1}^{\infty} a_j |z|^t} \right|.$$

The last expression is less than $1 - \delta$ if

$$\sum_{j=t+1}^{\infty} \frac{(j-\delta)}{1-\delta} a_j |z|^t < 1.$$

We use the fact that $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)} a_j < 1.$$

Equation (4.3) holds true if

$$\frac{j-\delta}{1-\delta} |z|^t < \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t < \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)(j-\delta)},$$

which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf'' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$|zf''(z)| < 1 - \delta. \tag{4.4}$$

We have

$$|zf''(z)| \leq \left| \sum_{j=t+1}^{\infty} j(j-1)a_j|z|^{t-1} \right| < 1 - \delta,$$

$$\sum_{j=t+1}^{\infty} \frac{j(j-1)}{1-\delta} a_j|z|^{t-1} < 1.$$

We use the fact that $f \in \mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1 + \mu(j-1)][1 + \beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)} a_j < 1.$$

Equation (4.4) holds true if

$$\frac{j(j-1)}{1-\delta} |z|^{t-1} < \frac{[1 + \mu(j-1)][1 + \beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)},$$

or, equivalently,

$$|z|^{t-1} < \frac{(1-\delta)[1 + \mu(j-1)][1 + \beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(j-1)(1-\alpha)},$$

which yields the convexity of the family. □

5 Neighborhood property

In this section we study neighborhood property for functions in the class $\mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$.

Definition 5.1 For functions f belong to \mathcal{A} of the form and $\varepsilon \geq 0$, we define $(\eta - \varepsilon)$ -neighborhood of f by

$$N_{\varepsilon}^{\eta}(f) = \left\{ g(z) \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \sum_{j=2}^{\infty} j^{\eta+1} |a_j - b_j| \leq \varepsilon \right\},$$

where η is a fixed positive integer.

By using the following lemmas we will investigate the $(\eta - \varepsilon)$ -neighborhood of function in $\mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$.

Lemma 5.1 Let $-1 \leq \beta < 1$, if $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ satisfies

$$\sum_{j=2}^{\infty} j^{\rho+1} |b_j| \leq \frac{2\beta\nu(1-\alpha)}{1 + \beta(2\nu-1)}$$

then $g(z) \in \mathcal{RT}(\gamma, \lambda, l, \alpha, \beta)$.

Proof By using of Theorem 2.1, it is sufficient to show that

$$\frac{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\}}{2\beta\nu(1 - \alpha)} = \frac{j^{\rho+1}}{2\beta\nu(1 - \alpha)} [1 + \beta(2\nu - 1)].$$

But

$$\frac{[1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\}}{2\beta\nu(1 - \alpha)} \leq \frac{j^{\rho+1}}{2\beta\nu(1 - \alpha)} [1 + \beta(2\nu - 1)].$$

Therefore it is enough to prove that

$$Q(j, \rho) = \frac{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}}{j^{\rho+1}} \leq 1,$$

the result follows because the last inequality holds for all $j \geq t + 1$. □

Lemma 5.2 Let $f(z) = z - \sum_{k=2}^\infty a_k z^k \in \mathcal{T}$, $\gamma, \lambda, l \geq 0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $\epsilon \geq 0$. If $\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, then

$$\sum_{j=t+1}^\infty j^{\rho+1} a_j \leq \frac{2\beta\nu(1 - \alpha)(1 + \epsilon)(t + 1)^{\rho+1}}{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^\rho + (1 - \gamma)\frac{(n+t)!}{n!t!}\}},$$

where either $\rho = 0$ or $\rho = 1$. The result is sharp with the extremal function

$$f(z) = z - \frac{2\beta\nu(1 - \alpha)(1 + \epsilon)}{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^\rho + (1 - \gamma)\frac{(n+t)!}{n!t!}\}} z^{t+1}, \quad z \in U.$$

Proof Letting $g(z) = \frac{f(z)+\epsilon z}{1+\epsilon}$ we have

$$g(z) = z - \sum_{j=t+1}^\infty \frac{a_j}{1 + \epsilon} z^j, \quad z \in U.$$

In view of Theorem 2.3, $g(z) = \sum_{j=1}^\infty \eta_j g_j(z)$ where $\eta_j \geq 0$, $\sum_{j=1}^\infty \eta_j = 1$,

$$g_1(z) = z$$

and

$$g_j(z) = z - \frac{2\beta\nu(1 - \alpha)(1 + \epsilon)}{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(n+j-1)!}{n!(j-1)!}\}} z^j, \quad j \geq t + 1.$$

So we obtain

$$\begin{aligned} g(z) &= \eta_1 z + \sum_{j=t+1}^\infty \eta_j \left[z - \frac{2\beta\nu(1 - \alpha)(1 + \epsilon)}{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(n+j-1)!}{n!(j-1)!}\}} z^j \right] \\ &= z - \sum_{j=t+1}^\infty \eta_j \frac{2\beta\nu(1 - \alpha)(1 + \epsilon)}{[1 + \mu(j - 1)][1 + \beta(2\nu - 1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^\rho + (1 - \gamma)\frac{(n+j-1)!}{n!(j-1)!}\}} z^j. \end{aligned}$$

Since $\eta_j \geq 0$ and $\sum_{j=2}^{\infty} \eta_j \leq 1$, it follows that

$$\sum_{j=t+1}^{\infty} a_k \leq \sup_{j \geq t+1} j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}.$$

Since whenever $\rho = 0$ or $\rho = 1$ we conclude that

$$W(j, \rho, \gamma, \alpha, \beta, \epsilon) = j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}$$

is a decreasing function of j , the result will follow. The proof is complete. \square

Theorem 5.1 *Let $\rho = 0$ or $\rho = 1$ and suppose $0 \leq \beta < 1$ and*

$$\begin{aligned} -1 &\leq \theta \\ &< \frac{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\} - 2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}, \end{aligned}$$

$f(z) \in \mathcal{T}$ and $\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, then the $(\eta - \epsilon)$ -neighborhood of f is the subset of $\mathcal{RI}(\lambda, \lambda, l, \alpha, \beta)$, where

$$\begin{aligned} \epsilon &\leq 2(1-\alpha) \left\{ \theta \gamma [1+\mu(t-1)][1+\beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda t+l}{l+1} \right)^n + (1-\gamma) \frac{(n+t)!}{n!t!} \right\} \right. \\ &\quad \left. - \beta \gamma [1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1} \right\} / \left([1+\theta(2\nu-1)][1+\mu(t-1)] \right. \\ &\quad \left. \times [1+\beta(2\nu-1)] \left\{ \gamma \left(\frac{1+\lambda t+l}{l+1} \right)^n + (1-\gamma) \frac{(n+t)!}{n!t!} \right\} \right). \end{aligned}$$

The result is sharp.

Proof For $f(z) = z - \sum_{j=2}^{\infty} |a_j|z^j$, let $g(z) = z + \sum_{j=2}^{\infty} b_jz^j$ be in $N_{\epsilon}^{\eta}(f)$. So by Lemma 5.2, we have

$$\begin{aligned} \sum_{j=2}^{\infty} j^{\eta+1} |b_j| &= \sum_{j=2}^{\infty} j^{\eta+1} |a_j - b_j - a_j| \\ &\leq \epsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}. \end{aligned}$$

By using Lemma 5.1, $g(z) \in \mathcal{L}(\gamma, \alpha, \beta)$ if

$$\epsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} \leq \frac{2\theta\nu(1-\alpha)}{1+\theta(2\nu-1)},$$

that is, $\epsilon \leq \frac{2(1-\alpha)\{\theta\gamma[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\} - \beta\gamma[1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1}}{[1+\theta(2\nu-1)][1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}$, and the proof is complete. \square

Competing interests

The author declares that she has no competing interests.

Author's contributions

The author drafted the manuscript, read and approved the final manuscript.

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