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Stability and superstability of ternary homomorphisms and ternary derivations on ternary quasi-Banach algebras

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Abstract

In this article, we investigate the generalized Hyers-Ulam-Rassias stability, Isac-Rassias type stability and superstability of ternary homomorphisms and ternary derivations associated to the generalized m-variables Cauchy-Jensen functional equation

$$\sum_{i=1}^{m} f(x_i) - \frac{1}{2m}\left[\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right)\right] = 0$$

for a fixed positive integer m with $m \geq 3$ on ternary quasi-Banach algebras.
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1. Introduction

A functional equation (ξ) is stable if any function g satisfying the equation (ξ) approximately is near to a true solution of (ξ). A functional equation (ξ) is superstable if any function g satisfying the equation (ξ) approximately is a true solution of (ξ).

It is of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation.

The first stability problem was raised by Ulam [1] during his talk at the University of Wisconsin in 1940. The stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Aoki [3] and Bourgin [4] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [5] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. It was shown by Gajda [6], as well as by Rassias and Šemrl [7] that one cannot prove a stability theorem of the additive equation for a specific function. Găvruta [8] obtained generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function.

Bourgin [4] is the first mathematician dealing with stability of (ring) ho-momorphism $f(xy) = f(x)f(y)$. The topic of approximate homomorphisms and approximate derivations was studied by a number of mathematicians (see [9-13], and references therein).

We refer the readers to [2,5-8,11-51] and references therein for more detailed results on the stability problems of various functional equations.

We note that a quasi-norm is a real-valued function on a vector space X satisfying the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda.x\| = |\lambda|.\ \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$. The pair $(X, \|.\|)$ is called a quasi-normed space if $\|.\|$ is a quasi-norm on X . A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|.\|$ is called a p-norm $(0 \leq p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [52] who introduced the notion of cubic matrix which in turn was generalized by Kapranov et al. [39]. There are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation. The comments on physical applications of ternary structures can be found in [37,39,40,43,44,52-59].

Let A be a linear space over a complex field equipped with a mapping []: $A^3 = A \times A \times A \to A$ with $(x, y, z) \mapsto [x, y, z]$ that is linear in variables x, y, z and satisfies the associative identity $[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]]$ for all x, y, z, u, v in A. The pair $(A, [\ ])$ is called a ternary algebra.

Assume that A is a ternary algebra. We say A has a unit if there exist an element $e \in A$ such that $[e, e, a] = [eae] = [a, e, e] = a$ for all $a \in A$.

Let A be a ternary algebra and let $(A, \|.\|)$ be a quasi-Banach space (p-Banach space) (with constant $K \geq 1$). Then A is called a ternary quasi-Banach algebra (ternary p-Banach algebra) if $\|[x, y, z]\| \leq K\|x\|\|y\|\|z\|$ for all $x, y, z \in A$.

Let $\mathcal{A}$ and $\mathcal{B}$ be ternary algebras. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a ternary homomorphism if

$$H([abc]) = [H(a)H(b)H(c)]$$

for all $a, b, c \in \mathcal{A}$. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a ternary derivation if

$$\delta([abc]) = [\delta(a)bc] + [a\delta(b)c] + [ab\delta(c)]$$

for all $a, b, c \in \mathcal{A}$ (see [25-31,46,60]).

Recently, Ebadian and et al. [61] investigated the solution and stability of functional equation

$$\sum_{i=1}^{m} f(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + f(\sum_{i=1}^{m} x_i) = 2m \sum_{i=1}^{m} f(x_i) \tag{1.1}$$

for a fixed positive integer m with $m \geq 2$ in quasi-Banach spaces. In this paper, we establish the generalized Hyers-Ulam-Rassias stability of ternary homomorphisms and ternary derivations on ternary quasi-Banach algebras. Moreover, by using the main theorems, we prove the superstability of ternary homomorphisms and ternary derivations on ternary quasi Banach algebras.

Throughout this article, we assume that A is a ternary quasi-Banach algebra with quasi-norm $\|.\|_A$ and B is a ternary p-Banach algebra with quasi-norm $\|.\|_B$

2. Ternary homomorphisms

From now on, we assume that $m, n_0 \in \mathbb{N}$ are positive integers $m \geq 3$, and suppose that $\mathbb{T}^1_{\frac{1}{n_o}} := \{e^{i\theta}; 0 \leq \theta \leq \frac{2\pi}{n_o}\}$. Moreover, we will use the following abbreviation for a given mapping $f : A \to B$:

$$D_\mu f(x_1, x_2, \ldots, x_m, a, b, c, u) := \sum_{i=1}^{m} f(mx_i + \sum_{j=1, j\neq i}^{m} x_j) + f(\sum_{i=1}^{m} x_i) - 2m \sum_{i=1}^{m} f(x_i)$$
$$+ f([abc]) - [f(a)f(b)f(c)] + f(\mu u) - \mu f(u)$$

for all $a, b, c, u, x_1, x_2, \ldots, x_m \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$

Theorem 2.1. *Let* $\phi : A \underbrace{\times \cdots \times A}_{m+4-times} \to [0, \infty)$ *be a function satisfying*

$$\Psi(x) = \sum_{i=1}^{\infty} (\frac{1}{m})^{ip} (\phi(m^{i-1}x, 0, \ldots, 0))^p < \infty$$

for all $x \in A$, *and*

$$\lim_{n\to\infty} \frac{1}{m^n} \phi(m^n x_1, \ldots, m^n x_m, m^n a, m^n b, m^n c, m^n u) = 0 \tag{2.1}$$

for all $u, a, b, c, x_j \in A$ $(1 \leq j \leq m)$. *Let* $f : A \to B$ *be a mapping such that* $f(0) = 0$ *and that*

$$\left\| D_\mu f(x_1, \ldots, x_m, a, b, c, u) \right\| \leq \phi(x_1, \ldots, x_m, a, b, c, u) \tag{2.2}$$

for all $u, a, b, c, x_j \in A$ $(1 \leq j \leq m)$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then there exists a unique ternary homomorphism* $T : A \to B$ *such that inequality*

$$\left\| f(x) - T(x) \right\| \leq [\Psi(x)]^{\frac{1}{p}} \tag{2.3}$$

for all $x \in A$.

Proof: Putting $\mu = 1$, $a = b = c = u = 0$ *in* (2.2), then we have

$$\left\| D_1 f(x_1, \ldots, x_m, 0, 0, 0, 0) \right\| \leq \phi(x_1, \ldots, x_m, 0, 0, 0, 0)$$

for all $x_1, x_2, \ldots, x_m \in A$. By using the Theorem 2.2 of [61], the limit

$$\lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$$

exists for all $x \in A$ and the mapping

$$T(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \quad (x \in A)$$

is a unique additive function which satisfies (2.3). Moreover, one can show that $T(x) = \frac{1}{m^n} T(m^n x) = \frac{1}{m^{2n}} T(m^{2n} x)$ for all $x \in A$. Putting $a = b = c = x_1 = x_2 = \ldots = x_m = 0$ in (2.2) to get

$$\left\| f(\mu u) - \mu f(u) \right\| = \left\| D_\mu f(0, 0, \ldots, 0, u) \right\| \leq \phi(0, 0, \ldots, 0, u)$$

for all $u \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then by definition of T and (2.1), we have

$$\left\| T(\mu u) - \mu T(u) \right\| = \lim_{n\to\infty} \frac{1}{m^n} \left\| f(m^n \mu u) - u f(m^n u) \right\| \leq \lim_{n\to\infty} \frac{1}{m^n} \phi(0, 0, \ldots, 0, m^n u) = 0$$

for all $u \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. This means that

$$T(\mu u) = \mu T(u)$$

for all $u \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. By the same reasoning as that in the proof of Theorem 2.1 of [19], one can show that $T : A \to B$ is $\mathbb{C}$-linear. On the other hand, by putting $u = x_1 = x_2 = \ldots = x_m = 0$ in (2.2), we have

$$\left\| f([abc]) - [f(a) f(b) f(c)] \right\| \leq \phi(0, 0, \ldots, 0, a, b, c, 0)$$

for all $a, b, c \in A$. It follows that

$$\begin{aligned}
\left\| T([abc]) - [T(a)T(b)T(c)] \right\| &= \left\| \frac{1}{m^{2n}} T([m^{2n} abc]) - [T(a) \frac{1}{m^n} T(m^n b) \frac{1}{m^n} T(m^n c)] \right\| \\
&= \lim_{n\to\infty} \left\| \frac{1}{m^3 n} f([(m^n a)(m^n b)(m^n c)]) - [(\frac{1}{m^n} f(m^n a))(\frac{1}{m^n} f(m^n b))(\frac{1}{m^n} f(m^n c))] \right\| \\
&= \lim_{n\to\infty} \frac{1}{m^3 n} \left\| f([(m^n a)(m^n b)(m^n c)]) - [(f(m^n a))(f(m^n b))(f(m^n c))] \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{m^3 n} \phi(0, 0, \ldots 0, m^n a, m^n b, m^n c, 0) \\
&= 0
\end{aligned}$$

for all $a, b, c \in A$. This means that $T : A \to B$ is a ternary homomorphism. The uniqueness of T follows from Theorem 2.2 of [61].

Corollary 2.2. *Let* θ, r, r_j $(1 \leq j \leq m)$ *be non-negative real numbers such that* $0 < r, r_j < 1$. *Suppose that a mapping* $f : A \to B$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|D_\mu f(x_1, \ldots, x_m, a, b, c, u)\right\|_B \le \theta \sum_{j=1}^{m} \left\|x_j\right\|_A^{r_j} + \|a\|_A^r + \|b\|_A^r + \|c\|_A^r + \|u\|_A^r$$

for all $a, b, c, u, x_1, x_1, \ldots, x_m \in A$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then there exists a unique ternary homomorphism* $T: A \to B$ *such that*

$$\left\|f(x) - T(x)\right\|_B \le \frac{\theta \|x\|^{r_1}}{m^{r_1}} \left\{\frac{m^{(1-r_1)p}}{m^{(1-r_1)p} - 1}\right\}^{\frac{1}{p}}$$

for all $x \in A$

Proof: *It follows from Theorem* 2.1 *by putting*

$$\phi(x_1, \ldots, x_m, a, b, c, u) = \theta \sum_{j=1}^{m} \left\|x_j\right\|_A^{r_j} + \|a\|_A^r + \|b\|_A^r + \|c\|_A^r + \|u\|_A^r$$

for all $a, b, c, u, x_1, x_1, \ldots, x_m \in A$.

Now, we investigate the Hyers-Ulam type stability of ternary homomor-phisms on ternary quasi Banach algebras as follows.

Corollary 2.3. *Let* θ *be non-negative real number. Suppose that a mapping* $f: A \to B$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|D_\mu f(x_1, \ldots, x_m, a, b, c, u)\right\|_B \le \theta$$

for all $a, b, c, u, x_1, x_1, \ldots, x_m \in A$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then there exists a unique ternary homomorphism* $T: A \to B$ *such that*

$$\left\|f(x) - T(x)\right\|_B \le \theta \left\{\frac{1}{m^p - 1}\right\}^{\frac{1}{p}}$$

for all $x \in A$.

Proof. It follows from Theorem 2.1, by putting

$$\phi(x_1, x_2, \ldots x_m, a, b, c, u) := \theta$$

for all $u, a, b, c, x_1, x_2, \ldots, x_m \in A$.

Isac and Rassias [38] generalized the Hyers' theorem by introducing a mapping $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions:

1) $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$,
2) $\psi(ts) \le \psi(t)\psi(s)$; $s, t > 0$,
3) $\psi(t) < t$; $t > 1$.

These stability results can be applied in stochastic analysis [38], financial and actuarial mathematics, as well as in psychology and sociology. The following corollary is Isac-Rassias type stability of ternary homomorphisms on ternary quasi-Banach algebras.

Corollary 2.4. *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a mapping such that*

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0,$$
$$\psi(ts) \le \psi(t)\psi(s) \quad s, t > 0,$$
$$\psi(t) < t \quad t > 1.$$

Let θ, r, r_j $(1 \le j \le m)$ *be non-negative real numbers. Let* $f : A \to B$ *be a mapping such that* $f(0) = 0$ *and that*

$$\left\|D_\mu f(x_1, \ldots, x_m, a, b, c, u)\right\|_B \le \theta \left(\sum_{j=1}^{m} \psi(\|x_j\|_A) + \psi(\|a\|_A) + \psi(\|b\|_A) + \psi(\|c\|_A) + \psi(\|u\|_A) \right)$$

for all $u, a, b, c, x_1, x_2, \ldots, x_m \in A$. *Then there exists a unique ternary homo-morphism* $T : A \to B$ *such that*

$$\left\|f(x) - T(x)\right\|_B \le k\theta\psi(m^{-1})\psi(\|x\|)$$

for all $x \in A$, *where* $k = \frac{\psi(m)}{m-\psi(m)}$.

Proof: *The proof follows from Theorem* 2.1 *by taking*

$$\phi(x_1, \ldots, x_m, a, b, c, u) := \theta \left(\sum_{j=i}^{m} \psi(\|x_j\|_A) + \psi(\|a\|_A) + \psi(\|b\|_A) + \psi(\|c\|_A) + \psi(\|u\|_A) \right)$$

for all $u, a, b, c, x_1, x_2, \ldots, x_m \in A$.

Moreover, we have the superstability of ternary homomorphisms on ternary quasi Banach algebras as follows.

Corollary 2.5. *Let* θ, r, r_j $(1 \le j \le m)$ *be non-negative real numbers such that* $0 < r, r_j < 1$. *Suppose that a mapping* $f : A \to B$ *with* $f(0) = 0$ *satisfies the inequality*

$$\left\|D_\mu f(x_1, \ldots, x_m, a, b, c, u)\right\|_B \le \theta \left(\prod_{j=1}^{m} \left\|x_j\right\|_A^{r_j} \right) \|a\|_A^r \, \|b\|_A^r \, \|c\|_A^r \, \|u\|_A^r$$

for all $a, b, c, u, x_1, x_1, \ldots, x_m \in A$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then* $f : A \to B$ *is a ternary homomorphism.*

Proof: *Putting*

$$\phi(x_1, \ldots, x_m, a, b, c, u) := \theta \left(\prod_{j=1}^{m} \left\|x_j\right\|_A^{r_j} \right) \|a\|_A^r \, \|b\|_A^r \, \|c\|_A^r \, \|u\|_A^r$$

for all $a, b, c, u, x_1, x_1, \ldots, x_m \in A$. *Then we have* $\varphi(x, 0, 0, \ldots, 0) = 0$. *By Theorem 2.1, there exists a unique ternary homomorphism* $T : A \to B$ *such that*

$$\left\|f(x) - T(x)\right\|_B \le [\Psi(x)]^{\frac{1}{p}} = 0$$

for all $x \in A$. *This means that* $f(x) = T(x)$ *for all* $x \in A$. *Hence* $f : A \to B$ *is a ternary homomorphism.*

3. Ternary derivations

In this section, we use the following abbreviation for a given mapping $f : A \to A$:

$$\Delta_\mu f(x_1, x_2, \ldots, x_m, a, b, c, u) := \sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) - 2m \sum_{i=1}^{m} f(x_i)$$
$$+ f([abc]) - [f(a)bc] - [af(b)c] - [abf(c)] + f(\mu u) - \mu f(u)$$

for all $a, b, c, u, x_1, x_2, \ldots, x_m \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$.

Theorem 3.1. *Let* $\phi : \underbrace{A \times \cdots \times A}_{m+4-times} \to [0, \infty)$ *be a function satisfying*

$$\Psi(x) = \sum_{i=1}^{\infty} \left(\frac{1}{m}\right)^{ip} (\phi(m^{i-1}x, 0, \ldots, 0))^p < \infty$$

for all $x \in A$, *and*

$$\lim_{n\to\infty} \frac{1}{m^n} \phi(m^n x_1, \ldots, m^n x_m, m^n a, m^n b, m^n c, m^n u) = 0$$

for all $u, a, b, c, x_j \in A$ $(1 \le j \le m)$. *Let* $f : A \to B$ *be a mapping such that* $f(0) = 0$ *and that*

$$\left\| \Delta_\mu f(x_1, \ldots, x_m, a, b, c, u) \right\| \le \phi(x_1, \ldots, x_m, a, b, c, u) \tag{3.1}$$

for all $u, a, b, c, x_j \in A$ $(1 \le j \le m)$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then there exists a unique ternary derivation* $D : A \to B$ *such that*

$$\left\| f(x) - D(x) \right\| \le [\Psi(x)]^{\frac{1}{p}}$$

for all $x \in A$.

Proof: By using the same technique of proving Theorem 2.1, the limit

$$\lim_{n\to\infty} \frac{1}{m^n} f(m^n x)$$

exists for all $x \in A$ and the mapping

$$D(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \quad (x \in A)$$

is a unique $\mathbb{C}$-linear function which satisfies (2.3). On the other hand, by putting $u = x_1 = x_2 = \cdots = x_m = 0$ in (3.1), we have

$$\left\| f([abc]) - [f(a)bc] - [af(b)c] - [abf(c)] \right\| \le \phi(0, 0, \ldots, 0, a, b, c, 0)$$

for all a, b, $c \in A$. It follows that

$$\begin{aligned}
&\|D([abc]) - [D(a)bc] - [aD(b)c] - [abD(c)]\| \\
&= \left\| \frac{1}{m^{2n}} D([m^{2n}abc]) - \left[D(a) \frac{1}{m^n}(m^n b) \frac{1}{m^n}(m^n c) \right] \right. \\
&\quad \left. - \left[\frac{1}{m^n}(m^n a) D(b) \frac{1}{m^n}(m^n c) \right] - \left[\frac{1}{m^n} D(m^n a) \frac{1}{m^n}(m^n b) D(c) \right] \right\| \\
&= \lim_{n\to\infty} \left\| \frac{1}{m^3 n} f([(m^n a)(m^n b)(m^n c)]) - \left[\left(\frac{1}{m^n} f(m^n a) \right) \left(\frac{1}{m^n}(m^n b) \right) \left(\frac{1}{m^n}(m^n c) \right) \right] \right. \\
&\quad \left. - \left[\left(\frac{1}{m^n}(m^n a) \right) \left(\frac{1}{m^n} f(m^n b) \right) \left(\frac{1}{m^n}(m^n c) \right) \right] - \left[\left(\frac{1}{m^n}(m^n a) \right) \left(\frac{1}{m^n}(m^n b) \right) \left(\frac{1}{m^n} f(m^n c) \right) \right] \right\| \\
&= \lim_{n\to\infty} \frac{1}{m^3 n} \| f([(m^n a)(m^n b)(m^n c)]) - [(f(m^n a))((m^n b))((m^n c))] \\
&\quad - [((m^n a))(f(m^n b))((m^n c))] - [((m^n a))((m^n b))(f(m^n c))] \| \\
&\le \lim_{n\to\infty} \frac{1}{m^3 n} \phi(0, 0, \ldots, 0, m^n a, m^n b, m^n c, 0) \\
&= 0
\end{aligned}$$

for all a, b, $c \in A$. This means that $D : A \to B$ is a ternary derivation.

Corollary 3.2. *Let θ, r, r_j $(1 \le j \le m)$ be non-negative real numbers such that $0 < r$, $r_j < 1$. Suppose that a mapping $f : A \to B$ with $f(0) = 0$ satisfies the inequality*

$$\left\| \Delta_\mu f(x_1, \ldots, x_m, a, b, c, u) \right\|_A \le \theta (\sum_{j=1}^{m} \left\| x_j \right\|_A^{r_j} + \|a\|_A^r + \|b\|_A^r + \|c\|_A^r + \|u\|_A^r$$

for all a, b, c, u, x_1, x_1, ..., $x_m \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then there exists a unique ternary derivation $D : A \to B$ such that

$$\left\| f(x) - D(x) \right\|_A \le \frac{\theta \|x\|^{r_1}}{m^{r_1}} \left\{ \frac{m^{(1-r_1)p}}{m^{(1-r_1)p} - 1} \right\}^{\frac{1}{p}}$$

for all $x \in A$.

Proof: *It follows from Theorem* 3.1 *by putting*

$$\phi(x_1, \ldots, x_m, a, b, c, u) = \theta \sum_{j=1}^{m} \left\| x_j \right\|_A^{r_j} + \|a\|_A^r + \|b\|_A^r + \|c\|_A^r + \|u\|_A^r$$

for all a, b, c, u, x_1, x_1, ..., $x_m \in A$.

We have the Hyers-Ulam type stability of ternary derivations on ternary quasi Banach algebras as follows.

Corollary 3.3. *Let θ be non-negative real number. Suppose that a mapping $f : A \to B$ with $f(0) = 0$ satisfies the inequality*

$$\left\| \Delta_\mu f(x_1, \ldots, x_m, a, b, c, u) \right\|_A \le \theta$$

for all a, b, c, u, x_1, x_1, ..., $x_m \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. Then there exists a unique ternary derivation $D : A \to B$ such that

$$\left\| f(x) - D(x) \right\|_A \le \theta \left\{ \frac{1}{m^p - 1} \right\}^{\frac{1}{p}}$$

for all $x \in A$.

Proof. It follows from Theorem 3.1, by putting

$$\phi(x_1, x_2, \ldots, x_m, a, b, c, u) := \theta$$

for all u, a, b, c, x_1, x_2, ..., $x_m \in A$.

By using the same technique of proving Corollary 2.4, we can prove the Isac-Rassias type stability of ternary derivations on ternary quasi-Banach algebras as follows.

Corollary 3.4. *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a mapping such that*

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0,$$
$$\psi(ts) \le \psi(t)\psi(s) \quad s, t > 0,$$
$$\psi(t) < t \quad t > 1.$$

Let θ, r, r_j $(1 \le j \le m)$ *be non-negative real numbers. Let* $f : A \to A$ *be a mapping such that* $f(0) = 0$ *and that*

$$\|\Delta_\mu f(x_1, \ldots, x_m, a, b, c, u)\|_A \le \theta \left(\sum_{j=1}^{m} \psi(\|x_j\|_A) + \psi(\|a\|_A) + \psi(\|b\|_A) + \psi(\|c\|_A) + \psi(\|u\|_A) \right)$$

for all u, a, b, c, x_1, x_2, ..., $x_m \in A$. *Then there exists a unique ternary derivation* D: $A \to A$ *such that*

$$\|f(x) - D(x)\|_A \le k\theta\psi(m^{-1})\psi(\|x\|)$$

for all $x \in A$, *where* $k = \frac{\psi(m)}{m-\psi(m)}$.

Similar to Corollary 2.5, we can prove the superstability of ternary derivations on ternary quasi-Banach algebras as follows.

Corollary 3.5. *Let* θ, r, r_j $(1 \le j \le m)$ *be non-negative real numbers such that* $0 < r$, $r_j < 1$. *Let* f: $A \to A$ *be a mapping such that* $f(0) = 0$ *and that*

$$\|\Delta_\mu f(x_1, \ldots, x_m, a, b, c, u)\|_A \le \theta \left(\prod_{j=1}^{m} \|x_j\|_A^{r_j} \right) \|a\|_A^r \, \|b\|_A^r \, \|c\|_A^r \, \|u\|_A^r$$

for all a, b, c, u, x_1, x_1, ..., $x_m \in A$ *and all* $\mu \in \mathbb{T}^1_{\frac{1}{n_o}}$. *Then* $f : A \to A$ *is a ternary derivation.*

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Authors' contributions

All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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