

Research Article

Uniqueness and Parameter Dependence of Positive Solution of Fourth-Order Nonhomogeneous BVPs

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We investigate the following fourth-order four-point nonhomogeneous Sturm-Liouville boundary value problem: $u^{(4)} = f(t, u)$, $t \in [0, 1]$, $\alpha u(0) - \beta u'(0) = \lambda_1$, $\gamma u(1) + \delta u'(1) = \lambda_2$, $au''(\xi_1) - bu'''(\xi_1) = -\lambda_3$, $cu''(\xi_2) + du'''(\xi_2) = -\lambda_4$, where $0 \leq \xi_1 < \xi_2 \leq 1$ and λ_i ($i = 1, 2, 3, 4$) are nonnegative parameters. Some sufficient conditions are given for the existence and uniqueness of a positive solution. The dependence of the solution on the parameters λ_i ($i = 1, 2, 3, 4$) is also studied.

1. Introduction

Boundary value problems (BVPs for short) consisting of fourth-order differential equation and four-point homogeneous boundary conditions have received much attention due to their striking applications. For example, Chen et al. [1] studied the fourth-order nonlinear differential equation

$$u^{(4)} = f(t, u), \quad t \in (0, 1), \quad (1.1)$$

with the four-point homogeneous boundary conditions

$$u(0) = u(1) = 0, \quad (1.2)$$

$$au''(\xi_1) - bu'''(\xi_1) = 0, \quad cu''(\xi_2) + du'''(\xi_2) = 0, \quad (1.3)$$

where $0 \leq \xi_1 < \xi_2 \leq 1$. By means of the upper and lower solution method and Schauder fixed point theorem, some criteria on the existence of positive solutions to the BVP (1.1)–(1.3) were

established. Bai et al. [2] obtained the existence of solutions for the BVP (1.1)–(1.3) by using a nonlinear alternative of Leray-Schauder type. For other related results, one can refer to [3–5] and the references therein.

Recently, nonhomogeneous BVPs have attracted many authors' attention. For instance, Ma [6, 7] and L. Kong and Q. Kong [8–10] studied some second-order multipoint nonhomogeneous BVPs. In particular, L. Kong and Q. Kong [10] considered the following second-order BVP with multipoint nonhomogeneous boundary conditions

$$\begin{aligned} u'' + a(t)f(u) &= 0, \quad t \in (0, 1), \\ u(0) &= \sum_{i=1}^m a_i u(t_i) + \lambda, \quad u(1) = \sum_{i=1}^m b_i u(t_i) + \mu, \end{aligned} \quad (1.4)$$

where λ and μ are nonnegative parameters. They derived some conditions for the above BVP to have a unique solution and then studied the dependence of this solution on the parameters λ and μ . Sun [11] discussed the existence and nonexistence of positive solutions to a class of third-order three-point nonhomogeneous BVP. The authors in [12] studied the multiplicity of positive solutions for some fourth-order two-point nonhomogeneous BVP by using a fixed point theorem of cone expansion/compression type. For more recent results on higher-order BVPs with nonhomogeneous boundary conditions, one can see [13–16].

Inspired greatly by the above-mentioned excellent works, in this paper we are concerned with the following Sturm-Liouville BVP consisting of the fourth-order differential equation:

$$u^{(4)} = f(t, u), \quad t \in [0, 1] \quad (1.5)$$

and the four-point nonhomogeneous boundary conditions

$$\alpha u(0) - \beta u'(0) = \lambda_1, \quad \gamma u(1) + \delta u'(1) = \lambda_2, \quad (1.6)$$

$$au''(\xi_1) - bu'''(\xi_1) = -\lambda_3, \quad cu''(\xi_2) + du'''(\xi_2) = -\lambda_4, \quad (1.7)$$

where $0 \leq \xi_1 < \xi_2 \leq 1$ and λ_i ($i = 1, 2, 3, 4$) are nonnegative parameters. Under the following assumptions:

- (A1) $\alpha, \beta, \gamma, \delta, a, b, c$, and d are nonnegative constants with $\beta > 0, \delta > 0, \rho_1 := \alpha\gamma + \alpha\delta + \gamma\beta > 0, \rho_2 := ad + bc + ac(\xi_2 - \xi_1) > 0, -a\xi_1 + b > 0$, and $c(\xi_2 - 1) + d > 0$;
- (A2) $f(t, u) : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and monotone increasing in u for every $t \in [0, 1]$;
- (A3) there exists $0 \leq \theta < 1$ such that

$$f(t, ku) \geq k^\theta f(t, u) \quad \text{for any } t \in [0, 1], \quad k \in (0, 1), \quad u \in [0, +\infty), \quad (1.8)$$

we prove the uniqueness of positive solution for the BVP (1.5)–(1.7) and study the dependence of this solution on the parameters λ_i ($i = 1, 2, 3, 4$).

2. Preliminary Lemmas

First, we recall some fundamental definitions.

Definition 2.1. Let X be a Banach space with norm $\|\cdot\|$. Then

- (1) a nonempty closed convex set $P \subseteq X$ is said to be a cone if $mP \subseteq P$ for all $m \geq 0$ and $P \cap (-P) = \{0\}$, where 0 is the zero element of X ;
- (2) every cone P in X defines a partial ordering in X by $u \leq v \Leftrightarrow v - u \in P$;
- (3) a cone P is said to be normal if there exists $M > 0$ such that $0 \leq u \leq v$ implies that $\|u\| \leq M\|v\|$;
- (4) a cone P is said to be solid if the interior $\overset{\circ}{P}$ of P is nonempty.

Definition 2.2. Let P be a solid cone in a real Banach space X , $T : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ an operator, and $0 \leq \theta < 1$. Then T is called a θ -concave operator if

$$T(ku) \geq k^\theta Tu \quad \text{for any } k \in (0, 1), u \in \overset{\circ}{P}. \quad (2.1)$$

Next, we state a fixed point theorem, which is our main tool.

Lemma 2.3 (see [17]). *Assume that P is a normal solid cone in a real Banach space X , $0 \leq \theta < 1$, and $T : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ is a θ -concave increasing operator. Then T has a unique fixed point in $\overset{\circ}{P}$.*

The following two lemmas are crucial to our main results.

Lemma 2.4. *Assume that ρ_1 and ρ_2 are defined as in (A1) and $\rho_1 \rho_2 \neq 0$. Then for any $h \in C[0, 1]$, the BVP consisting of the equation*

$$u^{(4)}(t) = h(t), \quad t \in [0, 1] \quad (2.2)$$

and the boundary conditions (1.6) and (1.7) has a unique solution

$$u(t) = \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) h(\tau) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t), \quad t \in [0, 1], \quad (2.3)$$

where

$$\begin{aligned}
 G_1(t, s) &= \frac{1}{\rho_1} \begin{cases} (\alpha s + \beta)(\gamma + \delta - \gamma t), & 0 \leq s \leq t \leq 1, \\ (\alpha t + \beta)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1, \end{cases} \\
 G_2(t, s) &= \frac{1}{\rho_2} \begin{cases} (a(s - \xi_1) + b)(c(\xi_2 - t) + d), & s \leq t, \xi_1 \leq s \leq \xi_2, \\ (a(t - \xi_1) + b)(c(\xi_2 - s) + d), & t \leq s, \xi_1 \leq s \leq \xi_2, \end{cases} \\
 \phi_1(t) &= \frac{1}{\rho_1}(\gamma + \delta - \gamma t), \quad t \in [0, 1], \\
 \phi_2(t) &= \frac{1}{\rho_1}(\alpha t + \beta), \quad t \in [0, 1], \\
 \phi_3(t) &= \frac{1}{\rho_2} \int_0^1 (c(\xi_2 - s) + d) G_1(t, s) ds, \quad t \in [0, 1], \\
 \phi_4(t) &= \frac{1}{\rho_2} \int_0^1 (a(s - \xi_1) + b) G_1(t, s) ds, \quad t \in [0, 1].
 \end{aligned} \tag{2.4}$$

Proof. Let

$$u''(t) = v(t), \quad t \in [0, 1]. \tag{2.5}$$

Then

$$v''(t) = h(t), \quad t \in [0, 1]. \tag{2.6}$$

By (2.5) and (1.6), we know that

$$u(t) = - \int_0^1 G_1(t, s) v(s) ds + \frac{1}{\rho_1} (\alpha \lambda_2 - \gamma \lambda_1) t + \frac{1}{\rho_1} ((\gamma + \delta) \lambda_1 + \beta \lambda_2), \quad t \in [0, 1]. \tag{2.7}$$

On the other hand, in view of (2.5) and (1.7), we have

$$av(\xi_1) - bv'(\xi_1) = -\lambda_3, \quad cv(\xi_2) + dv'(\xi_2) = -\lambda_4. \tag{2.8}$$

So, it follows from (2.6) and (2.8) that

$$v(t) = - \int_{\xi_1}^{\xi_2} G_2(t, s) h(s) ds + \frac{1}{\rho_2} (c \lambda_3 - a \lambda_4) t + \frac{1}{\rho_2} ((a \xi_1 - b) \lambda_4 - (c \xi_2 + d) \lambda_3), \quad t \in [0, 1], \tag{2.9}$$

which together with (2.7) implies that

$$u(t) = \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) h(\tau) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t), \quad t \in [0, 1]. \quad (2.10)$$

□

Lemma 2.5. Assume that (A1) holds. Then

- (1) $G_1(t, s) > 0$ for $(t, s) \in [0, 1] \times [0, 1]$;
- (2) $G_2(t, s) > 0$ for $(t, s) \in [0, 1] \times [\xi_1, \xi_2]$;
- (3) $\phi_i(t) > 0$ for $t \in [0, 1]$, $i = 1, 2, 3, 4$.

3. Main Result

For convenience, we denote $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$. In the remainder of this paper, the following notations will be used:

- (1) $\lambda \rightarrow \infty$ if at least one of λ_i ($i = 1, 2, 3, 4$) approaches ∞ ;
- (2) $\lambda \rightarrow \mu$ if $\lambda_i \rightarrow \mu_i$ for $i = 1, 2, 3, 4$;
- (3) $\lambda > \mu$ if $\lambda_i \geq \mu_i$ for $i = 1, 2, 3, 4$ and at least one of them is strict.

Let $X = C[0, 1]$. Then $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined as usual by the sup norm.

Our main result is the following theorem.

Theorem 3.1. Assume that (A1)–(A3) hold. Then the BVP (1.5)–(1.7) has a unique positive solution $u_\lambda(t)$ for any $\lambda > \mathbf{0}$, where $\mathbf{0} = (0, 0, 0, 0)$. Furthermore, such a solution $u_\lambda(t)$ satisfies the following properties:

- (P1) $\lim_{\lambda \rightarrow \infty} \|u_\lambda\| = \infty$;
- (P2) $u_\lambda(t)$ is strictly increasing in λ , that is,

$$\lambda > \mu > \mathbf{0} \implies u_\lambda(t) > u_\mu(t), \quad t \in [0, 1]; \quad (3.1)$$

- (P3) $u_\lambda(t)$ is continuous in λ , that is, for any given $\mu > \mathbf{0}$,

$$\lambda \rightarrow \mu \implies \|u_\lambda - u_\mu\| \rightarrow 0. \quad (3.2)$$

Proof. Let $P = \{u \in X \mid u(t) \geq 0, \quad t \in [0, 1]\}$. Then P is a normal solid cone in X with $\overset{\circ}{P} = \{u \in X \mid u(t) > 0, \quad t \in [0, 1]\}$. For any $\lambda > \mathbf{0}$, if we define an operator $T_\lambda : \overset{\circ}{P} \rightarrow X$ as follows:

$$T_\lambda u(t) = \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, u(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t), \quad t \in [0, 1], \quad (3.3)$$

then it is not difficult to verify that u is a positive solution of the BVP (1.5)–(1.7) if and only if u is a fixed point of T_λ .

Now, we will prove that T_λ has a unique fixed point by using Lemma 2.3.

First, in view of Lemma 2.5, we know that $T_\lambda : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$.

Next, we claim that $T_\lambda : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ is a θ -concave operator.

In fact, for any $k \in (0, 1)$ and $u \in \overset{\circ}{P}$, it follows from (3.3) and (A3) that

$$\begin{aligned} T_\lambda(ku)(t) &= \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, ku(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \\ &\geq k^\theta \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, u(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \\ &\geq k^\theta \left(\int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, u(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \right) \\ &= k^\theta T_\lambda u(t), \quad t \in [0, 1], \end{aligned} \quad (3.4)$$

which shows that T_λ is θ -concave.

Finally, we assert that $T_\lambda : \overset{\circ}{P} \rightarrow \overset{\circ}{P}$ is an increasing operator.

Suppose that $u, v \in \overset{\circ}{P}$ and $u \leq v$. By (3.3) and (A2), we have

$$\begin{aligned} T_\lambda u(t) &= \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, u(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \\ &\leq \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, v(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \\ &= T_\lambda v(t), \quad t \in [0, 1], \end{aligned} \quad (3.5)$$

which indicates that T_λ is increasing.

Therefore, it follows from Lemma 2.3 that T_λ has a unique fixed point $u_\lambda \in \overset{\circ}{P}$, which is the unique positive solution of the BVP (1.5)–(1.7). The first part of the theorem is proved.

In the rest of the proof, we will prove that such a positive solution $u_\lambda(t)$ satisfies properties (P1), (P2), and (P3).

First,

$$\begin{aligned} u_\lambda(t) &= T_\lambda u_\lambda(t) \\ &= \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, u_\lambda(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t), \quad t \in [0, 1], \end{aligned} \quad (3.6)$$

which together with $\phi_i(t) > 0$ ($i = 1, 2, 3, 4$) for $t \in [0, 1]$ implies (P1).

Next, we show (P2). Assume that $\lambda > \mu > 0$. Let

$$\bar{\chi} = \sup\{\chi > 0 : u_\lambda(t) \geq \chi u_\mu(t), t \in [0, 1]\}. \quad (3.7)$$

Then $u_\lambda(t) \geq \bar{\chi} u_\mu(t)$ for $t \in [0, 1]$. We assert that $\bar{\chi} \geq 1$. Suppose on the contrary that $0 < \bar{\chi} < 1$. Since T_λ is a θ -concave increasing operator and for given $u \in \overset{\circ}{P}$, $T_\lambda u$ is strictly increasing in λ , we have

$$\begin{aligned} u_\lambda(t) &= T_\lambda u_\lambda(t) \geq T_\lambda(\bar{\chi} u_\mu)(t) > T_\mu(\bar{\chi} u_\mu)(t) \\ &\geq (\bar{\chi})^\theta T_\mu u_\mu(t) = (\bar{\chi})^\theta u_\mu(t) > \bar{\chi} u_\mu(t), \quad t \in [0, 1], \end{aligned} \quad (3.8)$$

which contradicts the definition of $\bar{\chi}$. Thus, we get $u_\lambda(t) \geq u_\mu(t)$ for $t \in [0, 1]$. And so,

$$u_\lambda(t) = T_\lambda u_\lambda(t) \geq T_\lambda u_\mu(t) > T_\mu u_\mu(t) = u_\mu(t), \quad t \in [0, 1], \quad (3.9)$$

which indicates that $u_\lambda(t)$ is strictly increasing in λ .

Finally, we prove (P3). For any given $\mu > 0$, we first suppose that $\lambda \rightarrow \mu$ with $\mu/2 < \lambda < \mu$. From (P2), we know that

$$u_\lambda(t) < u_\mu(t), \quad t \in [0, 1]. \quad (3.10)$$

Let

$$\bar{\sigma} = \sup\{\sigma > 0 : u_\lambda(t) \geq \sigma u_\mu(t), t \in [0, 1]\}. \quad (3.11)$$

Then $0 < \bar{\sigma} < 1$ and $u_\lambda(t) \geq \bar{\sigma} u_\mu(t)$ for $t \in [0, 1]$. If we define

$$\omega(\lambda) = \min\left\{\frac{\lambda_i}{\mu_i} : \mu_i > 0\right\}, \quad (3.12)$$

then $0 < \omega(\lambda) < 1$ and

$$\begin{aligned}
 u_\lambda(t) &= T_\lambda u_\lambda(t) \\
 &\geq T_\lambda(\bar{\sigma} u_\mu)(t) \\
 &= \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, \bar{\sigma} u_\mu(\tau)) d\tau ds + \sum_{i=1}^4 \lambda_i \phi_i(t) \\
 &\geq \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, \bar{\sigma} u_\mu(\tau)) d\tau ds + \omega(\lambda) \sum_{i=1}^4 \mu_i \phi_i(t) \\
 &\geq \omega(\lambda) \left(\int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) f(\tau, \bar{\sigma} u_\mu(\tau)) d\tau ds + \sum_{i=1}^4 \mu_i \phi_i(t) \right) \\
 &= \omega(\lambda) T_\mu(\bar{\sigma} u_\mu)(t) \\
 &\geq \omega(\lambda) (\bar{\sigma})^\theta T_\mu u_\mu(t) \\
 &= \omega(\lambda) (\bar{\sigma})^\theta u_\mu(t), \quad t \in [0, 1],
 \end{aligned} \tag{3.13}$$

which together with the definition of $\bar{\sigma}$ implies that

$$\omega(\lambda) (\bar{\sigma})^\theta \leq \bar{\sigma}. \tag{3.14}$$

So,

$$\bar{\sigma} \geq (\omega(\lambda))^{1/(1-\theta)}. \tag{3.15}$$

Therefore,

$$u_\lambda(t) \geq \bar{\sigma} u_\mu(t) \geq (\omega(\lambda))^{1/(1-\theta)} u_\mu(t), \quad t \in [0, 1]. \tag{3.16}$$

In view of (3.10) and (3.16), we obtain that

$$\|u_\lambda - u_\mu\| \leq \left(1 - (\omega(\lambda))^{1/(1-\theta)}\right) \|u_\mu\|, \tag{3.17}$$

which together with the fact that $\omega(\lambda) \rightarrow 1$ as $\lambda \rightarrow \mu$ shows that

$$\|u_\lambda - u_\mu\| \rightarrow 0 \quad \text{as } \lambda \rightarrow \mu \text{ with } \lambda < \mu. \tag{3.18}$$

Similarly, we can also prove that

$$\|u_\lambda - u_\mu\| \rightarrow 0 \quad \text{as } \lambda \rightarrow \mu \text{ with } \lambda > \mu. \tag{3.19}$$

Hence, (P3) holds. \square

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