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Existence and stability of solitary waves for the generalized Korteweg-de Vries equations

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Abstract

In this paper, we consider the fractional Korteweg-de Vries equations with general nonlinearities. By studying constrained minimization problems and applying the method of concentration-compactness, we obtain the existence of solitary waves for the generalized Korteweg-de Vries equations under some assumptions of the nonlinear term. Moreover, we prove that the set of minimizers is a stable set for the initial value problem of the equations, in the sense that a solution which starts near the set will remain near it for all time.

Keywords: generalized Korteweg-de Vries equations; constrained minimization problems; concentration-compactness; stability

1 Introduction

This paper is devoted to studying the existence and stability of solitary wave solutions of the generalized Korteweg-de Vries equation

$$u_t + \big(f(u)\big)_x - \big(L(u)\big)_x = 0 \quad \text{in } \mathbb{R}, \tag{1.1}$$

where $f(u)$ satisfies the following assumption:

(A) $f(u) \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $\lim_{u\to 0} \frac{f(u)}{|u|} = 0$ and $\lim_{|u|\to\infty} \frac{f(u)}{|u|^\gamma} = 0$ for some $1 < \gamma < 1 + 4\alpha$,

$$\widehat{L(u)}(\xi) = |\xi|^{2\alpha}\widehat{u}(\xi),$$

$0 < \alpha \leq 1$, the Fourier transform $\mathcal{F}\psi(\xi) = \widehat{\psi}(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} u(x)e^{-i\xi\cdot x}\,dx$.

When $f(u) = \frac{1}{2}u^2$ and $\alpha = 1$, equation (1.1) is the well-known Korteweg-de Vries equation, introduced by Korteweg and de Vries in 1895 (*cf.* [1]). The existence and stability of solitary waves of the Korteweg-de Vries equation is considered by Benjamin in [2]. Recently, in [3], Pelinovsky obtained a Korteweg-de Vries equation with a forcing term, which is a simple analytical model of tsunami generation by submarine landslides.

Here, we consider the generalized Korteweg-de Vries equation (1.1). Let $F(u) = \int_0^u f(s)\,ds$. Since the functionals

$$E(u) = \int_{-\infty}^{+\infty}\left[\frac{1}{2}uL(u) - F(u)\right]dx$$

and

$$Q(u) = \frac{1}{2}\int_{-\infty}^{+\infty} u^2\,dx$$

are two conserved quantities with (1.1), for studying the existence of solitary wave solutions to (1.1), by the variational methods, the solitary wave solutions to equation (1.1) will be founded as minimizers of

$$I_q := \inf\bigl\{E(u);\ u \in H^1(\mathbb{R}), Q(u) = q\bigr\}, \tag{I_q}$$

where $q > 0$. Denote the set of minimizers of the problem (I_q) by

$$G_q := \bigl\{u; u \in H^1(\mathbb{R}), E(u) = I_q, Q(u) = q\bigr\}. \tag{G_q}$$

Inspired by the methods used in [4, 5], by studying the problem (I_q), we obtain the existence of solitary waves for equation (1.1) with some special nonlinearities $f(u) = \frac{1}{p}u^p$, where $1 < p < 1 + 4\alpha$, and general nonlinearities satisfying the assumption (A). Moreover, we prove that the set G_q of minimizers is a stable set for the initial value problem of equation (1.1) in the sense that a solution which starts near the set will remain near it for all time. In order to obtain those results, we have to overcome one main difficulty: the minimization problem (I_q) is given in the unbounded domain $\mathbb{R}$ which results in the loss of compactness. As is done in [4, 6], we overcome the difficulty of loss of compactness by the method of concentration-compactness introduced by Lions in [7, 8] for solving some minimization problems in unbounded domains.

Now we give our main results.

Theorem 1.1 *Suppose that* $\alpha = 1$ *and* $f(u)$ *satisfies condition* (A) *and* $I_{q_0} < 0$ *for some* $q_0 > 0$. *Then there exists* $0 < q^* \leq q_0$ *such that* G_{q^*} *is not empty. Moreover, if* $\{u_n\}$ *is a minimizing sequence for the problem* (I_{q^*}), *then there exist a sequence* $\{y_n\} \subset \mathbb{R}$ *and* $g \in G_{q^*}$ *such that* $\{u_n(\cdot + y_n)\}$ *contains a subsequence converging strongly in* $H^1(\mathbb{R})$ *to* g, *and*

$$\lim_{n\to+\infty} \inf_{g\in G_{q^*}} \|u_n - g\| = 0,$$

where $\|\cdot\|$ *is the norm of* $H^1(\mathbb{R})$.

Theorem 1.2 *Under the assumptions of Theorem* 1.1, *the set* G_{q^*} *is* $H^1(\mathbb{R})$*-stable with respect to equation* (1.1), *i.e., for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if*

$$\inf_{g\in G_{q^*}} \|u_0 - g\| < \delta,$$

then the solution $u(x,t)$ *to equation* (1.1) *with initial data* u_0 *satisfies*

$$\inf_{g\in G_{q^*}} \bigl\|u(t,\cdot) - g\bigr\| < \varepsilon$$

for any $t \in [0, T)$.

Theorem 1.3 *Suppose that* $0 < \alpha < 1$ *and* $f(u)$ *satisfies condition* (A) *and* $I_{q_0} < 0$ *for some* $q_0 > 0$. *Then there exists* $0 < q^* \leq q_0$ *such that* G_{q^*} *is not empty. Moreover, if* $\{u_n\}$ *is a minimizing sequence for the problem* (I_{q^*}), *then there exists a sequence* $\{y_n\} \subset \mathbb{R}$ *and* $g \in G_{q^*}$ *such that* $\{u_n(\cdot + y_n)\}$ *contains a subsequence converging strongly in* $H^\alpha(\mathbb{R})$ *to* g, *and*

$$\lim_{n\to+\infty} \inf_{g\in G_{q^*}} \|u_n - g\|_{\alpha,2} = 0,$$

where $\|\cdot\|_{\alpha,2}$ *is the norm of* $H^\alpha(\mathbb{R})$ *given in Section* 5.

Theorem 1.4 *Under the assumptions of Theorem* 1.1, *the set* G_{q^*} *is* $H^\alpha(\mathbb{R})$*-stable with respect to equation* (1.1), *i.e., for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if*

$$\inf_{g\in G_{q^*}} \|u_0 - g\|_{\alpha,2} < \delta,$$

then the solution $u(x,t)$ *to equation* (1.1) *with initial data* u_0 *satisfies*

$$\inf_{g\in G_{q^*}} \big\|u(t,\cdot) - g\big\|_{\alpha,2} < \varepsilon$$

for any $t \in [0, T)$.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we study the existence and stability of solitary waves of equation (1.1) with some special nonlinearities $f(u) = \frac{1}{p}u^p$. Section 4 is devoted to studying equation (1.1) with general nonlinearities $f(u)$ satisfying the assumption (A). We shall consider the existence and stability of solitary waves of equation (1.1) with $0 < \alpha < 1$ in Section 5.

2 Some preliminaries

At first, we give some notations. The set of all integers and the set of all real numbers are written as $\mathbb{Z}$ and $\mathbb{R}$, respectively. And all the integrals will be taken over $\mathbb{R}$ unless specified. $L^p(\mathbb{R})$ denotes the usual Lebesgue space with the norm $|\cdot|_p$ given by

$$|\cdot|_p = \left(\int |u|^p \, dx\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < +\infty.$$

The Sobolev space $H^1(\mathbb{R})$ is defined by

$$H^1(\mathbb{R}) := \big\{u : u \in L^2(\mathbb{R}) \text{ and } u_x \in L^2(\mathbb{R})\big\},$$

whose norm is given by

$$\|\cdot\| = \left(\int \big(|u_x|^2 + |u|^2\big) \, dx\right)^{\frac{1}{2}}.$$

Now, we give Lemma 2.1 and Lemma 2.2 which will be used to study the behavior of the minimizing sequence for the problem (I_q). Lemma 2.2 is due to Lions [7, 8].

Lemma 2.1 *Suppose that $B > 0$ and $\delta > 0$ are given. Then there exists $\eta = \eta(B,\delta)$ such that if $u \in H^1(\mathbb{R})$ with $\|u\| \le B$ and $|u|_{p+1} \ge \delta$, then*

$$\sup_{y\in\mathbb{R}} \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} |u|^{p+1}\,dx \ge \eta.$$

Proof We have

$$\sum_{j\in\mathbb{Z}} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \bigl[(u_x)^2 + u^2\bigr]\,dx = \|u\|^2 \le \frac{B^2}{|u|_{p+1}^{p+1}} |u|_{p+1}^{p+1} = \sum_{j\in\mathbb{Z}} \frac{B^2}{|u|_{p+1}^{p+1}} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} |u|^{p+1}\,dx.$$

Therefore, there exists some $j_0 \in \mathbb{Z}$ such that

$$\int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} \bigl[(u_x)^2 + u^2\bigr]\,dx \le \frac{B^2}{|u|_{p+1}^{p+1}} \int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} |u|^{p+1}\,dx.$$

Applying the Sobolev embedding theorem [9], there exists a constant A such that

$$\begin{aligned}\left(\int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} \bigl[(u_x)^2 + u^2\bigr]\,dx\right)^{\frac{1}{p+1}} &\le A\left(\int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} \bigl[(u_x)^2 + u^2\bigr]\,dx\right)^{\frac{1}{2}} \\ &\le \frac{AB}{|u|_{p+1}^{\frac{p+1}{2}}} \left(\int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} |u|^{p+1}\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, we obtain

$$\int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} |u|^{p+1}\,dx \ge \left(\frac{|u|_{p+1}^{\frac{p+1}{2}}}{AB}\right)^{\frac{2(p+1)}{p-1}} \ge \frac{\delta^{\frac{(p+1)^2}{p-1}}}{(AB)^{\frac{2(p+1)}{p-1}}}.$$

Taking $\eta = \frac{\delta^{\frac{(p+1)^2}{p-1}}}{(AB)^{\frac{2(p+1)}{p-1}}}$, it follows that

$$\sup_{y\in\mathbb{R}} \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} |u|^{p+1}\,dx \ge \int_{j_0-\frac{1}{2}}^{j_0+\frac{1}{2}} |u|^{p+1}\,dx \ge \eta.$$

□

Lemma 2.2 *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R})$ such that*

$$\sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u_n|^2\,dx \to 0 \quad \text{as } n \to +\infty$$

for some $r > 0$. Then $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$.

Proof Let $2 < s < \infty$. Without loss of generality, we may assume $r = 1$. It follows from the interpolation inequalities that

$$|u|_{L^s(B(y,1))} \le A|u|_{L^2(B(y,1))}^{\frac{s+2}{2}} \|u\|_{H^1(B(y,1))}^{\frac{s-2}{2}},$$

where $A > 0$ is a constant independent of u.

Covering $\mathbb{R}$ by a family of intervals $(y_i - 1, y_i + 1)$ such that each point of $\mathbb{R}$ is contained in at most two such intervals and summing this inequality over this family of intervals, we get

$$|u|_{L^s}^s \le 2A\left(\sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u|^2\,dx\right)^{\frac{s+2}{4}} \|u\|^{\frac{s-2}{2}}.$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R})$ and $\sup_{y\in\mathbb{R}} \int_{y-1}^{y+1} |u_n|^2\,dx \to 0$ as $n \to +\infty$, applying the above inequality, we know that $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$. □

Next, we establish a convergence result that will be used in the proof of Theorem 1.3.

Lemma 2.3 *Let $f \in C(\mathbb{R}, \mathbb{R})$ and suppose that*

$$\bigl|f(t)\bigr| \le C\bigl(|t| + |t|^{p_1}\bigr) \quad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

where $1 < p_1 < \infty$. If $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R})$ and $u_n \to u_0$ a.e. on $\mathbb{R}$, then

$$\lim_{n\to\infty}\left[\int_{-\infty}^{+\infty} F(u_n)\,dx - \int_{-\infty}^{+\infty} F(u_0)\,dx - \int_{-\infty}^{+\infty} F(u_n - u_0)\,dx\right] = 0,$$

where $F(u) = \int_0^u f(s)\,ds$.

Proof Let $R > 0$. Applying the mean value theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} F(u_n)\,dx \\
&\quad = \int_{|x|<R} F(u_n)\,dx + \int_{|x|\ge R} F\bigl(u_0 + (u_n - u_0)\bigr)\,dx \\
&\quad = \int_{|x|<R} F(u_n)\,dx + \int_{|x|\ge R} \bigl[F(u_n - u_0)\,dx + f(u_n - u_0 + \theta u_0)u_0\bigr]\,dx,
\end{aligned}$$

where $0 < \theta < 1$ is dependent on x and R. Now we write

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} F(u_n)\,dx - \int_{-\infty}^{+\infty} F(u_0)\,dx - \int_{-\infty}^{+\infty} F(u_n - u_0)\,dx\right| \\
&\quad \le \left|\int_{|x|<R} \bigl[F(u_n) - F(u_0)\bigr]\,dx\right| + \left|\int_{|x|\ge R} F(u_0)\,dx\right| + \left|\int_{|x|<R} F(u_n - u_0)\,dx\right| \\
&\qquad + \left|\int_{|x|\ge R} f(u_n - u_0 + \theta u_0)u_0\,dx\right|.
\end{aligned} \tag{2.2}$$

It follows from (2.1), the mean value theorem and the Hölder inequality that

$$\begin{aligned}
&\left|\int_{|x|<R} \bigl[F(u_n) - F(u_0)\bigr]\,dx\right| \\
&\quad = \left|\int_{|x|<R} f\bigl(u_0 + \theta(u_n - u_0)\bigr)(u_n - u_0)\,dx\right|
\end{aligned}$$

$$\begin{aligned}
&\le C\int_{|x|<R}\bigl|u_0+\theta(u_n-u_0)\bigr||u_n-u_0|\,dx \\
&\quad + C\int_{|x|<R}\bigl|u_0+\theta(u_n-u_0)\bigr|^{p_1}|u_n-u_0|\,dx \\
&\le C\int_{|x|<R}|u_0||u_n-u_0|\,dx + C\int_{|x|<R}|u_n-u_0|^2\,dx \\
&\quad + \varepsilon\int_{|x|<R}|u_0|^{p_1}|u_n-u_0|\,dx + C_\varepsilon\int_{|x|<R}|u_n-u_0|^{p_1+1}\,dx \\
&\le C\biggl(\int_{|x|<R}|u_0|^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{|x|<R}|u_n-u_0|^2\,dx\biggr)^{\frac{1}{2}} + C\int_{|x|<R}|u_n-u_0|^2\,dx \\
&\quad + \varepsilon\biggl(\int_{|x|<R}|u_0|^{2p_1}\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{|x|<R}|u_n-u_0|^2\,dx\biggr)^{\frac{1}{2}} \\
&\quad + C_\varepsilon\int_{|x|<R}|u_n-u_0|^{p_1+1}\,dx
\end{aligned} \tag{2.3}$$

and

$$\biggl|\int_{|x|<R}F(u_n-u_0)\,dx\biggr| \le C\int_{|x|<R}|u_n-u_0|^2\,dx + C\int_{|x|<R}|u_n-u_0|^{p_1+1}\,dx. \tag{2.4}$$

Since the embedding $H^1(\mathbb{R}) \hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R})$ $(2 \le s < \infty)$ is compact, the inequalities (2.3) and (2.4) imply that

$$\biggl|\int_{|x|<R}\bigl[F(u_n)-F(u_0)\bigr]\,dx\biggr| \to 0 \quad \text{as } n\to\infty, \tag{2.5}$$

$$\biggl|\int_{|x|<R}F(u_n-u_0)\,dx\biggr| \to 0 \quad \text{as } n\to\infty. \tag{2.6}$$

Similarly, by the Hölder inequality, the Sobolev embedding theorem and (2.1), we get

$$\begin{aligned}
&\biggl|\int_{|x|\ge R}f(u_n-u_0+\theta u_0)u_0\,dx\biggr| \\
&\quad \le C\int_{|x|\ge R}|u_n-u_0+\theta u_0||u_0|\,dx + C\int_{|x|\ge R}|u_n-u_0+\theta u_0|^{p_1}|u_0|\,dx \\
&\quad \le C\biggl(\int_{|x|\ge R}|u_n|^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{|x|\ge R}|u_0|^2\,dx\biggr)^{\frac{1}{2}} + C\int_{|x|\ge R}|u_0|^2\,dx \\
&\qquad + \varepsilon\biggl(\int_{|x|\ge R}|u_n|^{2p_1}\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{|x|\ge R}|u_0|^2\,dx\biggr)^{\frac{1}{2}} + C_\varepsilon\int_{|x|\ge R}|u_0|^{p_1+1}\,dx \\
&\quad \le C\|u_n\|\biggl(\int_{|x|\ge R}|u_0|^2\,dx\biggr)^{\frac{1}{2}} + C\int_{|x|\ge R}|u_0|^2\,dx + \varepsilon\|u_n\|^{p_1}\biggl(\int_{|x|\ge R}|u_0|^2\,dx\biggr)^{\frac{1}{2}} \\
&\qquad + C_\varepsilon\int_{|x|\ge R}|u_0|^{p_1+1}\,dx.
\end{aligned}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R})$, we see that

$$\left|\int_{|x|\geq R} f(u_n - u_0 + \theta u_0)u_0\,dx\right| \to 0 \quad \text{as } R \to \infty. \tag{2.7}$$

Hence, combining (2.2), (2.5), (2.6) and (2.7), we obtain

$$\int_{-\infty}^{+\infty} F(u_n)\,dx - \int_{-\infty}^{+\infty} F(u_0)\,dx - \int_{-\infty}^{+\infty} F(u_n - u_0)\,dx \to 0 \quad \text{as } n \to \infty. \qquad \square$$

3 The case of special nonlinearity

In this section, we only consider the case of $\alpha = 1$ and $f(u) = \frac{1}{p}u^p$, where $1 < p < 5$. Correspondingly,

$$E(u) = \int_{-\infty}^{+\infty} \left[\frac{1}{2}(u_x)^2 - \frac{1}{p}\frac{1}{p+1}u^{p+1}\right] dx.$$

At first, we commence by studying some properties of the functional $I_q : (0, +\infty) \to \mathbb{R}$ and the behavior of the minimizing sequences for the problem (I_q).

Lemma 3.1 *For any $q > 0$,*

(i) $-\infty < I_q < 0$;

(ii) *If $\{u_n\}$ is a minimizing sequence for the problem (I_q), there exists a constant $B > 0$ such that $\|u_n\| \leq B$ for all n;*

(iii) *If $\{u_n\}$ is a minimizing sequence for the problem (I_q), there exist a positive constant δ and a sequence $\{y_n\}$ of real numbers such that*

$$\int_{y_n-\frac{1}{2}}^{y_n+\frac{1}{2}} |u_n|^{p+1}\,dx \geq \delta$$

for sufficiently large n.

Proof (i) Choose any function $u \in H^1(\mathbb{R})$ such that $Q(u) = q$ and $\int u^{p+1}\,dx \neq 0$. For any $\theta > 0$, define $u_\theta(x) = \sqrt{\theta}u(\theta x)$. Then we have

$$Q(u_\theta) = q$$

and

$$E(u_\theta) = \frac{1}{2}\theta^2 \int (u_x)^2\,dx - \frac{1}{p(p+1)}\theta^{\frac{p-1}{2}} \int u^{p+1}\,dx.$$

For $1 < p < 5$, by taking $\theta > 0$ sufficiently small, we get $I_q \leq E(u_\theta) < 0$.

Next we prove $I_q > -\infty$. Let $u \in H^1(\mathbb{R})$ such that $Q(u) = q$. By the Sobolev embedding theorems and interpolation inequalities, we get

$$\left|\int u^{p+1}\,dx\right| \leq |u|_{p+1}^{p+1} \leq A|u|_{\frac{P-1}{2(P+1)}}^{P+1} \leq A\|u\|^{\frac{p-1}{2}}|u|_2^{\frac{p+3}{2}}, \tag{3.1}$$

where A denotes various constants which are independent of u. Using the Young inequality, we derive from (3.1)

$$\left|\int u^{p+1}\,dx\right| \le \varepsilon\|u\|^2 + A_\varepsilon|u|_2^{\frac{2p+6}{5-p}} \le \varepsilon\|u\|^2 + A_{\varepsilon,q},$$

where $\varepsilon > 0$ is arbitrary and $A_{\varepsilon,q}$ depends on ε and q, but not on u. Therefore,

$$\begin{aligned} E(u) &= E(u) + Q(u) - Q(u) \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{p(p+1)}\int u^{p+1}\,dx - \frac{1}{2}\int u^2\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{p(p+1)}\|u\|^2 - \frac{1}{p(p+1)}A_{\varepsilon,q} - q. \end{aligned}$$

Choosing $\varepsilon < \frac{p(p+1)}{2}$, we obtain the lower bound of the functional E

$$E(u) \ge -\frac{1}{p(p+1)}A_{\varepsilon,q} - q,$$

which implies $I_q \ge -\frac{1}{p(p+1)}A_{\varepsilon,q} - q > -\infty$.

(ii) Let $\{u_n\}$ be a minimizing sequence for the problem (I_q). Then, by (3.1), we have

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 &= E(u_n) + Q(u_n) + \frac{1}{p(p+1)}\int u_n^{p+1}\,dx \\ &\le \sup_n E(u_n) + q + \frac{1}{p(p+1)}|u_n|_{p+1}^{p+1} \\ &\le A + q + A\|u\|^{\frac{p-1}{2}}|u|_2^{\frac{p+3}{2}} \le A\bigl(1 + \|u\|^{\frac{p-1}{2}}\bigr), \end{aligned}$$

where A denotes various constants which are independent of n. Since $1 < p < 5$, the existence of the desired bound B follows.

(iii) Let $\{u_n\}$ be a minimizing sequence for the problem (I_q). Then we claim: there exists a constant $\eta > 0$ such that $|u_n|_{p+1} \ge \eta$ for all sufficiently large n. We argue by contradiction: if no such $\eta > 0$ exists, then $\liminf_{n\to\infty}\int |u_n|^{p+1}\,dx \le 0$. Hence

$$I_q = \lim_{n\to\infty} E(u_n) \ge -\liminf_{n\to\infty}\int |u_n|^{p+1}\,dx \ge 0,$$

which contradicts (i). So, the claim is achieved.

Combining (ii) and Lemma 2.1, there exist a positive constant δ and a sequence $\{y_n\}$ of real numbers such that

$$\int_{y_n-\frac{1}{2}}^{y_n+\frac{1}{2}} |u_n|^{p+1}\,dx \ge \delta$$

for sufficiently large n. The proof of Lemma 3.1 is completed. □

The next lemma will establish a subadditivity inequality which will be a crucial step in the proof of the existence minimizer for the problem (I_q).

Lemma 3.2 *For all* $q_1, q_2 > 0$, $I_{q_1+q_2} < I_{q_1} + I_{q_2}$.

Proof For given $u \in H^1(\mathbb{R})$, $|u|_2^2 = q_1$, $\theta > 0$, let $u_\theta(x) = \theta^{\frac{2}{p+1}} u(\theta^{\frac{p-1}{p+1}} x)$, where $\theta = (\frac{q_2}{q_1})^{\frac{p+1}{5-p}}$. Then it follows that

$$Q(u_\theta) = \frac{q_2}{q_1} Q(u) = q_2,$$

and

$$E(u_\theta) = \left(\frac{q_2}{q_1}\right)^{\frac{p+3}{5-p}} E(u).$$

Hence we get

$$I_{q_2} = \inf\left\{\left(\frac{q_2}{q_1}\right)^{\frac{p+3}{5-p}} E(u) : Q(u) = q_1\right\} = \left(\frac{q_2}{q_1}\right)^{\frac{p+3}{5-p}} I_{q_1}. \tag{3.2}$$

Now, from (3.2) and Lemma 3.1, we obtain for all $q_1, q_2 > 0$,

$$I_{q_1+q_2} = (q_2 + q_1)^{\frac{p+3}{5-p}} I_1 < \left(q_2^{\frac{p+3}{5-p}} + q_1^{\frac{p+3}{5-p}}\right) I_1 = I_{q_1} + I_{q_2}. \qquad \square$$

Now we formulate the following two theorems, which are special cases corresponding to Theorem 1.1 and Theorem 1.2, and give their proof with the aim of Lemma 3.1 and Lemma 3.2.

Theorem 3.1 *Let $\alpha = 1$ and $f(u) = \frac{1}{p} u^p$, where $1 < p < 5$. For any $q > 0$, the set G_q is not empty. Moreover, if $\{u_n\}$ is a minimizing sequence for the problem (I_q), then there exist a sequence $\{y_n\} \subset \mathbb{R}$ and $g \in G_q$ such that $\{u_n(\cdot + y_n)\}$ contains a subsequence converging strongly in $H^1(\mathbb{R})$ to g, and*

$$\lim_{n\to+\infty} \inf_{g\in G_q} \|u_n - g\| = 0.$$

Theorem 3.2 *Let $\alpha = 1$ and $f(u) = \frac{1}{p} u^p$, where $1 < p < 5$. For any $q > 0$, the set G_q is $H^1(\mathbb{R})$-stable with respect to equation* (1.1), *i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$\inf_{g\in G_q} \|u_0 - g\| < \delta,$$

then the solution $u(x,t)$ to equation (1.1) *with initial data u_0 satisfies*

$$\inf_{g\in G_q} \left\|u(t,\cdot) - g\right\| < \varepsilon$$

for any $t \in [0, T)$.

Proof of Theorem 3.1 From (3.2), it is easy to check that I_q is continuous on $(0,\infty)$. Let $\{u_n\}$ be a minimizing sequence for the problem (I_q). By Lemma 3.1, there exist a positive constant δ and a sequence $\{y_n\}$ of real numbers such that

$$\int_{y_n-\frac{1}{2}}^{y_n+\frac{1}{2}} |u_n|^{p+1}\,dx \geq \delta$$

for sufficiently large n.

Let us define $v_n = u_n(x + y_n)$. Hence $Q(V_n) = Q(u_n) = q$, $E(v_n) = E(u_n) \to I_q$, as $n \to \infty$, and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |v_n|^{p+1}\,dx = \int_{y_n-\frac{1}{2}}^{y_n+\frac{1}{2}} |u_n|^{p+1}\,dx \geq \delta > 0. \tag{3.3}$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R})$, by Lemma 3.1, we may assume going, if necessary, to a subsequence

$$\begin{aligned} &v_n \rightharpoonup g \quad \text{in } H^1(\mathbb{R}),\\ &v_n \to g \quad \text{in } L^{p+1}_{\text{loc}}(\mathbb{R}),\\ &v_n \to g \quad \text{a.e. on } \mathbb{R}. \end{aligned} \tag{3.4}$$

Hence, by (3.3), we get $g \not\equiv 0$. And applying the Brezis-Lieb lemma [10], we have

$$|v_n|_2^2 = |v_n - g|_2^2 + |g|_2^2, \tag{3.5}$$

$$|v_n|_{p+1}^{p+1} = |v_n - g|_{p+1}^{p+1} + |g|_{p+1}^{p+1}. \tag{3.6}$$

Now we show that $Q(g) = \frac{1}{2}\int g^2\,dx = q$. In the contrary case, $0 < Q(g) = \lambda < q$. By (3.5), we obtain $\lim_{n\to\infty} q_n = \lim_{n\to\infty} Q(v_n - g) = q - \lambda$. Then it follows from (3.5) and (3.6) that

$$\begin{aligned} I_q = E(v_n) + o(1) &\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{2}|v_n|_2^2 - \frac{1}{p(p+1)}|v_n|_{p+1}^{p+1} + o(1)\\ &= E(v_n - g) + E(g) + o(1) \geq I_{q_n} + I_\lambda + o(1). \end{aligned} \tag{3.7}$$

Since I_q is continuous on $(0,\infty)$, letting $n \to \infty$, we get $I_q \geq I_{q-\lambda} + I_\lambda$, which contradicts Lemma 3.2. Therefore $Q(g) = q$. It then follows from (3.5) that

$$v_n \to g \quad \text{in } L^2(\mathbb{R}). \tag{3.8}$$

Applying the interpolation inequality, (3.4) and (3.8), we get

$$v_n \to g \quad \text{in } L^{p+1}(\mathbb{R}). \tag{3.9}$$

Using the weak low semi-continuity of the norm in $H^1(\mathbb{R})$, we know that

$$\begin{aligned} I_q &\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{2}|v_n|_2^2 - \frac{1}{p(p+1)}|v_n|_{p+1}^{p+1} + o(1)\\ &\geq \frac{1}{2}\|g\|^2 - \frac{1}{2}|v_n - g|_2^2 - \frac{1}{2}|g|_2^2 - \frac{1}{p(p+1)}|v_n - g|_{p+1}^{p+1} - \frac{1}{p(p+1)}|g|_{p+1}^{p+1} + o(1)\\ &= E(g) - \frac{1}{2}|v_n - g|_2^2 - \frac{1}{p(p+1)}|v_n - g|_{p+1}^{p+1} + o(1). \end{aligned}$$

Letting $n \to \infty$, by (3.8) and (3.9), we obtain $E(g) \leq I_q$. On the other hand, it follows from $Q(g) = q$ that $E(g) \geq I_q$. Therefore $E(g) = I_q$, which implies that g is a minimizer of the problem (I_q) (*i.e.*, $G_q \neq \emptyset$). Then it follows from (3.7), (3.8) and (3.9) that

$$v_n = u_n(\cdot + y_n) \to g \quad \text{in } H^1(\mathbb{R}).$$

We prove $\lim_{n\to+\infty} \inf_{g\in G_q} \|u_n - g\| = 0$ with an argument by contradiction. Assume that there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\inf_{g\in G_q} \|u_{n_k} - g\| \geq \varepsilon_0 > 0 \tag{3.10}$$

for all n_k. With the result of the above proof, we obtain that there exist a subsequence of $\{u_{n_k}\}$, denoted again by $\{u_{n_k}\}$, $\{y_{n_k}\} \subset \mathbb{R}$ and $g \in G_q$ such that

$$u_{n_k}(\cdot + y_{n_k}) \to g \quad \text{in } H^1(\mathbb{R}).$$

Since $g(\cdot - y_{n_k}) \in G_q$,

$$\left\| u_{n_k} - g(\cdot - y_{n_k})\right\| = \left\| u_{n_k}(\cdot + y_{n_k}) - g\right\| \to 0 \quad \text{as } n_k \to +\infty,$$

which contradicts (3.10). □

An immediate consequence of Theorem 3.1 is that G_q forms a stable set for the initial-value problem for equation (1.1).

Proof of Theorem 3.2 We prove Theorem 3.2 with an argument by contradiction. Assume that the set G_q is not $H^1(\mathbb{R})$-stable. Then there exist $\varepsilon_0 > 0$, $\{\psi_n\} \subset H^1(\mathbb{R})$ and a sequence of times $\{t_n\}$ such that

$$\inf_{g\in G_q} \|\psi_n - g\| < \frac{1}{n}, \tag{3.11}$$

and

$$\inf_{g\in G_q} \left\| u_n(\cdot, t_n) - g\right\| \geq \varepsilon_0 \tag{3.12}$$

for all n, where $u_n(x,t)$ solves equation (1.1) with $u_n(x,0) = \psi_n$.

Equation (3.11) implies that

$$E(\psi_n) \to I_q, \qquad Q(\psi_n) \to q.$$

Choose $\{\mu_n\} \subset \mathbb{R}$ such that $Q(\mu_n\psi_n) = q$ for all n. Thus $\mu_n \to 1$ as $n \to +\infty$. Hence the sequence $v_n = \mu_n u_n(\cdot, t_n)$ satisfies $Q(v_n) = q$ and

$$\lim_{n\to\infty} E(v_n) = \lim_{n\to\infty} E\big(u_n(\cdot, t_n)\big) = \lim_{n\to\infty} E(\psi_n) = I_q.$$

Therefore $\{v_n\}$ is a minimizing sequence for the problem (I_q). By Theorem 1.1, there exists $\{g_{n_k}\} \subset G_q$ such that

$$\|v_{n_k} - g_{n_k}\| < \frac{\varepsilon_0}{2} \tag{3.13}$$

for sufficiently large n_k. Since $\mu_n \to 1$ and $\|u_n(\cdot,t_n)\|$ is bounded, we derive from (3.12) and (3.13)

$$\begin{aligned}\varepsilon_0 &\le \|u_{n_k}(\cdot,t_{n_k}) - g_{n_k}\| \\ &\le \|u_{n_k}(\cdot,t_{n_k}) - \mu_{n_k}u_{n_k}(\cdot,t_{n_k})\| + \|\mu_{n_k}u_{n_k}(\cdot,t_{n_k}) - g_{n_k}\| \\ &\le \bigl(|\mu_{n_k} - 1|\bigr)\|u_{n_k}(\cdot,t_{n_k})\| + \frac{\varepsilon_0}{2} \le \frac{3}{4}\varepsilon_0 \end{aligned} \tag{3.14}$$

for sufficiently large n_k. (3.14) is a contradiction. Therefore, the set G_q is $H^1(\mathbb{R})$-stable with respect to equation (1.1). □

4 The case for more general nonlinearities

In this section, we consider (1.1) with $\alpha = 1$ and more general nonlinearities f satisfying condition (A). At first, we study the properties of the functional $I_q : (0,\infty) \to \mathbb{R}$ and the minimizing sequence of the problem (I_q).

Lemma 4.1

(i) *For any* $q > 0$, I_q *is finite and continuous on* $(0,\infty)$. *Moreover, each minimizing sequence for* (I_q) *is bounded*;

(ii) $I_q \le 0$ *for any* $q > 0$.

Proof (i) According to assumption (A), we observe that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\bigl|F(u)\bigr| \le \varepsilon|u|^2 + \varepsilon|u|^{\gamma+1} + C_\varepsilon|u|^\alpha, \tag{4.1}$$

where $2 < \alpha < \gamma + 1$. By the Sobolev embedding theorems and interpolation inequalities, we obtain

$$\int_{-\infty}^{+\infty} |u|^\alpha\,dx \le A\|u\|^{\frac{\alpha-2}{2}}|u|_2^{\frac{\alpha+2}{2}} \tag{4.2}$$

and

$$\int_{-\infty}^{+\infty} |u|^{\gamma+1}\,dx \le A\|u\|^{\frac{\gamma-1}{2}}|u|_2^{\frac{\gamma+3}{2}}, \tag{4.3}$$

where $A > 0$ is independent of u. Then using the Young inequality, we can derive from (4.2) and (4.3) that for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$\int_{-\infty}^{+\infty} |u|^\alpha\,dx \le \eta\|u\|^2 + C_\eta|u|_2^{\frac{2(\alpha+2)}{6-\alpha}} \tag{4.4}$$

and

$$\int_{-\infty}^{+\infty} |u|^{\gamma+1}\,dx \le \eta\|u\|^2 + C_\eta|u|_2^{\frac{2\gamma+6}{5-\gamma}}. \tag{4.5}$$

Let $u \in H^1(\mathbb{R})$ such that $Q(u) = q$. It follows from (4.1), (4.4) and (4.5) that

$$\begin{aligned} E(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{-\infty}^{+\infty} u^2\,dx - \int_{-\infty}^{+\infty} F(u)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}|u|_2^2 - \varepsilon|u|_2^2 - \varepsilon\eta\|u\|^2 - \varepsilon C_\eta |u|_2^{\frac{2\gamma+6}{5-\gamma}} - C_\varepsilon\eta\|u\|^2 - C_\varepsilon C_\eta |u|_2^{\frac{2(\alpha+2)}{6-\alpha}} \\ &\geq \left(\frac{1}{2} - \varepsilon\eta - C_\varepsilon\eta\right)\|u\|^2 - C_{\varepsilon,\eta,q}, \end{aligned} \tag{4.6}$$

where $C_{\varepsilon,\eta,q}$ is a positive constant dependent only on ε and η for given $q > 0$. Choosing $\varepsilon > 0$ and $\eta > 0$ such that $\frac{1}{2} - \varepsilon\eta - C_\varepsilon\eta > 0$, we see that $I_q > -\infty$.

Since $|\theta u|_2^2 = \theta^2|u|_2^2$ for $\theta > 0$, it is easy to check that I_q is continuous on $(0, \infty)$.

Let $\{u_n\}$ be a minimizing sequence for the problem (I_q). From (4.6) and the fact that I_q is finite, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R})$.

(ii) For given $u \in H^1(\mathbb{R})$ such that $Q(u) = q$, let $u_\theta(x) = \frac{1}{\sqrt{\theta}}u(\frac{x}{\theta})$ for $\theta > 0$. We obtain that

$$|u_\theta|_2^2 = |u|_2^2 = q \tag{4.7}$$

and

$$E(u_\theta) = \frac{1}{2\theta^2}\int_{-\infty}^{+\infty}(u_x)^2\,dx - \theta\int_{-\infty}^{+\infty} F\left(\frac{1}{\sqrt{\theta}}u\right)dx. \tag{4.8}$$

Combining (4.1), (4.7) and (4.8), we obtain

$$I_q \leq E(u_\theta) \to 0 \quad \text{as } \theta \to +\infty. \qquad \square$$

Lemma 4.2 *Suppose that $I_{q_0} < 0$ for some $q_0 > 0$. Then the following two properties hold*:

(i) *$\frac{I_q}{q}$ is non-increasing on $(0, +\infty)$ and $\lim_{q\to 0^+}\frac{I_q}{q} = 0$*;

(ii) *there exists $q_1 \leq q_0$ such that*

$$\frac{I_q}{q} > \frac{I_{q_0}}{q_0} \quad \textit{for } q \in (0, q_1).$$

Proof First we observe that if $\sigma > 0$ and $\beta > 0$ with $Q(u) = \beta$ and $u_\sigma(x) = u(\frac{1}{\sigma}x)$, then $Q(u_\sigma) = \sigma\beta$ and

$$E(u_\sigma) = \frac{1}{2\sigma}\int_{-\infty}^{+\infty}(u_x)^2\,dx - \sigma\int_{-\infty}^{+\infty} F(u)\,dx.$$

Consequently, for $q_1 > 0$ and $q_2 > 0$, we have

$$I_{q_2} = \inf\left\{\frac{q_1}{2q_2}\int_{-\infty}^{+\infty}(u_x)^2\,dx - \frac{q_2}{q_1}\int_{-\infty}^{+\infty} F(u)\,dx, Q(u) = q_1\right\}.$$

If $q_1 > q_2 > 0$, then for each $\varepsilon > 0$, there exists $u \in H^1(\mathbb{R})$ with $Q(u) = q_1$ such that

$$I_{q_2} + \varepsilon > \frac{1}{2}\frac{q_1}{q_2}\int_{-\infty}^{+\infty}(u_x)^2\,dx - \frac{q_2}{q_1}\int_{-\infty}^{+\infty} F(u)\,dx > \frac{q_2}{q_1}E(u) \geq \frac{q_2}{q_1}I_{q_1}. \tag{4.9}$$

This inequality yields $\frac{I_{q_2}}{q_2} \geq \frac{I_{q_1}}{q_1}$ for $0 < q_2 < q_1$.

Since $I_q \le 0$ for all $q > 0$, we see that

$$\lim_{q\to 0^+} \frac{I_q}{q} = A \le 0.$$

We claim that $A = 0$. Letting $\varepsilon = q^2$, $0 < q \le q_0$, from (4.9), there exists $u^{(q)} \in H^1(\mathbb{R})$ with $Q(u^{(q)}) = q_0$ such that

$$\begin{aligned} I_q + q^2 &\ge \frac{1}{2}\frac{q_0}{q}\int_{-\infty}^{+\infty} \bigl(u_x^{(q)}\bigr)^2\,dx - \frac{q}{q_0}\int_{-\infty}^{+\infty} F\bigl(u^{(q)}\bigr)\,dx \\ &\ge \frac{q}{q_0}\biggl[\frac{1}{2}\int_{-\infty}^{+\infty} \bigl(u_x^{(q)}\bigr)^2\,dx - \int_{-\infty}^{+\infty} F\bigl(u^{(q)}\bigr)\,dx\biggr]. \end{aligned} \tag{4.10}$$

It follows from (4.6) and (4.10) that

$$I_q + q^2 \ge \frac{q}{q_0}\biggl(C_1(q_0)\int_{-\infty}^{+\infty} \bigl(u_x^{(q)}\bigr)^2\,dx - C_2(q_0)\biggr),$$

where $C_1(q_0) > 0$ and $C_2(q_0) > 0$ are constants independent of q_0. Hence we obtain $q_0^2 \ge C_1(q_0)\int_{-\infty}^{+\infty}(u^{(q)})_x^2\,dx - C_2(q_0)$, which implies

$$\int_{-\infty}^{+\infty} \bigl(u_x^{(q)}\bigr)^2\,dx \le C_3(q_0), \tag{4.11}$$

where $C_3(q_0)$ is dependent only on q_0. Combining (4.1), (4.4), (4.5) and (4.11), we also get

$$\left|\int_{-\infty}^{+\infty} F\bigl(u^{(q)}\bigr)\,dx\right| \le C_4(q_0), \tag{4.12}$$

where $C_4(q_0)$ is dependent only on q_0.

We claim that

$$\int_{-\infty}^{+\infty} \bigl(u_x^{(q)}\bigr)^2\,dx \to 0 \quad \text{as } q \to 0^+. \tag{4.13}$$

Indeed, if there exist $\varepsilon_0 > 0$ and $q_n \to 0^+$ such that $\int_{-\infty}^{+\infty}(u_x^{(q_n)})^2\,dx \ge \varepsilon_0$, then by (4.11) and (4.12), we obtain

$$\frac{I_{q_n}}{q_n} + q_n \ge \frac{q_0\varepsilon_0}{2}\frac{1}{q_n^2} - \frac{1}{q_0}C_4(q_0) \to +\infty \quad \text{as } q_n \to 0^+,$$

which contradicts $\lim_{q\to 0^+}\frac{I_q}{q} = A \le 0$. Therefore (4.13) is achieved and this implies that $\lim_{q\to 0^+}\int_{-\infty}^{+\infty} F(u^{(q)})\,dx = 0$ and, consequently,

$$\frac{I_q}{q} + q \ge -\frac{1}{q_0}\int_{-\infty}^{+\infty} F\bigl(u^{(q)}\bigr)\,dx \to 0 \quad \text{as } q \to 0^+.$$

This shows that $\lim_{q\to 0^+}\frac{I_q}{q} = 0$.

(2) We observe that $\lim_{q\to 0^+}\frac{I_q}{q} = 0 > \frac{I_{q_0}}{q_0}$, which implies (ii). $\square$

Then we establish a subadditivity inequality similar to Lemma 3.2 with the aim of Lemma 4.2.

Lemma 4.3 *Suppose that* $I_{q_0} < 0$ *for some* $q_0 > 0$. *Then there exists* $0 < q^* \leq q_0$ *such that* $I_{q^*} < I_{q^*-q} + I_q$ *for* $0 < q < q^*$.

Proof According to Lemma 4.2, the set

$$\left\{q_1 | q_1 \leq q_0 \text{ and } \frac{I_q}{q} > \frac{I_{q_0}}{q_0} \text{ or each } q \in (0, q_1)\right\}$$

is nonempty. We define

$$q^* = \sup\left\{q_1 | q_1 \leq q_0 \text{ and } \frac{I_q}{q} > \frac{I_{q_0}}{q_0} \text{ for each } q \in (0, q_1)\right\}. \tag{4.14}$$

It follows from the continuity of I_q and $\lim_{q\to 0^+} \frac{I_q}{q} = 0$ that $0 < q^* \leq q_0$,

$$I_{q^*} = \frac{q^*}{q_0} I_{q_0} < 0, \tag{4.15}$$

$$I_q > \frac{q}{q_0} I_{q_0} \quad \text{for all } q \in \bigl(0, q^*\bigr). \tag{4.16}$$

Therefore,

$$I_{q^*} = \frac{q^*}{q_0} I_{q_0} = \frac{q^* - q}{q_0} I_{q_0} + \frac{q}{q_0} I_{q_0} < I_{q^*-q} + I_q,$$

for all $q \in (0, q^*)$. □

Now we give the proof of Theorem 1.1 with the aim of Lemma 4.1, Lemma 4.2 and Lemma 4.3. Since the proof of Theorem 1.1 is similar to that of Theorem 3.1, we only give the sketch of the proof.

Proof of Theorem 1.1 Let $\{u_n\}$ be a minimizing sequence of I_{q^*}, where q^* is defined in (4.14). Since $\{u_n\}$ is bounded, we may assume

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}),$$

$$u_n \to u \quad \text{a.e. on } \mathbb{R}.$$

First, we consider the case $u = 0$. In this case, by Lemma 2.2, either

(a) $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$, or

(b) there exists a sequence $\{y_n\} \subset \mathbb{R}$ such that

$$\upsilon_n(x) = u_n(x + y_n) \rightharpoonup g \neq 0 \quad \text{in } H^1(\mathbb{R}).$$

In the case (a), combining Lemma 2.2 and condition (A), we obtain

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} F(u_n)\, dx = 0$$

and, consequently,

$$I_{q^*} = \lim_{n\to\infty} E(u_n) = \lim_{n\to\infty}\left[\frac{1}{2}\int_{-\infty}^{+\infty}(u_{nx})^2\,dx - \int_{-\infty}^{+\infty}F(u_n)\,dx\right] \geq 0,$$

which contradicts Lemma 4.1. Hence (b) holds. Then it follows from Lemma 2.3 and Lemma 4.3 that g is the minimizer for the problem (I_{q^*}) (*i.e.*, $G_{q^*} \neq \emptyset$) and the result of Theorem 1.1 holds. The proof is similar to that of Theorem 3.1, we omit the details.

If $u \neq 0$, we repeat the previous argument in the case (b) to obtain the result of Theorem 1.1. □

Proof of Theorem 1.2 Theorem 1.2 is an immediate result of Theorem 1.1. We can prove it with an argument similar to that of Theorem 3.2. Here, we omit the details of the proof. □

5 The case for $0 < \alpha < 1$

In this section, we only consider the case of $0 < \alpha < 1$, *i.e.*, we consider the existence and stability of solitary waves for the fractional Korteweg-de Vries equations with general nonlinearities. At first, we give the definition of $H^\alpha(\mathbb{R})$. The fractional order Sobolev space $H^\alpha(\mathbb{R})$ is defined by

$$H^\alpha(\mathbb{R}) := \left\{u; u:\mathbb{R}\to\mathbb{C},\ u\in L^2(\mathbb{R}) \text{ and } \mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{\frac{\alpha}{2}}\mathcal{F}u\right]\in L^2(\mathbb{R})\right\},$$

whose norm is given by

$$\|\cdot\|_{\alpha,2} = \left|\mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{\frac{\alpha}{2}}\mathcal{F}\cdot\right]\right|_2.$$

Since the functionals

$$E(u) = \int_{-\infty}^{+\infty}\left[\frac{1}{2}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2 - F(u)\right]dx$$

and

$$Q(u) = \frac{1}{2}\int_{-\infty}^{+\infty}u^2\,dx$$

are two conserved quantities with (1.1), for studying the existence of solitary wave solutions to (1.1), by the variational methods, the solitary wave solutions to the equation (1.1) will be founded as minimizers of

$$I_q := \inf\left\{E(u); u\in H^1(\mathbb{R}), Q(u) = q\right\}, \tag{I_q}$$

where $q > 0$. Denote the set of minimizers of the problem (I_q) by

$$G_q := \left\{u; u\in H^1(\mathbb{R}), E(u) = I_q,\ Q(u) = q\right\}. \tag{G_q}$$

Similar to Lemma 4.1 and Lemma 4.3, we obtain the following two lemmas.

Lemma 5.1

(i) *I_q^∞ and I_q are finite and continuous on $(0,+\infty)$; moreover, for any $q > 0$, each minimizing sequence for the problem (I_q^∞) or (I_q) is bounded in $H^\alpha(\mathbb{R})$;*

(ii) *$I_q^\infty \le 0$ for any $q > 0$.*

Lemma 5.2 *If $I_{q_0}^\infty < 0$ for some $q_0 > 0$, then there exists q_1, $0 < q_1 \le q_0$, such that*

$$I_{q_1}^\infty < I_{q_1-q}^\infty + I_q^\infty \quad \text{for } 0 < q < q_1.$$

Applying the above two lemmas and commutator estimates [5, Lemma 2.5], we prove Theorem 1.3 and Theorem 1.4 by similar steps to those given in Section 4. Here we omit the details of Theorem 1.3 and Theorem 1.4.

Competing interests
The author declares that they have no competing interests.

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