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Research article

Identification of Source Terms in a Coupled Age-structured Population Model with Discontinuous Diffusion Coefficients

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Abstract: This article concerns the inverse problem of the coupled age-structured population dynamics system with discontinuous diffusion coefficients. The internal observations with two measurements are allowed to obtain the stability result for the inverse problem consisting of simultaneously retrieving two space dependent source terms in the given parabolic system. The proof of the result relies on Carleman estimates and certain energy estimates for parabolic system.

Keywords: Stability, Source term; Age-structured population model; Carleman estimate

1. Introduction

Population dynamics is the branch of life sciences that studies short-term and long-term changes in the size and age composition of populations, and the biological and environmental processes influencing those changes. To study the basic ideas of modeling a population dynamics system, one can refer to [5]. In [10, 15], the authors studied the existence and uniqueness results along with regularity results of population dynamics model with age-dependent diffusion coefficient. Several studies were made on controllability of the model in the past. For instance, Ainseba [1] proved the exact and approximate controllability for a linear age-dependent and spatially structured population dynamics problem. Ainseba and Anitha [3] discussed the local exact controllability of a linear age and space population dynamics model where the birth process is nonlocal, whereas in [2], the authors studied internal exact controllability of a linear age and space structured population model. The null controllability of a linear age-structured model with degenerate dispersion coefficient in population dynamics was studied in [4, 19]. Uesaka and Yamamoto [25] considerd a time-dependent structured population model equation and established unique continuation results using Carleman estimate. As for as inverse problems for a population model is concerned, only fewer works were done in the past. One such noted work was done by Blasio and Lorenzi [10].

In this paper we focus on an inverse problem of reconstructing the source terms in the coupled age-structured population model from the partial knowledge of a solution of the system. These kind of inverse problems for the reaction diffusion system with discontinuous coefficients has already been investigated by several authors [8, 9, 17, 18, 24]. In this context, we consider the following linear coupled age-structured model:

$$\left.\begin{aligned} Du &= div(k(x)\nabla u) + \mu(x)u + \alpha(x)v + h_1(t,a,x), \quad (t,a,x) \in Q \\ Dv &= div(\tilde{k}(x)\nabla v) + \tilde{\mu}(x)u + \tilde{\alpha}(x)v + h_2(t,a,x), \quad (t,a,x) \in Q \end{aligned}\right\} \tag{1.1}$$

with the initial/boundary conditions

$$\begin{aligned} &u(\theta_1,a,x) := u(\theta_1)(a,x),\ v(\theta_1,a,x) := v(\theta_1)(a,x), \quad (a,x) \in Q_A \\ &u(t,\theta_2,x) := u(\theta_2)(t,x),\ v(t,\theta_2,x) := v(\theta_2)(t,x), \quad (t,x) \in Q_T \\ &u(t,a,x) = v(t,a,x) = 0 \text{ on } \Sigma \end{aligned}$$

and transmission conditions

$$\begin{aligned} u|_{(0,T)\times(0,A)\times B^+} &= u|_{(0,T)\times(0,A)\times B^-} \\ v|_{(0,T)\times(0,A)\times B^+} &= v|_{(0,T)\times(0,A)\times B^-} \end{aligned}$$

where $u(t,a,x)$ and $v(t,a,x)$ be the densities of individuals at time t with age a and at a point x, A be the life expectancy of an individual and T be a positive constant, k and $\tilde{k}$ represent the diffusivity of the species u and v, μ and $\tilde{\mu}$ represents the natural growth rate of the species u corresponding to the species u and v, α and $\tilde{\alpha}$ represents the natural growth rate of v corresponding to the species u and v, h_1 and h_2 represent the corresponding source terms. We have used the notations $Du := u_t + u_a$, $Q = (0,T)\times(0,A)\times\Omega$, $Q_T = (0,T)\times\Omega$, $Q_A = (0,A)\times\Omega$ and $\Sigma = (0,T)\times(0,A)\times\Gamma$ for some bounded, connected open subset Ω of $\mathbb{R}^n$, for $n \leq 3$, with boundary Γ of class C^2 and some fixed $\theta_1 \in (0,T)$ and $\theta_2 \in (0,A)$. Let Ω_0 and Ω_1 be a partition of Ω into two nonempty open sets such that $\overline{\Omega_0} \subset \Omega$, $\Omega_1 = \Omega \setminus \overline{\Omega_0}$.

We denote by $B = \overline{\Omega_0} \cap \overline{\Omega_1}$, the interface, which will be supposed of class C^2 and by $\vec{n}$, the outward unit normal to Ω_1 at the points of B and also the outward unit normal to Ω at the points of Γ. Let B^+ and B^-, respectively, be the parts of B corresponding to the positive and negative direction of the normal $\vec{n}$.

The diffusion coefficients $k, \tilde{k}$ are assumed to be piecewise regular such that

$$k(x) = \begin{cases} k_0(x) \text{ if } x \in \Omega_0 \\ k_1(x) \text{ if } x \in \Omega_1 \end{cases} \quad \tilde{k}(x) = \begin{cases} \tilde{k}_0(x) \text{ if } x \in \Omega_0 \\ \tilde{k}_1(x) \text{ if } x \in \Omega_1 \end{cases} \tag{1.2}$$

Let $\mu, \tilde{\mu}, \alpha, \tilde{\alpha} \in L^\infty(\Omega)$ and h_i's be decomposed as $h_1(t,a,x) = f(x)R(t,a,x)$ and $h_2(t,a,x) = g(x)\mathcal{R}(t,a,x)$. Further assume that

$$\left|\frac{\partial h_i}{\partial t}(t,a,x)\right| \leq |g_i(t)|\,|h_i(\theta_1,a,x)|\,, \quad \left|\frac{\partial h_i}{\partial a}(t,a,x)\right| \leq |j_i(a)|\,|h_i(t,\theta_2,x)|\,,\ i = 1,2 \tag{1.3}$$

for all $(t,a,x) \in [0,T]\times[0,A]\times\overline{\Omega}$, where $g_i(t) \in L^2(0,T)$ and $j_i(a) \in L^2(0,A), i = 1,2$.

Now, let us give some assumptions on the parameters involved in (1.1).

Assumption: 1.1. *The coefficients* $k_i, \tilde{k}_i, i = 0, 1$ *satisfy the following:*

- $k_i, \tilde{k}_i \in C^2(\overline{\Omega}_i), \;\; i = 0, 1$
- $k_0|_{B^+} \neq k_1|_{B^-}, \; \tilde{k}_0|_{B^+} \neq \tilde{k}_1|_{B^-}$

Assumption: 1.2. *The coefficients* $k, \tilde{k}, \mu, \tilde{\mu}, \alpha, \tilde{\alpha}$ *satisfy the following:*

- *Suppose* $0 < r_0 \leq k(x), \; 0 < r_2 \leq \tilde{k}(x)$ *in* Ω *exists and the functions* $k(x), \tilde{k}(x)$ *and all their first derivatives are respectively bounded by the positive constants* r_1 *and* r_3. *And suppose* $0 < \mu_0 \leq \mu(x) \leq \mu_1 < \infty, \; 0 < \mu_2 \leq \tilde{\mu}(x) \leq \mu_3 < \infty, \; 0 < \alpha_0 \leq \alpha(x) \leq \alpha_1 < \infty, \; 0 < \alpha_2 \leq \tilde{\alpha}(x) \leq \alpha_3 < \infty$ *a.e. in* Ω

The inverse problems will be studied in the following context:

Is it possible to determine the space dependent source terms $f(x)$ *and* $g(x)$ *from the measurements of Du and Dv on a nonempty open subset* ω *of* Ω *along with the measurements of the solutions* u, v *and its derivatives at some fixed time* θ_1 *and fixed age* θ_2*?*

In order to get the basic ideas about bounded estimates, one can refer [20] . As far as the stability estimate of an inverse problems for parabolic equations via Carleman estimates is concerned, there exist a vast number of publications [6, 11, 12]. Referencing all these works is beyond the scope of this paper. So let us first recall briefly the initial results based on Carleman estimates. The theory of Carleman inequality is one of the fastest developing areas of partial differential equations(PDEs); in particular, after the pioneering work of Carleman in 1939, the theory of inequalities of Carleman type has been rapidly developed and now many general results are available for partial differential equations. For the first time, the method of Carleman estimates was introduced in the field of inverse problems by Bukhgeim and Klibanov [13, 14, 22]. The paper by Klibanov [23] presents a brief review of the applications of Carleman estimates to inverse problems for PDEs with respect to three fundamental issues, namely, uniqueness, stability and numerical methods. After these fundamental contributions to the study of inverse problems there have been abundant papers appearing in various dimensions of scope. Secondly, let us recall some interesting results based on the carleman estimates for partial differential equations with discontinuous diffusion coefficients. Benabdallah et al. [9] gave uniqueness and stability results for both the diffusion coefficients and the initial condition for the heat equation with a discontinuous diffusion coefficient. Doubova et al. [18] found an exact controllability result for a semi-linear heat equation with discontinuous diffusion coefficient. Golgeleyen [21] discussed the inverse problems for source term and coefficient of a potential term in a transport equation. Poisson [24] considered the heat equation with a discontinuous coefficient in three connected situations and gave the uniqueness and stability results for the diffusion coefficient in the main case from measurements of the solution on an arbitrary part of the boundary and at a fixed time in the whole spatial domain. Baudouin and Mercado [7] established the inverse problem of retrieving a stationary potential for the Schrödinger equation in a bounded domain with Dirichlet data and discontinuous principal coefficient from a single time-dependent Neumann boundary measurement.

The main objective of our work can be briefly described as follows. Let (u, v) be the solution of (1.1) associated with zero Dirichlet boundary conditions and the known semi-initial conditions $u(\theta_1)(a, x), u(\theta_2)(t, x), \; v(\theta_1)(a, x), \; v(\theta_2)(t, x)$ with the discontinuous diffusion coefficient $k(x), \tilde{k}(x)$ and unknown source terms $f(x)$ and $g(x)$. Then for sufficiently smooth $u(\theta_i), v(\theta_i), \; i = 1, 2$ there exists a constant $C > 0$ depending on $\Omega, \omega, r_0, r_1, r_2, r_3, \mu_1, \mu_3, \alpha_1, \alpha_3, p_1, p_2, l_1$ and l_2 (p_1, p_2, l_1 and l_2 will be defined later) satisfying

$$\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \leq C\left(\mathcal{A}(\omega) + \mathcal{B}(\theta_1) + \mathcal{E}(\theta_2)\right) \tag{1.4}$$

where

$$\begin{aligned}
\mathcal{A}(\omega) &= s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3 \left(|Du|^2 + |Dv|^2\right) dQ \\
\mathcal{B}(\theta_1) &= s\int_{Q_A} e^{-2s\eta(\theta_1)} \left(|div(k\nabla u(\theta_1))|^2 + |div(\tilde{k}\nabla v(\theta_1))|^2 + |u(\theta_1)|^2 + |v(\theta_1)|^2\right) dQ_A \\
\mathcal{E}(\theta_2) &= s\int_{Q_T} e^{-2s\eta(\theta_2)} \left(|div(k\nabla u(\theta_2))|^2 + |div(\tilde{k}\nabla v(\theta_2))|^2 + |u(\theta_2)|^2 + |v(\theta_2)|^2\right) dQ_T
\end{aligned}$$

It should be emphasized that to the best of our knowledge, as far as the inverse problem of a system of parabolic equations with discontinuous coefficients is concerned, there are few papers appeared; for instance see, Cristofol et al. [17] in which they have discussed the simultaneous reconstruction of the discontinuous diffusion coefficients for the parabolic equations and Baudouin and Mercado [7] established the inverse problem of retrieving a stationary potential for the Schrödinger equation with discontinuous principal coefficient from a single measurement whereas our work establishes the simultaneous identification of two source terms in a coupled age-structured population system with discontinuous diffusion coefficients from the knowledge of solutions on an arbitrary interior domain and at some arbitrary positive time and age. Further, it should be noted that, as far as the inverse problems for a system of age-structured model is concerned, there is no paper available in the literature for discontinuous diffusion coefficients.

The outline of this paper is as follows: In Section 2 we deduce a Carleman estimate for the system (1.1) with two observation which can be obtained from the classical Carleman estimates for parabolic system [18]. This estimate is applied successfully in Section 3 to derive an estimate for the source terms with the known observations.

2. Carleman Estimate

In this section, we quote a Carleman type estimate which is useful for further proceedings. But to get such a estimate, it is necessary to multiply the solution by some suitable weight functions, thus we need to introduce the following functions to express the Carleman estimate in the desired form. Let $\omega_0 \Subset \omega \Subset \Omega_0$. Let us define a function $\widetilde{\beta} \in C^2(\overline{\Omega}), \widetilde{\beta}_i = \widetilde{\beta}|_{\Omega_i},\ \ i = 0, 1$, such that

$$\left.\begin{array}{l}
\widetilde{\beta} > 0 \text{ in } \Omega,\ \ \widetilde{\beta} = 0 \text{ on } \Gamma \\
\partial_{\vec{n}}\widetilde{\beta} < 0 \text{ on } \Gamma,\ \ \widetilde{\beta} = 1 \text{ on } B \\
\partial_{\vec{n}}\widetilde{\beta}_0 > 0,\quad \partial_{\vec{n}}\widetilde{\beta}_1 > 0 \text{ on } B \\
k_0\partial_{\vec{n}}\widetilde{\beta}_0 = k_1\partial_{\vec{n}}\widetilde{\beta}_1 \text{ on } B \\
|\nabla\widetilde{\beta}| > 0 \text{ in } \overline{\Omega} \setminus \omega_0
\end{array}\right\} \tag{2.1}$$

The existence of such a function is referred in [18].

Let us consider the functions

$$\beta = \widetilde{\beta} + K, \ \text{ and } \ \overline{\beta} = \frac{5}{4}\max_{\overline{\Omega}} \beta \tag{2.2}$$

with $K > 0$ such that $K \geq 5 \max_{\overline{\Omega}} \widetilde{\beta}$.

Let λ be a sufficiently large positive constant that depends only on Ω and ω and will be defined later. For $t \in (0, T),\ a \in (0, A)$, let us define

$$\phi(t, a, x) = \frac{e^{\lambda\beta(x)}}{t(T-t)a(A-a)}, \qquad \eta(t, a, x) = \frac{e^{\lambda\bar{\beta}} - e^{\lambda\beta(x)}}{t(T-t)a(A-a)}. \tag{2.3}$$

Observe that the function ϕ and η are positive and we have the following relations

$$\nabla\phi = \lambda\phi\nabla\beta, \quad \nabla\eta = -\lambda\phi\nabla\beta. \tag{2.4}$$

Now, let us define

$$\begin{aligned} Z_k := \Big\{ q : q \in C^2([0,T] \times [0,A] \times \overline{\Omega}_i), \quad i = 0, 1, \quad & q|_{(0,T)\times(0,A)\times B^+} = q|_{(0,T)\times(0,A)\times B^-}, \\ & k_0 \partial_{\vec{n}} q|_{(0,T)\times(0,A)\times B^+} = k_1 \partial_{\vec{n}} q|_{(0,T)\times(0,A)\times B^-}, \quad q = 0 \ \text{ on } \ \Sigma \} \end{aligned}$$

As a first step in our analysis, we apply the classical Carleman estimates [18] derived for general parabolic type equations with discontinuous coefficient to the following operator

$$\mathcal{L}q := Dq - div(k(x)\nabla q), \quad (t, a, x) \in Q \tag{2.5}$$

Theorem: 2.1. *(Carleman estimate) Let β, ϕ and η be defined as in* (2.2)-(2.3) *and assume that $\omega \cap \Omega_0$ is nonempty. Suppose that Assumptions 1.1, 1.2 on the coefficient $k(x)$ holds. Then there exist parameters $\lambda_0 > 0$ and $s_0 > 0$ and a positive constant C that only depends on Ω, ω, r_0 and r_1 such that, for all $\lambda > \lambda_0$ and for all $s \geq s_0$, the following inequality holds*

$$\mathcal{I}(q; k) \leq C\left(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3|q|^2 \, dQ + \int_Q e^{-2s\eta}|\mathcal{L}q|^2 \, dQ \right) \tag{2.6}$$

where

$$\begin{aligned} \mathcal{I}(q; k) \ &= \ (s\lambda)^{-1} \int_Q \phi^{-1} e^{-2s\eta}(|Dq|^2 + |div(k(x)\nabla q)|^2) dQ \\ &\quad + s\lambda^2 \int_Q \phi e^{-2s\eta}|\nabla q|^2 dQ + s^3\lambda^4 \int_Q \phi^3 e^{-2s\eta}|q|^2 dQ. \end{aligned}$$

for all $q \in Z_k$, where s_0 and λ_0 will be defined later.

Proof. Now let us make the change of variable for the unknown function $q(t, a, x) = e^{s\eta}\psi(t, a, x)$ in (2.5) along with the conditions $\psi(0, \cdot, \cdot) = \psi(T, \cdot, \cdot) = \psi(\cdot, 0, \cdot) = \psi(\cdot, A, \cdot) = 0$. Then we write the resulting equation in terms of the two operators $M_1\psi$ and $M_2\psi$ as

$$M_1\psi + M_2\psi = f_s \tag{2.7}$$

where

$$M_1\psi \ = \ -div(k\nabla\psi) - s^2\lambda^2\phi^2|\nabla\beta|^2 k\psi + s\eta_t\,\psi + s\eta_a\,\psi,$$

$$\begin{aligned} M_2\psi &= \psi_t + \psi_a + 2s\lambda\phi\, k\nabla\beta\cdot\nabla\psi + s\lambda^2\phi|\nabla\beta|^2 k\psi, \\ f_s &= e^{-s\eta}\mathcal{L}q - s\lambda\phi\, div(k\nabla\beta)\psi. \end{aligned}$$

Then we have

$$\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2(M_1\psi, M_2\psi) = \|f_s\|_2^2 \tag{2.8}$$

where $(\cdot,\cdot)$ denote the scalar product in $L^2(Q)$ and the norms are defined in $L^2(Q)$. Now let us estimate all the terms appearing in the inner product. As a first step, let us split the inner product as a sum of the terms $I_{ij},\ i,j = 1,2,3,4$, where I_{ij} is the inner product of the ith term in the expression of $M_1\psi$ with jth term in the expression of $M_2\psi$ above. Now we shall simplify and estimate each of these integrals by using Green's theorem and usual integration by parts.

Now the terms $I_{1j},\ j = 1,2,3,4$ become, with an integration by parts

$$\begin{aligned} I_{11} &= 0, \text{ and similarly } I_{12} = 0 \\ I_{13} &= -s\lambda\int_\Sigma \phi|k\partial_{\vec n}\psi|^2\,(\partial_{\vec n}\beta)\,d\Sigma + s\lambda\int_{Q_B}\phi|k\partial_{\vec n}\psi|^2\,[\partial_{\vec n}\beta]_B\,dQ_B \\ &\quad + s\lambda^2\int_Q \phi k^2|\nabla\beta|^2|\nabla\psi|^2\,dQ + s\lambda\int_Q \phi k^2\Delta\beta|\nabla\psi|^2\,dQ \\ I_{14} &= s\lambda^2\int_{Q_B}\phi(k\partial_{\vec n}\beta)(k\partial_{\vec n}\psi)\,[\partial_{\vec n}\beta]_B\,\psi\,dQ_B + s\lambda^2\int_Q \phi k\, div(k|\nabla\beta|^2)\nabla\psi\,\psi\,dQ \\ &\quad + s\lambda^2\int_Q \phi k^2|\nabla\beta|^2|\nabla\psi|^2\,dQ + s\lambda^3\int_Q \phi k^2\nabla\beta|\nabla\beta|^2\nabla\psi\,\psi\,dQ \end{aligned}$$

where $Q_B := (0,T)\times(0,A)\times B$. Computations corresponding to the scalar product of the second term in $M_1\psi$ with $M_2\psi$ gives

$$\begin{aligned} I_{21} &= s^2\lambda^2\int_Q \phi\,\phi_t k|\nabla\beta|^2|\psi|^2\,dQ \\ I_{22} &= s^2\lambda^2\int_Q \phi\,\phi_a k|\nabla\beta|^2|\psi|^2\,dQ \\ I_{23} &= s^3\lambda^3\int_{Q_B}\phi^3|k\partial_{\vec n}\beta|^2\,[\partial_{\vec n}\beta]_B\,|\psi|^2\,dQ_B + s^3\lambda^3\int_Q \phi^3 div(k^2\nabla\beta|\nabla\beta|^2)\,|\psi|^2\,dQ \\ &\quad + 3s^3\lambda^4\int_Q \phi^3k^2|\nabla\beta|^4|\psi|^2\,dQ \\ I_{24} &= -s^3\lambda^4\int_Q \phi^3k^2|\nabla\beta|^4|\psi|^2\,dQ \end{aligned}$$

Calculating the scalar products I_{3j} and $I_{4j},\ j = 1,2,3,4$,

$$\begin{aligned} I_{31} &= -\frac{s}{2}\int_Q \eta_{tt}|\psi|^2\,dQ \text{ and similarly } I_{41} = -\frac{s}{2}\int_Q \eta_{aa}|\psi|^2\,dQ \\ I_{32} &= I_{42} = -\frac{s}{2}\int_Q \eta_{ta}|\psi|^2\,dQ \end{aligned}$$

$$
\begin{aligned}
I_{33} &= s^2\lambda^2 \int_Q \phi\ \phi_t k|\nabla\beta|^2|\psi|^2\, dQ - s^2\lambda^2 \int_Q \phi\ \eta_t k|\nabla\beta|^2|\psi|^2\, dQ \\
&\quad - s^2\lambda \int_Q \phi\eta_t\ div(k\nabla\beta)|\psi|^2\, dQ \\
I_{43} &= s^2\lambda^2 \int_Q \phi\ \phi_a k|\nabla\beta|^2|\psi|^2\, dQ - s^2\lambda^2 \int_Q \phi\ \eta_a k|\nabla\beta|^2|\psi|^2\, dQ \\
&\quad - s^2\lambda \int_Q \phi\eta_a\ div(k\nabla\beta)|\psi|^2\, dQ \\
I_{34} &= s^2\lambda^2 \int_Q \phi\eta_t k|\nabla\beta|^2|\psi|^2\, dQ \\
I_{44} &= s^2\lambda^2 \int_Q \phi\eta_a k|\nabla\beta|^2|\psi|^2\, dQ
\end{aligned}
$$

where we have used the notation $[\ \cdot\ ]_B$ to denote the jump on B and $[k]_B = k_0 - k_1 \leq 0$ on B and $[\partial_{\vec{n}}\beta]_B = \partial_{\vec{n}}\beta_0 - \partial_{\vec{n}}\beta_1 \geq 0$ on B, where $\vec{n}$ is the outward unit normal to Ω_1 and also $\partial_{\vec{n}}\beta \leq 0$ on Γ, using (2.1) and (2.2). Also note that $k_0\partial_{\vec{n}}\beta_0|_{B^+} - k_1\partial_{\vec{n}}\beta_1|_{B^-} = 0$.

Substituting all the preceding equalities in (2.8), we obtain

$$
\begin{aligned}
&\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2s^3\lambda^3 \int_{Q_B} \phi^3|k\partial_{\vec{n}}\beta|^2\ [\partial_{\vec{n}}\beta]_B\ |\psi|^2\, dQ_B \\
&\quad + 2s\lambda \int_{Q_B} \phi|k\partial_{\vec{n}}\psi|^2\ [\partial_{\vec{n}}\beta]_B\, dQ_B - 2s\lambda \int_\Sigma \phi|k\partial_{\vec{n}}\psi|^2\ (\partial_{\vec{n}}\beta)\, d\Sigma \\
&\quad + 3s^3\lambda^4 \int_Q \phi^3 k^2|\nabla\beta|^4|\psi|^2\, dQ + 2s\lambda^2 \int_Q \phi k^2|\nabla\beta|^2|\nabla\psi|^2\, dQ \\
&= \|f_s\|_2^2 - 2X \qquad (2.9)
\end{aligned}
$$

where

$$
\begin{aligned}
X = s\lambda &\int_Q \phi k^2\Delta\beta|\nabla\psi|^2\, dQ + s\lambda^2 \int_{Q_B} \phi(k\partial_{\vec{n}}\beta)(k\partial_{\vec{n}}\psi)\ [\partial_{\vec{n}}\beta]_B\ \psi\, dQ_B \\
&+ s\lambda^2 \int_Q \phi k\ div(k|\nabla\beta|^2)\nabla\psi\ \psi\, dQ + s\lambda^3 \int_Q \phi k^2\nabla\beta|\nabla\beta|^2\nabla\psi\ \psi\, dQ \\
&+ s^3\lambda^3 \int_Q \phi^3 div(k^2\nabla\beta|\nabla\beta|^2)\ |\psi|^2\, dQ + 2s^2\lambda^2 \int_Q \phi\ D\phi\ k|\nabla\beta|^2|\psi|^2\, dQ \\
&- s^2\lambda \int_Q \phi D\eta\ div(k\nabla\beta)|\psi|^2\, dQ - \frac{s}{2} \int_Q D^2\eta|\psi|^2\, dQ
\end{aligned}
$$

Making use of the estimates (2.3) and (2.4), we have

$$
\begin{aligned}
|X| &\leq s^2\lambda^4 \int_Q \phi^3|\psi|^2\, dQ + C\ s^3\lambda^3 \int_Q \phi^3|\psi|^2\, dQ \\
&\quad + s\lambda \int_Q \phi|\nabla\psi|^2\, dQ + C\ s\lambda \int_Q \phi|\nabla\psi|^2\, dQ \\
&\quad + \epsilon\ s\lambda^2 \int_Q \phi|\nabla\psi|^2\, dQ + \epsilon\ s\lambda \int_{Q_B} \phi|k\partial_{\vec{n}}\psi|^2\ [\partial_{\vec{n}}\beta]_B\, dQ_B
\end{aligned}
$$

$$+C_\epsilon\, s\lambda^3 A^4 T^4 \int_{Q_B} \phi^3 |k\partial_{\vec{n}}\beta|^2\, [\partial_{\vec{n}}\beta]_B |\psi|^2\, dQ_B$$

for any $s \geq s_1 = C(\Omega, r_1)\left(A^2T^2(A^2 + T^2 + A^2T^2 + A + T) + AT(T^2 + A^2)\right)$ and for any $\lambda \geq \lambda_1 = C(\Omega, r_1)\left(A\sqrt{T} + T\sqrt{A} + \sqrt[3]{A}\sqrt[6]{T} + \sqrt[3]{T}\sqrt[6]{A}\right)$.

On the other hand, from (2.7), we have

$$\|f_s\|_2^2 \leq \|e^{-s\eta}\mathcal{L}q\|_2^2 + C(\Omega, r_1, \mu_1)\left(s^2\lambda^2A^2T^2 \int_Q \phi^3|\psi|^2\, dQ\right) \tag{2.10}$$

for any $\lambda \geq 1$. Further, note that all the integrals in the left hand side of (2.9) are non-negative. Moreover, we know that (2.1) and (2.2) hold. Then, for some $\lambda_0 \geq 1$, we have

$$\left.\begin{array}{ll} \lambda^2\phi|k\nabla\beta|^2 & \geq C(\Omega, r_0)\lambda^2\phi \\ \lambda^4\phi^3k^2|\nabla\beta|^4 & \geq C(\Omega, r_0)\lambda^4\phi^3 \end{array}\right\} \tag{2.11}$$

for all $(t, a, x) \in (0, T) \times (0, A) \times (\Omega \setminus \omega_0)$ and $\lambda \geq \lambda_0$.

Using the estimates (2.10) and (2.11) in (2.9), we obtain

$$\begin{aligned} &\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + Cs^3\lambda^4 \int_0^T\int_0^A\int_{\Omega\setminus\omega_0} \phi^3|\psi|^2\, dx\, da\, dt + Cs\lambda^2 \int_0^T\int_0^A\int_{\Omega\setminus\omega_0} \phi|\nabla\psi|^2\, dx\, da\, dt \\ &+2s^3\lambda^3 \int_{Q_B} \phi^3|k\partial_{\vec{n}}\beta|^2[\partial_{\vec{n}}\beta]_B|\psi|^2\, dQ_B + 2s\lambda \int_{Q_B} \phi|k\partial_{\vec{n}}\psi|^2[\partial_{\vec{n}}\beta]_B\, dQ_B \\ &\leq \|f_s\|_2^2 + |X| \end{aligned}$$

for any $\lambda \geq 1$ and $s \geq s_2 = C((A + T)^4 + (A + T)^3)$.

For any sufficiently large $\lambda \geq \lambda_1$ and for any $s \geq s_3 = \max\{s_1, s_2\}$, all the upper bounds of X will be absorbed by one of the dominating term in the left hand side of the above inequality. For ϵ small enough, there exists a constant $C > 0$ such that

$$\begin{aligned} &\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + s\lambda^2 \int_Q \phi|\nabla\psi|^2\, dQ + s^3\lambda^4 \int_Q \phi^3|\psi|^2\, dQ \\ &\leq\ C\left[\|e^{-s\eta}\mathcal{L}q\|_2^2 + s\lambda^2 \int_{Q_{\omega_0}} \phi|\nabla\psi|^2\, dQ + s^3\lambda^4 \int_{Q_{\omega_0}} \phi^3|\psi|^2\, dQ\right] \end{aligned} \tag{2.12}$$

In order to obtain the Carleman estimate, it remains to obtain the first order derivative in time, age and second order derivative in space of the variable ψ in the left hand side. First one can be done using the expressions of $M_i\psi$ $(i = 1, 2)$. Indeed, from (2.7), we have

$$(s\lambda)^{-1} \int_Q \phi^{-1}|D\psi|^2\, dQ \leq C\left(s\lambda \int_Q \phi|\nabla\psi|^2\, dQ + s\lambda^3 \int_Q \phi|\psi|^2\, dQ + \|M_2\psi\|_2^2\right)$$

and

$$(s\lambda)^{-1} \int_Q \phi^{-1}|div(k\nabla\psi)|^2\, dQ \leq C\left(s^3\lambda^3 \int_Q \phi^3|\psi|^2\, d + \|M_1\psi\|_2^2\right)$$

for any $s \geq s_4 = C(\Omega, r_1)\,(AT(AT + A + T))$ and $\lambda \geq 1$. Thus we get

$$(s\lambda)^{-1}\int_Q \phi^{-1}\left(|D\psi|^2 + |div(k\nabla\psi)|^2\right)\,dQ + s^3\lambda^4\int_Q \phi^3|\psi|^2\,dQ + s\lambda^2\int_Q \phi|\nabla\psi|^2\,dQ$$
$$\leq C\left[\int_Q e^{-2s\eta}|\mathcal{L}q|^2\,dQ + s\lambda^2\int_{Q_{\omega_0}} \phi|\nabla\psi|^2\,dQ + s^3\lambda^4\int_{Q_{\omega_0}} \phi^3|\psi|^2\,dQ\right] \tag{2.13}$$

for any $s \geq s_5 = \max\{s_3, s_4\}$ and $\lambda \geq \lambda_0 = \max\{1, \lambda_1\}$.

Finally we turn back to our original variable by using the transformation $\psi = e^{-s\eta}q$. Noting that

$$\begin{aligned}
e^{-2s\eta}|q|^2 &= |\psi|^2,\\
e^{-2s\eta}|\nabla q|^2 &= |\nabla\psi + s\nabla\eta\psi|^2 \leq 2|\nabla\psi|^2 + 2s^2|\nabla\eta|^2|\psi|^2,\\
e^{-2s\eta}|Dq|^2 &\leq 2|D\psi|^2 + 2s^2|D\eta|^2|\psi|^2, \quad \text{and}\\
e^{-2s\eta}|div(k\nabla q)|^2 &\leq 4|div(k\nabla\psi)|^2 + 4s^2|div(k\nabla\eta)|^2|\psi|^2 + 8s^2|\nabla\eta|^2|k|^2|\nabla\psi|^2 + 4s^2|\nabla\eta|^4|k|^2|\psi|^2,
\end{aligned}$$

we have

$$(s\lambda)^{-1}\int_Q e^{-2s\eta}\phi^{-1}\left(|Dq|^2 + |div(k\nabla q)|^2\right)\,dQ + s^3\lambda^4\int_Q e^{-2s\eta}\phi^3|q|^2\,dQ + s\lambda^2\int_Q \phi|\nabla q|^2\,dQ$$
$$\leq C\left[\int_Q e^{-2s\eta}|\mathcal{L}q|^2\,dQ + s\lambda^2\int_{Q_{\omega_0}} e^{-2s\eta}\phi|\nabla q|^2\,dQ + s^3\lambda^4\int_{Q_{\omega_0}} e^{-2s\eta}\phi^3|q|^2\,dQ\right] \tag{2.14}$$

In order to conclude the proof of the Carleman estimate it is sufficient to derive the first order term in the right hand side of the above equation in terms of the zeroth order term of q in Q_ω.

In order to obtain, consider a function $\rho \in C_0^\infty(\omega)$ such that $\rho \equiv 1$ in ω_0 and $\rho \geq 0$. We consider $\omega_0 \subset \omega$ and the estimates obtained below remain true for larger ω.

Now multiplying the equation (2.5) by $s\lambda^2 e^{-2s\eta}\rho\phi q$ and integrating over $Q_\omega := (0, T) \times (0, A) \times \omega$, we obtain

$$s\lambda^2\int_{Q_\omega} e^{-2s\eta}\rho\phi q\, Dq\,dQ - s\lambda^2\int_{Q_\omega} e^{-2s\eta}\rho\phi q\, div(k\nabla q)\,dQ$$
$$+s\lambda^2\int_{Q_\omega} e^{-2s\eta}\rho\phi\mu|q|^2\,dQ = s\lambda^2\int_{Q_\omega} e^{-2s\eta}\rho\phi\,\mathcal{L}q\,q\,dQ$$

Using the definition of ρ along with the estimates (2.4), after some usual calculations, we finally obtain

$$s\lambda^2\int_{Q_{\omega_0}} e^{-2s\eta}\phi|\nabla q|^2\,dQ \leq C\left(\int_Q e^{-2s\eta}|\mathcal{L}q|^2 dQ + s^3\lambda^4\int_{Q_\omega} e^{-2s\eta}\phi^3|q|^2 dQ\right) \tag{2.15}$$

for any $s \geq s_6 = \sigma_1((A + T)^4 + (A + T)^3 + (A + T)^{8/3})$ and for any $\lambda \geq 1$. Substituting (2.15) in (2.14), we get

$$\mathcal{I}(q; k) \leq C\left[\int_Q e^{-2s\eta}|\mathcal{L}q|^2\,dQ + s^3\lambda^4\int_{Q_\omega} \phi^3 e^{-2s\eta}|q|^2\,dQ\right]$$

for all $s \geq s_0 = \max\{s_5, s_6\}$ and $\lambda \geq \lambda_0$. This completes the proof. □

Now we apply the above Carleman estimates derived for age-dependent diffusion model with discontinuous coefficient to the first equation in (1.1)(referred as (1.1a)). Let u be the solution of (1.1a) and suppose Assumptions 1.1, 1.2 hold true. Then for any $\lambda \geq \tilde{\lambda}_0 > 0$ and $s \geq \tilde{s}_0(\Omega, A, T) > 0$, there exists a constant $C(\Omega, \omega, r_0, r_1) > 0$ satisfying

$$\mathcal{I}(u;k) \leq C\left(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3|u|^2\, dQ + \int_Q e^{-2s\eta}|Du - div(k\nabla u)|^2\, dQ\right) \tag{2.16}$$

where $\mathcal{I}(u;k)$ is same as defined in (2.6). From (1.1b), we obtain for any $\lambda \geq \overline{\lambda}_0 > 0$ and $s \geq \overline{s}_0(\Omega, A, T)$, there exists a constant $C(\Omega, \omega, r_2, r_3) > 0$ satisfying

$$\mathcal{I}(v;\tilde{k}) \leq C\left(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3|v|^2\, dQ + \int_Q e^{-2s\eta}|Dv - div(\tilde{k}\nabla v)|^2\, dQ\right) \tag{2.17}$$

Now coupling the estimates (2.16)-(2.17), we get

$$\mathcal{I}(u;k) + \mathcal{I}(v;\tilde{k}) \leq C\left(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3\left(|u|^2 + |v|^2\right)\, dQ + \int_Q e^{-2s\eta}\left(|h_1|^2 + |h_2|^2\right)\, dQ\right) \tag{2.18}$$

for sufficiently large enough $s \geq \tilde{s} = \max\{\tilde{s}_0, \overline{s}_0, CA^2T^2\}$ and $\lambda \geq \tilde{\lambda} = \max\{\tilde{\lambda}_0, \overline{\lambda}_0\}$ with $C = C(\Omega, \omega, r_0, r_1, r_2, r_3, \mu_1, \mu_3, \alpha_1, \alpha_3) > 0$.

3. Stability Results

In this section, we establish a stability estimate using certain ideas from [9]. More precisely, we obtain an inequality which estimates the space dependent source terms $f(x)$ and $g(x)$ with an upper bound given by some Sobolev norm of the solution u, v and its derivative with respect to the time, age and certain spatial derivatives of u, v at time $\theta_1 \in (0, T)$ and at age $\theta_2 \in (0, A)$. In proving these kinds of stability estimates, the Carleman estimate obtained in the previous section will play a crucial part along with certain energy estimates.

Theorem: 3.1. *Suppose all the assumptions of Theorem 2.1 hold true with $s \geq \tilde{s}$ and $\lambda \geq \tilde{\lambda}$. Assume that $R, \mathcal{R} \in H^1(0, T; L^\infty(Q_A)) \cap H^1(0, A; L^\infty(Q_T))$ and $|R(\theta_1, a, x)| \geq l_1 > 0, |\mathcal{R}(\theta_1, a, x)| \geq l_2 > 0$ a.e. in Q_A, $|R(t, \theta_2, x)| \geq p_1 > 0, |\mathcal{R}(t, \theta_2, x)| \geq p_2 > 0$ a.e. in Q_T, Then there exists a constant $C = C(\Omega, \omega, T, A, r_0, r_1, r_2, r_3, \alpha_1, \alpha_3, \mu_1, \mu_3, l_1, l_2, p_1, p_2) > 0$ such that*

$$\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \leq C\left(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3\left(|Du|^2 + |Dv|^2\right)\, dQ + \mathcal{B}(\theta_1) + \mathcal{E}(\theta_2)\right).$$

where $\mathcal{B}(\theta_1)$ and $\mathcal{E}(\theta_2)$ are already defined in (1.4).

Proof. Let us set $y = Du$ and $z = Dv$. Then we have

$$\begin{cases} Dy = div(k\nabla y) + \mu y + \alpha z + Dh_1, & \text{in } Q \\ Dz = div(\tilde{k}\nabla z) + \tilde{\mu} y + \tilde{\alpha} z + Dh_2, & \text{in } Q \\ y(t, a, x) = 0, z(t, a, x) = 0 & \text{on } \Sigma \\ y(\theta_1, a, x) = y(\theta_1),\ z(\theta_1, a, x) = z(\theta_1), & \text{in } Q_A \\ y(t, \theta_2, x) = y(\theta_2),\ z(t, \theta_2, x) = z(\theta_2), & \text{in } Q_T \end{cases} \tag{3.1}$$

where

$$\begin{array}{rcl} y(\theta_1) &=& h_1(\theta_1)+div(k\nabla u)(\theta_1)-\mu(x)u(\theta_1)-\alpha(x)v(\theta_1),\\ z(\theta_1) &=& h_2(\theta_1)+div(\tilde{k}\nabla v)(\theta_1)-\tilde{\mu}(x)u(\theta_1)-\tilde{\alpha}(x)v(\theta_1),\\ y(\theta_2) &=& h_1(\theta_2)+div(k\nabla u)(\theta_2)-\mu(x)u(\theta_2)-\alpha(x)v(\theta_2),\\ \text{and } z(\theta_2) &=& h_2(\theta_2)+div(\tilde{k}\nabla v)(\theta_2)-\tilde{\mu}(x)u(\theta_2)-\tilde{\alpha}(x)v(\theta_2), \end{array}$$

Let θ_1 and θ_2 be some fixed points in $(0,T)$ and $(0,A)$ respectively, that is, a point at which $1/(t(T-t))$ and $1/(a(A-a))$ has its minimum value. In view of (3.1) and the estimate (2.18), we have

$$\mathcal{I}(y;k)+\mathcal{I}(z;\tilde{k})\leq C\left(s^3\lambda^4\int_{Q_\omega}e^{-2s\eta}\phi^3\left(|y|^2+|z|^2\right)dQ+\int_Q e^{-2s\eta}\left(|Dh_1|^2+|Dh_2|^2\right)dQ\right) \tag{3.2}$$

On the other hand, we have

$$\begin{aligned} &\int_{Q_A}sy(\theta_1,a,x)^2e^{-2s\eta(\theta_1,a,x)}dQ_A\\ &\qquad=\int_0^{\theta_1}\frac{\partial}{\partial t}\Big(\int_{Q_A}sy(t,a,x)^2e^{-2s\eta(t,a,x)}dQ_A\Big)dt\\ &\qquad\leq\int_Q-2s^2e^{-2s\eta}\eta_t|y|^2dQ+\int_Q 2syy_te^{-2s\eta}dQ \end{aligned}$$

Similarly, for any $\theta_2\in(0,A)$,

$$\begin{aligned} &\int_{Q_T}sy(t,\theta_2,x)^2e^{-2s\eta(t,\theta_2,x)}dQ_T\\ &\qquad=\int_0^{\theta_2}\frac{\partial}{\partial a}\Big(\int_{Q_T}sy(t,a,x)^2e^{-2s\eta(t,a,x)}dQ_T\Big)dt\\ &\qquad\leq\int_Q-2s^2e^{-2s\eta}\eta_a|y|^2dQ+\int_Q 2syy_ae^{-2s\eta}dQ \end{aligned}$$

Coupling the estimates

$$\begin{aligned} &\int_{Q_A}sy(\theta_1,a,x)^2e^{-2s\eta(\theta_1,a,x)}dQ_A+\int_{Q_T}sy(t,\theta_2,x)^2e^{-2s\eta(t,\theta_2,x)}dQ_T\\ &\leq\int_Q-2s^2e^{-2s\eta}D\eta|y|^2dQ+\int_Q 2syDye^{-2s\eta}dQ\\ &\leq C(TA^2+AT^2)s^2\int_Q\phi^2e^{-2s\eta}|y|^2dQ+2\int_Q\left(s\sqrt{s\lambda\phi}ye^{-s\eta}\right)\Big(\frac{1}{\sqrt{s\lambda\phi}}Dye^{-s\eta}\Big)dQ\\ &\leq C\left((T^3A^4+A^3T^4)s^2+T^4A^4s^3\lambda\int_Q\phi^3e^{-2s\eta}|y|^2dQ+\int_Q\frac{1}{s\lambda\phi}e^{-2s\eta}|Dy|^2dQ\right) \end{aligned}$$

$$\leq C\Big(s^3\lambda^4 \int_Q \phi^3 e^{-2s\eta}|y|^2 dQ + \int_Q (s\lambda\phi)^{-1} e^{-2s\eta}|Dy|^2 dQ\Big)$$
$$\leq C\mathcal{I}(y;k). \tag{3.3}$$

for any $\lambda \geq C(\Omega)AT(T^{\frac{1}{3}}A^{\frac{1}{3}} + T^{-\frac{1}{4}} + A^{-\frac{1}{4}})$ and $s \geq 1$. Similarly,

$$\int_{Q_A} sz(\theta_1)^2 e^{-2s\eta(\theta_1)} dQ_A + \int_{Q_T} sz(\theta_2)^2 e^{-2s\eta(\theta_2)} dQ_T \leq C\mathcal{I}(z;\tilde{k}). \tag{3.4}$$

And also, it is easy to see that, from (3.1)

$$\begin{aligned} s\int_{Q_A} e^{-2s\eta(\theta_1)} \sum_{i=1}^{2} h_i(\theta_1)^2 dQ_A + s\int_{Q_T} e^{-2s\eta(\theta_2)} \sum_{i=1}^{2} h_i(\theta_2)^2 dQ_T \\ \leq \; C\Big(\int_{Q_A} s(y(\theta_1)^2 + z(\theta_1)^2) e^{-2s\eta(\theta_1)} dQ_A \\ + \int_{Q_T} s(y(\theta_2)^2 + z(\theta_2)^2) e^{-2s\eta(\theta_2)} dQ_T + \mathcal{B}(\theta_1) + \mathcal{E}(\theta_2)\Big). \end{aligned} \tag{3.5}$$

where $\mathcal{B}(\theta_1), \mathcal{E}(\theta_2)$ are defined in (1.4) and C depends on $\Omega, \mu_1, \mu_3, \alpha_1, \alpha_3$. In view of (3.3)-(3.5), we have

$$s\int_{Q_A} e^{-2s\eta(\theta_1)} \sum_{i=1}^{2} h_i(\theta_1)^2 dQ_A + s\int_{Q_T} e^{-2s\eta(\theta_2)} \sum_{i=1}^{2} h_i(\theta_2)^2 dQ_T \leq C\Big(\mathcal{I}(y;k) + \mathcal{I}(z;\tilde{k}) + \mathcal{B}(\theta_1) + \mathcal{E}(\theta_2)\Big) \tag{3.6}$$

From the Carleman estimate derived in the previous section, we have

$$\begin{aligned} s\int_{Q_A} e^{-2s\eta(\theta_1)} \sum_{i=1}^{2} h_i(\theta_1)^2 dQ_A + s\int_{Q_T} e^{-2s\eta(\theta_2)} \sum_{i=1}^{2} h_i(\theta_2)^2 dQ_T \\ \leq \; C\Big(s^3\lambda^4 \int_{Q_\omega} e^{-2s\eta}\phi^3 \left(|y|^2 + |z|^2\right) dQ + \int_Q e^{-2s\eta} \left(|Dh_1|^2 + |Dh_2|^2\right) dQ \\ + \mathcal{B}(\theta_1) + \mathcal{E}(\theta_2)\Big). \end{aligned} \tag{3.7}$$

Now from the definition of the source terms (1.3) and also $R, \mathcal{R} \in H^1(0,T;L^\infty(Q_A)) \cap H^1(0,A;L^\infty(Q_T))$ we deduce that: there exist $g_i \in L^2(0,T)$ and $j_i \in L^2(0,A),\ i = 1,2$ so that

$$|DR(t,a,x)| \leq g_1(t)|R(\theta_1,a,x)| + j_1(a)|R(t,\theta_2,x)|, \quad \forall (t,a,x) \in Q.$$

$$|D\mathcal{R}(t,a,x)| \leq g_2(t)|\mathcal{R}(\theta_1,a,x)| + j_2(a)|\mathcal{R}(t,\theta_2,x)|, \quad \forall (t,a,x) \in Q.$$

Making use of the definition of the source terms, we get

$$\begin{aligned} s\int_{Q_A} \left(|f|^2|R(\theta_1)|^2 + |g|^2|\mathcal{R}(\theta_1)|^2\right) e^{-2s\eta(\theta_1)} dQ_A + s\int_{Q_T} \left(|f|^2|R(\theta_2)|^2 + |g|^2|\mathcal{R}(\theta_2)|^2\right) e^{-2s\eta(\theta_2)} dQ_T \\ \leq \; C\int_Q e^{-2s\eta} \left(|f|^2|g_1|^2|R(\theta_1)|^2 + |g|^2|g_2|^2|\mathcal{R}(\theta_1)|^2 + |f|^2|j_1|^2|R(\theta_2)|^2 + |g|^2|j_2|^2|\mathcal{R}(\theta_2)|^2\right) dQ \end{aligned}$$

$$+C\left(s^3\lambda^4\int_{Q_\omega} e^{-2s\eta}\phi^3\left(|y|^2+|z|^2\right)\,dQ+\mathcal{B}(\theta_1)+\mathcal{E}(\theta_2)\right) \tag{3.8}$$

Then, by virtue of the properties of η and ϕ, there exist m_0, m_1 and n_0, n_1 such that

$$\inf_{Q_A} e^{-2s\eta(\theta_1)} \geq m_0 > 0 \text{ and } \sup_{Q_A} e^{-2s\eta(\theta_1)} \leq m_1 < \infty, \quad \forall (a,x)\in Q_A,$$

and

$$\inf_{Q_T} e^{-2s\eta(\theta_2)} \geq n_0 > 0 \text{ and } \sup_{Q_T} e^{-2s\eta(\theta_2)} \leq n_1 < \infty, \quad \forall (t,x)\in Q_T,$$

But the functions $g_i \in L^2(0,T)$ and $j_i \in L^2(0,A),\ i=1,2$ implying that

$$\int_0^T |g_i|^2 dt \leq G_i < \infty, \int_0^A |j_i|^2 dt \leq K_i < \infty,\ i=1,2$$

For the choice of $s \geq s_0 = \max\{\tilde{s}, C(m_1,n_1)(G_1+G_2+K_1+K_2)\}$ and any $\lambda \geq \tilde{\lambda}$, we have

$$\begin{aligned}\int_{Q_A}\left(|f|^2|R(\theta_1)|^2+|g|^2|\mathcal{R}(\theta_1)|^2\right)dQ_A+\int_{Q_T}\left(|f|^2|R(\theta_2)|^2+|g|^2|\mathcal{R}(\theta_2)|^2\right)dQ_T \\ \leq\ C\left(s^3\lambda^4\int_{Q_\omega} e^{-2s\eta}\phi^3\left(|y|^2+|z|^2\right)\,dQ+\mathcal{B}(\theta_1)+\mathcal{E}(\theta_2)\right)\end{aligned} \tag{3.9}$$

where we have used the fact that $e^{-2s\eta} \leq e^{-2s\eta(\theta_1)}$ and $e^{-2s\eta} \leq e^{-2s\eta(\theta_2)}$ for all $(t,a,x)\in Q$.

Taking into account $|R(\theta_1,a,x)| \geq l_1 > 0, |\mathcal{R}(\theta_1,a,x)| \geq l_2 > 0$ a.e. in Q_A, $|R(t,\theta_2,x)| \geq p_1 > 0, |\mathcal{R}(t,\theta_2,x)| \geq p_2 > 0$ a.e. in Q_T, and set $r^2 = \min\{l_1^2, l_2^2\}$ and $p^2 = \min\{p_1^2, p_2^2\}$, we have

$$\|f\|^2_{L^2(\Omega)}+\|g\|^2_{L^2(\Omega)} \leq \frac{C}{(Ar^2+Tp^2)}\left(s^3\lambda^4\int_{Q_\omega} e^{-2s\eta}\phi^3\left(|y|^2+|z|^2\right)\,dQ+\mathcal{B}(\theta_1)+\mathcal{E}(\theta_2)\right). \tag{3.10}$$

Thus going back to the original variable $y = Du$ and $z = Dv$ one can complete the proof. □

4. Conclusion

In this paper we have proved the stability analysis of reconstructing the two space dependent source terms in the age-structured population model of two equations with discontinuous diffusion coefficients by two observations. It is observed that the results can be extended to the system consisting of m species and the reconstruction of m space dependent source term with m observations are possible. The reconstruction of all the source terms by a single observation (in general, reconstruction of m source terms with $m-1$ observations) would be an interesting work and as far as we know, it is very complicated due to the presence of source term.

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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