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On the convergence of fixed points for Lipschitz type mappings in hyperbolic spaces

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Abstract

In this paper, we prove strong and Δ -convergence theorems for a class of mappings which is essentially wider than that of asymptotically nonexpansive mappings on hyperbolic space through the S -iteration process introduced by Agarwal *et al.* (J. Nonlinear Convex Anal. 8:61-79, 2007) which is faster and independent of the Mann (Proc. Am. Math. Soc. 4:506-510, 1953) and Ishikawa (Proc. Am. Math. Soc. 44:147-150, 1974) iteration processes. Our results generalize, extend, and unify the corresponding results of Abbas *et al.* (Math. Comput. Model. 55:1418-1427, 2012), Agarwal *et al.* (J. Nonlinear Convex Anal. 8:61-79, 2007), Dhompongsa and Panyanak (Comput. Math. Appl. 56:2572-2579, 2008), and Khan and Abbas (Comput. Math. Appl. 61:109-116, 2011).

MSC: 47H10

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1 Introduction

The class of asymptotically nonexpansive mappings, introduced by Goebel and Kirk [1] in 1972, is an important generalization of the class of nonexpansive mapping and they proved that *if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping of C has a fixed point.*

There are numerous papers dealing with the approximation of fixed points of nonexpansive and asymptotically nonexpansive mappings in uniformly convex Banach spaces through modified Mann and Ishikawa iteration processes (see, *e.g.*, [2–9] and references therein). The class of Lipschitz mappings is larger than the classes of nonexpansive and asymptotically nonexpansive mappings. However, the theory of the computation of fixed points of non-Lipschitz mappings is equally important and interesting. There are few a results in this direction (see, *e.g.*, [10–13]).

In 1976, Lim [14] introduced a concept of convergence in a general metric space setting which he called ‘ Δ -convergence.’ In 2008, Kirk and Panyanak [15] specialized Lim’s concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Since then, the existence problem and the Δ -convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, nearly asymptotically nonexpansive, asymptotically nonexpansive mapping in intermediate sense, asymptotically nonexpansive nonself-mapping via Picard, Mann [16], Ishikawa [17], Agarwal *et al.* [18] in the framework of

CAT(0) space have been rapidly developed and many papers have appeared in this direction (see, e.g., [19–23]).

The purpose of the paper is to establish Δ -convergence as well as strong convergence through the S -iteration process for a class of mappings which is essentially wider than that of asymptotically nonexpansive mappings on a nonlinear domain, uniformly convex hyperbolic space which includes both uniformly convex Banach spaces and CAT(0) spaces. Therefore, our results extend and improve the corresponding ones proved by Abbas *et al.* [19], Dhompongsa and Panyanak [22], Khan and Abbas [23] and many other results in this direction.

2 Preliminaries

Let $F(T) = \{Tx = x : x \in C\}$ denotes the set of fixed point. We begin with the following definitions.

Definition 2.1 Let C be a nonempty subset of metric space X and $T : C \rightarrow C$ a mapping. A sequence $\{x_n\}$ in C is said to be an *approximating fixed point sequence* of T if

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Definition 2.2 Let C be a nonempty subset of a metric space X . The mapping $T : C \rightarrow C$ is said to be

- (1) *uniformly L -Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive number $L > 0$ such that

$$d(T^n x, T^n y) \leq Ld(x, y) \quad \text{for all } x, y \in C;$$

- (2) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + k_n)d(x, y) \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N};$$

- (3) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that

$$d(T^n x_n, p) \leq (1 + k_n)d(x_n, p) \quad \text{for all } x \in C, p \in F(T) \text{ and } n \in \mathbb{N}.$$

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu [11].

Let C be a nonempty subset of a metric space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$. A mapping $T : C \rightarrow C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n) \quad \text{for all } x, y \in C. \tag{2.1}$$

The infimum of the constants k_n for which (2.1) holds is denoted by $\eta(T^n)$ and is called the *nearly Lipschitz constant* of T^n .

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

- (4) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$;
- (5) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \eta(T^n) = 1$;
- (6) *nearly uniformly k -Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$.

Definition 2.3 Let C be a nonempty subset of a metric space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \rightarrow 0$. A mapping $T : C \rightarrow C$ is said to be *nearly asymptotically quasi-nonexpansive* with respect to $\{a_n\}$ if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that

$$d(T^n x, p) \leq (1 + u_n)d(x, p) + a_n$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.

In fact, if T is a nearly asymptotically nonexpansive mapping and $F(T)$ is nonempty, then T is a nearly asymptotically quasi-nonexpansive mapping. The following is an example of a nearly asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$.

Example 2.4 [19] Let $X = \mathbb{R}, C = (-\infty, 3]$ and $T : C \rightarrow C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2}x & \text{if } x \in (-\infty, 2], \\ x - 1 & \text{if } x \in (2, 3]. \end{cases}$$

Here, $F(T) = \{0\}$ and also, T is nearly asymptotically quasi-nonexpansive mapping with $\{u_n\} = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$ and $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$.

A nearly asymptotically quasi-nonexpansive mapping is called a nearly quasi-nonexpansive (asymptotically quasi-nonexpansive mapping) if $u_n = 0$ for all $n \in \mathbb{N}$ ($a_n = 0$ for all $n \in \mathbb{N}$). Notice that every nearly asymptotically quasi-nonexpansive mapping with bounded domain is nearly quasi-nonexpansive. Indeed, if C is a bounded subset of a metric space and $T : C \rightarrow C$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$, then

$$d(T^n x, p) \leq (1 + u_n)d(x, p) + a_n \leq d(x, p) + \left(u_n \sup_{x, y \in C} d(x, y) + a_n \right)$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.

The following example shows that T is a nearly quasi-nonexpansive mapping but not Lipschitzian and quasi-nonexpansive.

Example 2.5 [19] Let $X = \mathbb{R}, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $k \in (0, 1)$. Let $T : C \rightarrow C$ be a mapping defined by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ kx \sin \frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

Since $T : C \rightarrow C$ is obviously continuous, it easily follows that it is uniformly continuous. Note $F(T) = \{0\}$ and $T^n x \rightarrow 0$ uniformly, but T is not Lipschitzian. For each fixed $n \in \mathbb{N}$,

define

$$f_n(x) = |T^n x| - |x| \quad \text{for all } x \in C.$$

Fix a sequence $\{a_n\}$ in \mathbb{R} defined by

$$a_n = \max \left\{ \sup_{x \in C} f_n(x), 0 \right\} \quad \text{for all } n \in \mathbb{N}.$$

It is clear that $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \rightarrow 0$, since $T^n x \rightarrow 0$ uniformly. By the definition of $\{a_n\}$, we have

$$|T^n x| \leq |x| + a_n \quad \text{for all } x \in C \text{ and } n \in \mathbb{N}.$$

Clearly, T is a nearly quasi-nonexpansive mapping with respect to $\{a_n\}$ and it is not Lipschitz and not quasi-nonexpansive.

Lemma 2.6 [19, Lemma 2.11] *Let C be a nonempty subset of a metric space (X, d) and $T : C \rightarrow C$ a quasi- L -Lipschitzian, i.e., $F(T) \neq \emptyset$ and there exists a constant $L > 0$ such that*

$$d(Tx, Ty) \leq Ld(x, y) \quad \text{for all } x \in C \text{ and } y \in F(T).$$

If $\{x_n\}$ is a sequence in C such that $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ and $\lim_{n \rightarrow \infty} x_n = x \in C$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$, then x is a fixed point of T .

Throughout this paper we consider the following definition of a hyperbolic space introduced by Kohlenbach [24]. It is worth noting that they are different from the Gromov hyperbolic space [25] or from other notions of hyperbolic space that can be found in the literature (see, e.g., [26–28]).

Definition 2.7 A metric space (X, d) is a hyperbolic space if there exists a map $W : X^2 \times [0, 1] \rightarrow X$ satisfying

- (i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$,
- (ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$,
- (iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,
- (iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha)d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

An important example of a hyperbolic space is a CAT(0) space. It is nonlinear in nature and its brief introduction is as follows.

A metric space (X, d) is a *length space* if any two points of X are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is known as a *length metric* (otherwise an *inner metric* or *intrinsic metric*). In the case that no rectifiable path joins two points of the space, the distance between them is taken to be ∞ .

A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an

isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y . The space (X, d) is said to be a *geodesic space* if any two points of X are joined by a geodesic path and X is said to be *uniquely geodesic* if there is exactly one geodesic path denoted by $\alpha x \oplus (1 - \alpha)y$ joining x and y for each $x, y \in X$. The set $\{\alpha x \oplus (1 - \alpha)y : \alpha \in [0, 1]\}$ will be denoted by $[x, y]$, called the segment joining x to y . A subset C of a geodesic space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) is defined to be a collection of three points in X (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$ and such a triangle always exists (see [25]).

A geodesic metric space is a CAT(0) space if all geodesic triangles Δ in X with a comparison triangle $\bar{\Delta} \subset \mathbb{R}^2$ satisfy the CAT(0) inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

for all $x, y \in \Delta$ and for all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. Let X be a CAT(0) space. Define $W : X^2 \times [0, 1] \rightarrow X$ by $W(x, y, \alpha) = \alpha x \oplus (1 - \alpha)y$. Then W satisfies the four properties of a hyperbolic space. Also if X is a Banach space and $W(x, y, \alpha) = \alpha x + (1 - \alpha)y$, then X is a hyperbolic space. Therefore, our hyperbolic space represents a unified approach for both linear and nonlinear structures simultaneously.

To elaborate that there are hyperbolic spaces which are not imbedded in any Banach space, we give the following example.

Example 2.8 Let B be the open unit ball in complex Hilbert space with respect to the Poincaré metric (also called ‘Poincaré distance’)

$$d_B(x, y) = \arg \tanh \left| \frac{x - y}{1 - x\bar{y}} \right| = \arg \tanh(1 - \sigma(x, y))^{\frac{1}{2}},$$

where

$$\sigma(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - x\bar{y}|^2} \quad \text{for all } x, y \in B.$$

Then B is a hyperbolic space which is not imbedded in any Banach space.

A metric space (X, d) is called a convex metric space introduced by Takahashi [29] if it satisfies only (i). A subset C of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.

A hyperbolic space (X, d, W) is *uniformly convex* [30] if for any $u, x, y \in X$, $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ).

The hyperbolic space introduced by Kohlenbach [24] is slightly restrictive than the space of hyperbolic type [26] but general than hyperbolic space of [28]. Moreover, this class

of hyperbolic spaces also contains Hadamard manifolds, Hilbert balls equipped with the hyperbolic metric [31], \mathbb{R} -trees and Cartesian products of Hilbert balls as special cases.

Let C be a nonempty subset of hyperbolic space X . Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X . For $x \in X$, define a continuous functional $r_a(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ by

$$r_a(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x_n, x).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A_C(\{x_n\})$ of a bounded sequence of $\{x_n\}$ with respect to a subset of C of X is the set

$$A_C(\{x_n\}) = \{x \in X : r_a(x, \{x_n\}) \leq r_a(y, \{x_n\}) \text{ for any } y \in C\}.$$

This is the set of minimizers of the functional $r_a(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X , then it is simply denoted by $A(\{x_n\})$.

It is well known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets. The following lemma is due to Leustean [32] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.9 [32] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X .*

Recall that a sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$, if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Lemma 2.10 [33] *Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in C such that $A_C(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m \rightarrow \infty} r_a(y_m, \{x_n\}) = \rho$, then $\lim_{m \rightarrow \infty} y_m = y$.*

Lemma 2.11 [33] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $x \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n \rightarrow \infty} d(x_n, x) \leq c, \quad \limsup_{n \rightarrow \infty} d(y_n, x) \leq c, \quad \lim_{n \rightarrow \infty} d(W(x_n, y_n, t_n), x) = c$$

for some $c \geq 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 2.12 [5] *Let $\{\delta_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n \quad \text{for all } n \in \mathbb{N}.$$

If $\beta_n \geq 1$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$ and $\gamma_n < \infty$, then $\lim_{n \rightarrow \infty} \delta_n$ exists.

3 Strong and Δ -convergence theorems in hyperbolic space

In this section, we approximate fixed point for nearly asymptotically nonexpansive mappings in a hyperbolic space. More briefly, we established Δ -convergence and strong convergence theorems for iteration scheme (3.1).

First, we define the S -iteration process in hyperbolic space as follows.

Let C be a nonempty closed convex subset of a hyperbolic space X and $T : C \rightarrow C$ be a nearly asymptotically nonexpansive mapping. Then, for arbitrarily chosen $x_1 \in C$, we construct the sequence $\{x_n\}$ in C such that

$$\begin{cases} x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n), \\ y_n = W(x_n, T^n x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \quad (3.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ is called an S -iteration process.

Lemma 3.1 *Let C be a nonempty convex subset of a hyperbolic space X and $T : C \rightarrow C$ a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$.*

Proof First, we show that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T^n x_n, T^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(T^n x_n, p) + \alpha_n d(T^n y_n, p) \\ &\leq (1 - \alpha_n)((1 + u_n)d(x_n, p) + a_n) \\ &\quad + \alpha_n((1 + u_n)d(y_n, p) + a_n) \\ &\leq (1 + u_n)[(1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p)] + a_n \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T^n x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n[(1 + u_n)d(x_n, p) + a_n] \\ &\leq (1 + \beta_n u_n)d(x_n, p) + a_n \beta_n, \end{aligned} \quad (3.3)$$

from (3.2) and (3.3), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 + u_n)[(1 - \alpha_n)d(x_n, p) + \alpha_n((1 + \beta_n u_n)d(x_n, p) + a_n \beta_n) + a_n] \\ &\leq (1 + u_n(1 + \alpha_n \beta_n + \alpha_n \beta_n u_n^2))d(x_n, p) \\ &\quad + a_n(1 + (1 + u_n)\alpha_n \beta_n), \quad n \in \mathbb{N}. \end{aligned}$$

It follows that

$$d(x_{n+1}, p) \leq (1 + Mu_n)d(x_n, p) + a_n M_1 \quad (3.4)$$

for some $M, M_1 \geq 0$. $\{u_n\}$ is bounded. By Lemma 2.12, we find that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists. □

Lemma 3.2 *Let C be a nonempty and closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \rightarrow C$ be a nearly asymptotically quasi-nonexpansive mapping with sequences $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $F(T) \neq \emptyset$, then for the sequence $\{x_n\}$ in C defined by (3.1), we have $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$.*

Proof From Lemma 3.1, we find that $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$. We suppose that $\lim_{n \rightarrow \infty} d(x_n, p) = c \geq 0$. Since

$$d(T^n x_n, p) \leq (1 + u_n)d(x_n, p) + a_n \quad \text{for all } n \in \mathbb{N},$$

we have

$$\limsup_{n \rightarrow \infty} d(T^n x_n, p) \leq c.$$

Also

$$\begin{aligned} d(y_n, p) &\leq d(W(x_n, T^n x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1 + \beta_n u_n)d(x_n, p) + a_n \beta_n, \end{aligned}$$

which yields

$$\limsup_{n \rightarrow \infty} d(y_n, p) \leq c. \tag{3.5}$$

Hence

$$\limsup_{n \rightarrow \infty} d(T^n y_n, p) \leq \limsup_{n \rightarrow \infty} ((1 + u_n)d(y_n, p) + a_n) \leq c. \tag{3.6}$$

Since

$$c = \lim_{n \rightarrow \infty} d(x_{n+1}, p) = \lim_{n \rightarrow \infty} d(W(T^n x_n, T^n y_n, \alpha_n), p),$$

it follow from Lemma 2.11 that

$$\lim_{n \rightarrow \infty} d(T^n x_n, T^n y_n) = 0. \tag{3.7}$$

From (3.1) and (3.7), we have

$$\begin{aligned} d(x_{n+1}, T^n x_n) &= d(W(T^n x_n, T^n y_n, \alpha_n), T^n x_n) \\ &\leq \alpha_n d(T^n x_n, T^n y_n) \\ &\leq b d(T^n x_n, T^n y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{3.8}$$

Hence, from (3.7) and (3.8), we have

$$d(x_{n+1}, T^n y_n) \leq d(x_{n+1}, T^n x_n) + d(T^n x_n, T^n y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.9}$$

Now using (3.9), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, T^n y_n) + d(T^n y_n, p) \\ &\leq d(x_{n+1}, T^n y_n) + ((1 + u_n)d(y_n, p) + a_n), \end{aligned} \tag{3.10}$$

which gives from (3.10)

$$c \leq \liminf_{n \rightarrow \infty} d(y_n, p). \tag{3.11}$$

From (3.5) and (3.11), we obtain

$$c = \lim_{n \rightarrow \infty} d(y_n, p) = d(W(x_n, T^n x_n, \beta_n), p). \tag{3.12}$$

Apply Lemma 2.11 in (3.12), and we obtain

$$\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0. \quad \square$$

Theorem 3.3 *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \rightarrow C$ be a uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$. For arbitrary $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Then $\{x_n\}$ is Δ -convergent to an element of $F(T)$.*

Proof By Lemma 3.2, $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$. By uniform continuity of T , $d(x_n, T^n x_n) \rightarrow 0$ implies that $d(Tx_n, T^{n+1}x_n) \rightarrow 0$, observe that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(W(T^n x_n, T^n y_n, \alpha_n), x_n) \\ &\leq (1 - \alpha_n)d(x_n, T^n x_n) + \alpha_n d(T^n y_n, x_n) \\ &\leq d(x_n, T^n x_n) + \alpha_n d(T^n y_n, T^n x_n) \\ &\leq d(x_n, T^n x_n) + \eta(T^n)(d(x_n, y_n) + a_n) \\ &\leq d(x_n, T^n x_n) + \eta(T^n)(d(x_n, T^n x_n) + a_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Also

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq (1 + \eta(T^n)d(x_n, x_{n+1})) + d(x_{n+1}, T^n x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) + a_{n+1}, \end{aligned}$$

and hence

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \tag{3.13}$$

Next, we have to show that $\{x_n\}$ is Δ -convergent to an element of $F(T)$.

Since $\{x_n\}$ is bounded (by Lemma 3.1) therefore, Lemma 2.9 asserts that $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x\}$ (say). Let $A(\{y_n\}) = \{v\}$. Then by (3.13), $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$. T is a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$. By uniform continuity of T

$$\lim_{n \rightarrow \infty} d(T^i y_n, T^{i+1} y_n) = 0 \quad \text{for } i = 1, 2, 3, \dots \tag{3.14}$$

Now we claim that v is a fixed point of T . For this, we define a sequence $\{z_m\}$ in C by $z_m = T^m v$, $m \in \mathbb{N}$. For integers $m, n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_m, y_n) &\leq d(T^m v, T^m y_n) + d(T^m y_n, T^{m-1} y_n) + \dots + d(T y_n, y_n) \\ &\leq \eta(T^m)(d(v, y_n) + a_m) + \sum_{i=0}^{m-1} d(T^i y_n, T^{i+1} y_n). \end{aligned} \tag{3.15}$$

Then, by (3.14) and (3.15), we have

$$\begin{aligned} r_a(z_m, \{y_n\}) &= \limsup_{m \rightarrow \infty} d(z_m, y_n) \\ &\leq \eta(T^m)[r_a(v, \{y_n\}) + a_m]. \end{aligned}$$

Hence

$$\limsup_{m \rightarrow \infty} r_a(z_m, \{y_n\}) \leq r_a(v, \{y_n\}). \tag{3.16}$$

Since $A_C(\{y_n\}) = \{v\}$, by definition of asymptotic center $A_C(\{y_n\})$ of a bounded sequence $\{y_n\}$ with respect to $C \subset X$, we have

$$r_a(v, \{y_n\}) \leq r_a(y, \{y_n\}), \quad \forall y \in C.$$

This implies that

$$\liminf_{m \rightarrow \infty} r_a(z_m, \{y_n\}) \geq r_a(v, \{y_n\}), \tag{3.17}$$

therefore, from (3.16) and (3.17), we have

$$\lim_{m \rightarrow \infty} r_a(z_m, \{y_n\}) = r_a(v, \{y_n\}).$$

It follows from Lemma 2.10 that $T^m v \rightarrow v$. By uniform continuity of T , we have

$$Tv = T\left(\lim_{m \rightarrow \infty} T^m v\right) = T^{m+1} v = v,$$

which implies that v is a fixed point of T , i.e., $v \in F(T)$.

Next, we claim that v is the unique asymptotic center for each subsequence $\{y_n\}$ of $\{x_n\}$. Assume contrarily, that is, $x \neq v$. Since $\lim_{n \rightarrow \infty} d(x_n, v)$ exists by Lemma 3.1, therefore, by the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(y_n, v) &< \limsup_{n \rightarrow \infty} d(y_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(y_n, v), \end{aligned}$$

a contradiction and hence $x = v$. Since $\{y_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore, $A_C(\{y_n\}) = \{v\}$ for all subsequence of $\{y_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ -converges to a fixed point of T . \square

We now discuss the strong convergence for the S -iteration process defined by (3.1) for Lipschitzian type mappings in a uniformly convex hyperbolic space setting.

Theorem 3.4 *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \rightarrow C$ be a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Assume that $F(T)$ is a closed set. Let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Proof Necessity is obvious.

Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$. From (3.4), we have

$$d(x_{n+1}, F(T)) \leq (1 + Mu_n)d(x_n, F(T)) + M_1a_n, \quad n \in \mathbb{N},$$

so $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. It follows that $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. The following arguments are similar to those given in [34, Lemma 5] and [19, Theorem 4.3], and we obtain the following inequality:

$$d(x_{n+m}, p) \leq L \left[d(x_n, p) + \sum_{j=n}^{\infty} b_j \right]$$

for every $p \in F(T)$ and for all $m, n \geq 1$, where $L = e^{M(\sum_{j=n}^{n+m-1} u_j)} > 0$ and $b_j = M_1a_j$. As, $\sum_{n=1}^{\infty} u_n < \infty$ so $L^* = e^{M(\sum_{n=1}^{\infty} u_n)} \geq L = e^{M(\sum_{j=n}^{n+m-1} u_j)} > 0$. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} a_n < \infty$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4L^*} \quad \text{and} \quad \sum_{j=n_0}^{\infty} b_j < \frac{\epsilon}{6L^*}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\epsilon}{4L^*}$. Thus there must exist $p^* \in F$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{3L^*}.$$

Hence for $n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2L^* \left[d(x_{n_0}, p^*) + \sum_{j=n_0}^{\infty} b_j \right] \\ &< 2L^* \left(\frac{\epsilon}{3L^*} + \frac{\epsilon}{6L^*} \right) = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in closed subset C of a complete hyperbolic space and so it must converge strongly to a point q in C . Now, $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ gives $d(q, F(T)) = 0$. Since $F(T)$ is closed, we have $q \in F(T)$. \square

In the next result, the closedness assumption on $F(T)$ is not required.

Theorem 3.5 *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \rightarrow C$ an asymptotically quasi-nonexpansive mapping with sequence $\{u_n\}$ such that $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Proof Following an argument similar to those of Theorem 3.4, we see that $\{x_n\}$ is a Cauchy sequence in C . Let $\lim_{n \rightarrow \infty} x_n = x$. Since an asymptotically quasi-nonexpansive mapping is quasi- L -Lipschitzian, it follows from Lemma 2.6 that x is a fixed point of T . \square

Theorem 3.6 *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \rightarrow C$ a uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{a_n, \eta(T^n)\}$ such that $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$ and $a_n < \infty$. For arbitrary $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. If T is uniformly continuous and T^m is demicompact for some $m \in \mathbb{N}$, it follows that $\{x_n\}$ converges strongly to a fixed point of T .*

Proof By (3.13), we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. By the uniformly continuous of T , we have

$$d(x_n, Tx_n) \rightarrow 0 \quad \Rightarrow \quad d(Tx, T^2x_n) \rightarrow 0 \quad \Rightarrow \quad \dots \quad \Rightarrow \quad d(T^i x_n, T^{i+1} x_n) \rightarrow 0$$

for all $i \in \mathbb{N}$. It follows that

$$d(x_n, T^m x_n) \leq \sum_{i=0}^{m-1} d(T^i x_n, T^{i+1} x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $d(x_n, T^m x_n) \rightarrow 0$, and T^m is demicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j \rightarrow \infty} T^m x_{n_j} = x \in C$.

Note that

$$d(x_{n_j}, x) \leq d(x_{n_j}, T^m x_{n_j}) + d(T^m x_{n_j}, x) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, we get $x \in F(T)$. Since $\lim_{n \rightarrow \infty} d(x_n, x)$ exists by Lemma 3.1, and $\lim_{j \rightarrow \infty} d(x_n, x) = 0$, we conclude that $x_n \rightarrow x$. \square

Recall that a mapping T from a subset of a metric space (X, d) into itself with $F(T) \neq \emptyset$ is said to *satisfy condition (A)* (see [35]) if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0, f(t) > 0$ for $t \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \text{for all } x \in C.$$

Theorem 3.7 *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : C \rightarrow C$ a uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} \eta(T^n - 1) < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$. For arbitrary $x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Suppose that T satisfies the condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof By (3.13), we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Further, by condition (A),

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) \geq \lim_{n \rightarrow \infty} f(d(x_n, F(T))).$$

It follows that $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$. Therefore, the result follows from Theorem 3.4. \square

4 Conclusion

1. We prove strong and Δ -convergence of the S -iteration process, which is faster than the iteration processes used by Abbas *et al.* [19], Dhompongsa and Panyanak [22], and Khan and Abbas [23].
2. Theorem 3.3 extends Agarwal *et al.* [18, Theorem 3.8] from a uniformly convex Banach space to a uniformly convex hyperbolic space.
3. Theorem 3.3 extends Dhompongsa and Panyanak [22, Theorem 3.3] from the class of nonexpansive mappings to the class of mappings which are not necessarily Lipschitzian.
4. Theorem 3.6, extends corresponding results of Beg [36], Chang [37], Khan and Takahashi [4] and Osilike and Aniagbosor [5] for a more general class of non-Lipschitzian mappings in the framework of a uniformly convex hyperbolic space. It also extends the corresponding results of Dhompongsa and Panyanak [22] from the class of nonexpansive mappings to a more general class of non-Lipschitzian mappings in the same space setting.
5. Theorem 3.7 extends Sahu and Beg [12, Theorem 4.4] from a Banach to a uniformly convex hyperbolic space.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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