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# Solutions for a variational inclusion problem with applications to multiple sets split feasibility problems

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## Abstract

In this paper, we first study the set of common solutions for two variational inclusion problems in a real Hilbert space and establish a strong convergence theorem of this problem. As applications, we study unique minimum norm solutions of the following problems: multiple sets split feasibility problems, system of convex constrained linear inverse problems, convex constrained linear inverse problems, split feasibility problems, convex feasibility problems. We establish iteration processes of these problems and show the strong convergence theorems of these iteration processes.

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## 1 Introduction

Let  $C_1, C_2, \dots, C_m$  be nonempty closed convex subsets of a real Hilbert space  $\mathcal{H}$ . The well-known convex feasibility problem (CFP) is to find  $x^* \in \mathcal{H}$  such that

$$x^* \in C_1 \cap C_2 \cap \dots \cap C_m.$$

The convex feasibility problem has received a lot of attention due to its diverse applications in mathematics, approximation theory, communications, geophysics, control theory, biomedical engineering. One can refer to [1, 2].

If  $C_1$  and  $C_2$  are closed vector spaces of a real Hilbert space  $\mathcal{H}$ , in 1933, von Neumann showed that any sequence  $\{x_n\}$  is generated from the method of alternating projections:  $x_0 \in \mathcal{H}$ ,  $x_1 = P_{C_1}x_0$ ,  $x_2 = P_{C_2}x_1$ ,  $x_3 = P_{C_1}x_2$ ,  $\dots$ . Then  $\{x_n\}$  converges strongly to some  $\bar{x} \in C_1 \cap C_2$ . If  $C_1$  and  $C_2$  are nonempty closed convex subsets of  $\mathcal{H}$ , Bregman [3] showed that the sequence  $\{x_n\}$  generated from the method of alternating projection converges weakly to a point in  $C_1 \cap C_2$ . Hundal [4] showed that the strong convergence fails if  $C_1$  and  $C_2$  are nonempty closed convex subsets of  $\mathcal{H}$ . Recently, Boikanyo *et al.* [5] proposed the following process:

$$x_{2n+1} = J_{\beta_n}^{G_1}(\alpha_n u + (1 - \alpha_n)x_{2n} + e_n), \quad n = 0, 1, \dots \quad (1.1)$$

and

$$x_{2n} = J_{\rho_n}^{G_2} (\lambda_n u + (1 - \lambda_n)x_{2n-1} + e'_n), \quad n = 0, 1, \dots, \quad (1.2)$$

where  $u, x_0 \in \mathcal{H}$  are given arbitrarily,  $G_1$  and  $G_2$  are two set-valued maximal monotone operators with  $J_{\beta}^{G_1} = (I + \beta G_1)^{-1}$ , and  $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\rho_n\}, \{e_n\}, \{e'_n\}$  are sequences. Boikanyo *et al.* [5] proved that the sequence  $\{x_n\}$  converges strongly to a point  $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$  under suitable conditions.

The split feasibility problem (SFP) is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q,$$

where  $C, Q$  are nonempty closed convex subsets of real Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ , respectively.  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a bounded linear operator. The split feasibility problem (SFP) in finite dimensional real Hilbert spaces was first introduced by Censor and Elfving [6] for modeling inverse problems which arise from medical image reconstruction. Since then, the split feasibility problem (SFP) has received much attention due to its applications in signal processing, image reconstruction, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [1, 2, 6–17] and related literature. A special case of problem (SFP) is the convexly constrained linear inverse problem in a finite dimensional real Hilbert space [18]:

$$(\text{CLIP}) \quad \text{Find } \bar{x} \in C \text{ such that } A\bar{x} = b,$$

where  $C$  is a nonempty closed convex subset of a real Hilbert space  $\mathcal{H}_1$  and  $b$  is a given element of a real Hilbert space  $\mathcal{H}_2$ , which has extensively been investigated by using the Landweber iterative method [19]:

$$x_{n+1} := x_n + \gamma A^T(b - Ax_n), \quad n \in \mathbb{N}.$$

Let  $C_1, C_2, \dots, C_m$  be nonempty closed convex subsets of  $\mathcal{H}_1$ , let  $Q_1, Q_2, \dots, Q_m$  be nonempty closed convex subsets of  $\mathcal{H}_2$ , and let  $A_1, A_2, \dots, A_m : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be bounded linear operators. The well-known multiple sets split feasibility problem (MSSFP) is to find  $x^* \in \mathcal{H}_1$  such that

$$x^* \in C_i \text{ such that } A_i x^* \in Q_i \quad \text{for all } i = 1, 2, \dots, m.$$

The multiple sets split feasibility problem (MSSFP) contains convex feasibility problem (CFP) and split feasibility problem (SFP) as special cases [6, 10, 13]. Indeed, Censor *et al.* [11] first studied this type of problem. Xu [16] and Lopez *et al.* [13] also studied this type of problem. In 2011, Boikanyo and Moroşanu [20] gave the following algorithm:

$$\begin{cases} v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\alpha_n}^{G_1} v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\beta_n}^{G_2} v_{2n-1}, & n \in \mathbb{N}, \end{cases} \quad (1.3)$$

where  $u$  is given in  $\mathcal{H}$ , and  $G_1, G_2$  are two set-valued maximal monotone mappings on  $\mathcal{H}$ ,  $\{a_n\}, \{b_n\}, \{c_n\}, \{f_n\}, \{g_n\}$  and  $\{h_n\}$  are sequences in  $[0, 1]$ . Boikanyo and Moroşanu [20]

proved that  $\{v_n\}$  in (1.3) converges strongly to some  $\bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0)$  under suitable conditions.

Motivated by the above works, we consider the following algorithm:

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1} (I - \delta_n B_1) v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2} (I - \gamma_n B_2) v_{2n-1}, & n \in \mathbb{N}, \end{cases} \quad (1.4)$$

where  $G_1, G_2$  are two set-valued maximal monotone mappings on a real Hilbert space  $\mathcal{H}$ ,  $B_1, B_2 : C \rightarrow \mathcal{H}$  are two mappings,  $\{a_n\}, \{b_n\}, \{c_n\}, \{f_n\}, \{g_n\}$ , and  $\{h_n\}$  are sequences in  $[0, 1]$ . We show that the sequence  $\{v_n\}$  generated by (1.4) converges strongly to some  $\bar{x} \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$  under suitable conditions.

Algorithm (1.4) contains as special cases the inexact proximal point algorithm (e.g., [21, 22]) and the generalized contraction proximal point algorithm (e.g., [23]). Our conclusion extends and unifies Boikanyo and Moroşanu's result in [20], Wang and Cui's result in [24] becomes a special case. For details, one can see Section 3.

In this paper, we first study the set of common solutions for two variational inclusion problems in a Hilbert space and establish a strong convergence theorem of this problem. As applications, we study unique minimum norm solutions of the following problems: multiple sets split feasibility problems, system of convex constrained linear inverse problems, convex constrained linear inverse problems, split feasibility problems, convex feasibility problems. We establish iteration processes of these problems and show strong convergence theorems of these iteration processes.

## 2 Preliminaries

Throughout this paper, let  $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$  denote real Hilbert spaces with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\| \cdot \|$ ; let  $\mathbb{N}$  be the set of all natural numbers and  $\mathbb{R}^+$  be the set of all positive real numbers. A set-valued mapping  $A$  with domain  $\mathcal{D}(A)$  on  $\mathcal{H}$  is called monotone if  $\langle u - v, x - y \rangle \geq 0$  for any  $u \in Ax, v \in Ay$  and for all  $x, y \in \mathcal{D}(A)$ . A monotone operator  $A$  is called maximal monotone if its graph  $\{(x, y) : x \in \mathcal{D}(A), y \in Ax\}$  is not properly contained in the graph of any other monotone mapping. The set of all zero points of  $A$  is denoted by  $A^{-1}(0)$ , i.e.,  $A^{-1}(0) = \{x \in \mathcal{H} : 0 \in Ax\}$ . In what follows, we denote the strong convergence and the weak convergence of  $\{x_n\}$  to  $x \in \mathcal{H}$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. In order to facilitate our discussion, in the next section, we recall some facts. The following equality is easy to check:

$$\begin{aligned} & \|\alpha x + \beta y + \gamma z\|^2 \\ &= \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2 \end{aligned} \quad (2.1)$$

for each  $x, y, z \in \mathcal{H}$  and  $\alpha, \beta, \gamma \in [0, 1]$  with  $\alpha + \beta + \gamma = 1$ . Besides, we also have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad (2.2)$$

for each  $x, y \in \mathcal{H}$ . Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ , and a mapping  $T : C \rightarrow \mathcal{H}$ . We denote the set of all fixed points of  $T$  by  $\text{Fix}(T)$ . A mapping  $T : C \rightarrow \mathcal{H}$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for every  $x, y \in C$ . A mapping  $T : C \rightarrow \mathcal{H}$

is said to be quasi-nonexpansive if  $\text{Fix}(T) \neq \emptyset$  and  $\|Tx - y\| \leq \|x - y\|$  for all  $x \in C$  and  $y \in \text{Fix}(T)$ . A mapping  $T : C \rightarrow \mathcal{H}$  is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for every  $x, y \in C$ . Besides, it is easy to see that  $\text{Fix}(T)$  is a closed convex subset of  $C$  if  $T : C \rightarrow \mathcal{H}$  is a quasi-nonexpansive mapping. A mapping  $T : C \rightarrow \mathcal{H}$  is said to be  $\alpha$ -inverse-strongly monotone ( $\alpha$ -ism) if

$$\langle x - y, Tx - Ty \rangle \geq \alpha \|Tx - Ty\|^2$$

for all  $x, y \in \mathcal{H}$  and  $\alpha > 0$ .

The following lemmas are needed in this paper.

**Lemma 2.1** [25] *Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator, and let  $A^*$  be the adjoint of  $A$ . Suppose that  $C$  is a nonempty closed convex subset of  $\mathcal{H}_2$ , and that  $G : C \rightarrow \mathcal{H}_2$  is a firmly nonexpansive mapping. Then  $A^*(I - G)A$  is  $\frac{1}{\|A\|^2}$ -ism, that is,*

$$\frac{1}{\|A\|^2} \|A^*(I - G)Ax - A^*(I - G)Ay\|^2 \leq \langle x - y, A^*(I - G)Ax - A^*(I - G)Ay \rangle$$

for all  $x, y \in \mathcal{H}_1$ .

**Lemma 2.2** *Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ , and let  $G : C \rightarrow \mathcal{H}$  be a firmly nonexpansive mapping. Suppose that  $\text{Fix}(G)$  is nonempty. Then  $\langle x - Gx, Gx - w \rangle \geq 0$  for each  $x \in \mathcal{H}$  and each  $w \in \text{Fix}(G)$ .*

Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ . Then, for each  $x \in \mathcal{H}$ , there is a unique element  $\bar{x} \in C$  such that  $\|x - \bar{x}\| = \min_{y \in C} \|x - y\|$ . Here, we set  $P_C x = \bar{x}$  and  $P_C$  is said to be the metric projection from  $\mathcal{H}$  onto  $C$ .

**Lemma 2.3** [26] *Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ , and let  $P_C$  be the metric projection from  $\mathcal{H}$  onto  $C$ . Then  $\langle x - P_C x, P_C x - y \rangle \geq 0$  for each  $x \in \mathcal{H}$  and each  $y \in C$ .*

For a set-valued maximal monotone operator  $G$  on  $\mathcal{H}$  and  $r > 0$ , we may define an operator  $J_r^G : \mathcal{H} \rightarrow \mathcal{H}$  with  $J_r^G = (I + rG)^{-1}$  which is called the resolvent mapping of  $G$  for  $r$ .

**Lemma 2.4** [26, 27] *Let  $G : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be a set-valued maximal monotone mapping. Then we have that*

- (i) *for each  $\alpha > 0$ ,  $J_\alpha^G$  is a single-valued and firmly nonexpansive mapping.*
- (ii)  *$\mathcal{D}(J_\alpha^G) = \mathcal{H}$  and  $\text{Fix}(J_\alpha^G) = G^{-1}(0)$ .*
- (iii)  *$\|x - J_\alpha^G x\| \leq 2\|x - J_\beta^G x\|$  for each  $x \in \mathcal{H}$  and all  $\alpha, \beta \in (0, \infty)$  with  $0 < \alpha \leq \beta$ .*

**Lemma 2.5** [26] *Let  $G$  be a set-valued maximal monotone operator on  $\mathcal{H}$ . For  $a > 0$ , we define the resolvent  $J_a^G = (I + aG)^{-1}$ . Then the following holds:*

$$\|J_\alpha^G x - J_\beta^G x\|^2 \leq \frac{\alpha - \beta}{\alpha} \langle J_\alpha^G x - J_\beta^G x, J_\alpha^G x - x \rangle$$

for all  $\alpha, \beta > 0$  and  $x \in \mathcal{H}$ . In fact,

$$\|J_\alpha^G x - J_\beta^G x\| \leq \frac{|\alpha - \beta|}{\alpha} \|J_\alpha^G x - x\|$$

for all  $\alpha, \beta > 0$  and  $x \in \mathcal{H}$ .

**Lemma 2.6** [28] *Let  $\{S_n\}$  be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence  $\{S_{n_i}\}_{i \geq 0}$  of  $\{S_n\}$  such that  $S_{n_i} < S_{n_i+1}$  for all  $i \in \mathbb{N}$ . Consider the sequence  $\{\tau(n)\}_{n \geq n_0}$  defined by  $\tau(n) = \max\{n_0 \leq k \leq n : \tau_k < \tau_{k+1}\}$  for some sufficiently large number  $n_0 \in \mathbb{N}$ . Then  $\{\tau(n)\}_{n \geq n_0}$  is a nondecreasing sequence with  $\tau(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and for all  $n \geq n_0$ ,*

$$S_{\tau(n)} \leq S_{\tau(n)+1} \quad \text{and} \quad S_n \leq S_{\tau(n)+1}.$$

*In fact,  $\max\{S_{\tau(n)}, S_n\} \leq S_{\tau(n)+1}$ .*

**Lemma 2.7** [5] *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences in  $[0, 1]$ . Let  $\{e_n\}$  be a sequence of non-negative real numbers. Let  $\{t_n\}$  and  $\{k_n\}$  be two sequences of real numbers. Let  $\{S_n\}_{n \in \mathbb{N}}$  be a sequence of nonnegative real numbers with*

$$S_{n+1} \leq (1 - a_n)(1 - b_n)S_n + a_n t_n + b_n k_n + e_n$$

*for each  $n \in \mathbb{N}$ . Assume that:*

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \limsup_{n \rightarrow \infty} t_n \leq 0, \quad \limsup_{n \rightarrow \infty} k_n \leq 0, \quad \text{and} \quad \sum_{n=1}^{\infty} e_n = \infty.$$

*Then  $\lim_{n \rightarrow \infty} S_n = 0$ .*

A mapping  $T : \mathcal{H} \rightarrow \mathcal{H}$  is said to be averaged if  $T = (1 - \alpha)I + \alpha S$ , where  $S : \mathcal{H} \rightarrow \mathcal{H}$  is a nonexpansive mapping and  $\alpha \in (0, 1)$ .

**Lemma 2.8** [29] *Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$  and  $T : C \rightarrow \mathcal{H}$  be a mapping. Then the following hold:*

- (i)  *$T$  is a nonexpansive mapping if and only if  $I - T$  is  $\frac{1}{2}$ -inverse-strongly monotone ( $\frac{1}{2}$ -ism).*
- (ii) *If  $S$  is  $v$ -ism, then  $\gamma S$  is  $\frac{v}{\gamma}$ -ism.*
- (iii)  *$S$  is averaged if and only if  $I - S$  is  $v$ -ism for some  $v > \frac{1}{2}$ .  
Indeed,  $S$  is  $\alpha$ -averaged if and only if  $I - S$  is  $\frac{1}{(2\alpha)}$ -ism for  $\alpha \in (0, 1)$ .*
- (iv) *If  $S$  and  $T$  are averaged, then the composition  $ST$  is also averaged.*
- (v) *If the mappings  $\{T_i\}_{i=1}^n$  are averaged and have a common fixed point, then  $\bigcap_{i=1}^n \text{Fix}(T_i) = \text{Fix}(T_1 \cdots T_n)$  for each  $n \in \mathbb{N}$ .*

**Lemma 2.9** [30] *Let  $T$  be a nonexpansive self-mapping on a nonempty closed convex subset  $C$  of  $\mathcal{H}$ , and let  $\{x_n\}$  be a sequence in  $C$ . If  $x_n \rightharpoonup w$  and  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ , then  $Tw = w$ .*

Let  $C$  be a nonempty closed convex subset of  $\mathcal{H}$ . The indicator function  $\iota_C$  defined by

$$\iota_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C \end{cases}$$

is a proper lower semicontinuous convex function and its subdifferential  $\partial \iota_C$  defined by

$$\partial \iota_C x = \{z \in \mathcal{H} : \langle y - x, z \rangle \leq \iota_C(y) - \iota_C(x), \forall y \in \mathcal{H}\}$$

is a maximal monotone operator [31]. Furthermore, we also define the normal cone  $N_C u$  of  $C$  at  $u$  as follows:

$$N_C u = \{z \in \mathcal{H} : \langle z, v - u \rangle \leq 0, \forall v \in C\}.$$

We can define the resolvent  $J_\lambda^{\partial \iota_C}$  of  $\partial \iota_C$  for  $\lambda > 0$ , i.e.,

$$J_\lambda^{\partial \iota_C} x = (I + \lambda \partial \iota_C)^{-1} x$$

for all  $x \in \mathcal{H}$ . Since

$$\partial \iota_C x = \{z \in \mathcal{H} : \langle z, y - x \rangle \leq \iota_C(y) - \iota_C(x), \forall y \in \mathcal{H}\} = \{z \in \mathcal{H} : \langle z, y - x \rangle \leq 0, \forall y \in C\} = N_C x$$

for all  $x \in C$ , we have that

$$\begin{aligned} u = J_\lambda^{\partial \iota_C} x &\Leftrightarrow x \in u + \lambda \partial \iota_C u \\ &\Leftrightarrow x - u \in \lambda N_C u \\ &\Leftrightarrow \langle x - u, y - u \rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow u = P_C x. \end{aligned}$$

### 3 Main results

Let  $C$ ,  $Q$  and  $Q'$  be nonempty closed convex subsets of  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ , respectively. For each  $i = 1, 2$  and  $\kappa_i > 0$ , let  $B_i$  be a  $\kappa_i$ -inverse-strongly monotone mapping of  $C$  into  $\mathcal{H}_1$ , and let  $G_i$  be a set-valued maximal monotone mapping on  $\mathcal{H}_1$  such that the domain of  $G_i$  is included in  $C$  for each  $i = 1, 2$ . Let  $F_1$  be a firmly nonexpansive mapping of  $\mathcal{H}_2$  into  $\mathcal{H}_2$  and  $F_2$  be a firmly nonexpansive mapping of  $\mathcal{H}_3$  into  $\mathcal{H}_3$ . Note  $J_\lambda^{G_1} = (I + \lambda G_1)^{-1}$  and  $J_r^{G_2} = (I + r G_2)^{-1}$  for each  $\lambda > 0$  and  $r > 0$ . Let  $A_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator,  $A_2 : \mathcal{H}_1 \rightarrow \mathcal{H}_3$  be a bounded linear operator, and  $A_i^*$  be the adjoint of  $A_i$  for  $i = 1, 2$ . Throughout this paper, we use these notations unless specified otherwise.

**Theorem 3.1** Suppose that  $(B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$  is nonempty, and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{f_n\}$ ,  $\{g_n\}$ ,  $\{h_n\}$  are sequences in  $[0, 1]$  such that  $a_n + b_n + c_n = 1$ ,  $f_n + g_n + h_n = 1$ ,  $0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrarily fixed  $u \in \mathcal{H}$ , define a sequence  $\{v_n\}$  by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1} (I - \delta_n B_1) v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2} (I - \gamma_n B_2) v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n = P_{(B_1+G_1)^{-1}(0) \cap (B_2+G_2)^{-1}(0)} u$  provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \delta_n \leq b < 2\kappa_1$  and  $0 < f \leq \gamma_n \leq g < 2\kappa_2$  for each  $n \in \mathbb{N}$  and for some  $a, b, f, g \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Given any fixed  $\bar{v} := P_{(B_1+G_1)^{-1}(0) \cap (B_2+G_2)^{-1}(0)} u$ . Then  $\bar{v} = J_{\delta_n}^{G_1}(I - \delta_n B_1)\bar{v}$  and  $\bar{v} = J_{\gamma_n}^{G_2}(I - \gamma_n B_2)\bar{v}$ . Since  $B_1$  is  $\kappa_1$ -ism and  $\frac{\kappa_1}{\delta_n} > \frac{1}{2}$ , it follows from Lemma 2.8 that  $I - \delta_n B_1$  is averaged. Hence  $I - \delta_n B_1$  is nonexpansive. By Lemma 2.8,  $J_{\delta_n}^{G_1}(I - \delta_n G_1)$  and  $J_{\gamma_n}^{G_2}(I - \gamma_n G_2)$  are averaged. Hence  $J_{\delta_n}^{G_1}(I - \delta_n G_1)$  and  $J_{\gamma_n}^{G_2}(I - \gamma_n G_2)$  are nonexpansive. By condition (iv), we may assume that there exist positive real numbers  $c$  and  $h$  such that  $c_n \geq \ell > 0$  and  $h_n \geq h > 0$  for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , we have that

$$\begin{aligned}
 & \|v_{2n} - \bar{v}\| \\
 &= \|f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - \bar{v}\| \\
 &\leq f_n \|u - \bar{v}\| + g_n \|v_{2n-1} - \bar{v}\| + h_n \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - \bar{v}\| \\
 &\leq f_n \|u - \bar{v}\| + g_n \|v_{2n-1} - \bar{v}\| + h_n \|v_{2n-1} - \bar{v}\| \\
 &= f_n \|u - \bar{v}\| + (g_n + h_n) \|v_{2n-1} - \bar{v}\| \\
 &= f_n \|u - \bar{v}\| + (1 - f_n) \|v_{2n-1} - \bar{v}\|
 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
 & \|v_{2n+1} - \bar{v}\| \\
 &= \|a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v}\| \\
 &\leq a_n \|u - \bar{v}\| + b_n \|v_{2n} - \bar{v}\| + c_n \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v}\| \\
 &\leq a_n \|u - \bar{v}\| + b_n \|v_{2n} - \bar{v}\| + c_n \|v_{2n} - \bar{v}\| \\
 &= a_n \|u - \bar{v}\| + (b_n + c_n) \|v_{2n} - \bar{v}\| \\
 &\leq (1 - a_n) \{f_n \|u - \bar{v}\| + (1 - f_n) \|v_{2n-1} - \bar{v}\|\} + a_n \|u - \bar{v}\| \quad \text{by (3.1)} \\
 &= f_n (1 - a_n) \|u - \bar{v}\| + ((1 - a_n)(1 - f_n)) \|v_{2n-1} - \bar{v}\| + a_n \|u - \bar{v}\| \\
 &= (1 - (1 - a_n)(1 - f_n)) \|u - \bar{v}\| + (1 - a_n)(1 - f_n) \|v_{2n-1} - \bar{v}\| \\
 &\leq \max\{\|u - \bar{v}\|, \|v_{2n-1} - \bar{v}\|\} \\
 &\leq \max\{\|u - \bar{v}\|, \|v_1 - \bar{v}\|\}.
 \end{aligned}$$

By the mathematical induction method, we know that  $\{v_{2n-1}\}$ ,  $\{v_{2n}\}$  and  $\{v_n\}$  are bounded sequences. By Lemma 2.4, we have that

$$\begin{aligned}
 & \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v}\|^2 \\
 &= \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - J_{\delta_n}^{G_1}(I - \delta_n B_1)\bar{v}\|^2 \\
 &\leq \|(I - \delta_n B_1)v_{2n} - (I - \delta_n B_1)\bar{v}\|^2 - \|(I - J_{\delta_n}^{G_1}(I - \delta_n B_1))v_{2n} - (I - J_{\delta_n}^{G_1}(I - \delta_n B_1))\bar{v}\|^2
 \end{aligned}$$

$$\begin{aligned} &\leq \|v_{2n} - \bar{v}\|^2 - \|(I - J_{\delta_n}^{G_1}(I - \delta_n B_1))v_{2n} - (I - J_{\delta_n}^{G_1}(I - \delta_n B_1))\bar{v}\|^2 \\ &= \|v_{2n} - \bar{v}\|^2 - \|v_{2n} - J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n}\|^2 \end{aligned}$$

for each  $n \in \mathbb{N}$ . Hence,

$$\begin{aligned} &\|v_{2n+1} - \bar{v}\|^2 \\ &= \|a_n(u - \bar{v}) + b_n(v_{2n} - \bar{v}) + c_n(J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v})\|^2 \\ &\leq \|b_n(v_{2n} - \bar{v}) + c_n(J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v})\|^2 + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &= (1 - a_n)^2 \|b'_n(v_{2n} - \bar{v}) + c'_n(J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v})\|^2 \\ &\quad + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &= (1 - a_n)^2 \{b'_n\|v_{2n} - \bar{v}\|^2 + c'_n\|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v}\|^2 \\ &\quad - b'_n c'_n\|v_{2n} - J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n}\|^2\} + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &\leq b_n\|v_{2n} - \bar{v}\|^2 + c_n\|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - \bar{v}\|^2 + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &\leq b_n\|v_{2n} - \bar{v}\|^2 + c_n(\|v_{2n} - \bar{v}\|^2 - \|v_{2n} - J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n}\|^2) + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &= (b_n + c_n)\|v_{2n} - \bar{v}\|^2 + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle - c_n\|v_{2n} - J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n}\|^2, \quad (3.2) \end{aligned}$$

where

$$b'_n := \frac{b_n}{b_n + c_n}, \quad c'_n := \frac{c_n}{b_n + c_n}.$$

Similarly, we have

$$\begin{aligned} \|v_{2n} - \bar{v}\|^2 &\leq (g_n + h_n)\|v_{2n-1} - \bar{v}\|^2 + 2f_n\langle u - \bar{v}, v_{2n} - \bar{v} \rangle \\ &\quad - h_n\|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2. \end{aligned} \quad (3.3)$$

Consequently, it follows from (3.2) and (3.3) that

$$\begin{aligned} &\|v_{2n+1} - \bar{v}\|^2 \\ &\leq (b_n + c_n)\|v_{2n} - \bar{v}\|^2 + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle - c_n\|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2 \\ &\leq (1 - a_n)\{(1 - f_n)\|v_{2n-1} - \bar{v}\|^2 + 2f_n\langle u - \bar{v}, v_{2n} - \bar{v} \rangle \\ &\quad - h_n\|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2\} + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &\quad - c_n\|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2 \\ &= (1 - a_n)(1 - f_n)\|v_{2n-1} - \bar{v}\|^2 + 2f_n(1 - a_n)\langle u - \bar{v}, v_{2n} - \bar{v} \rangle \\ &\quad - h_n(1 - a_n)\|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2 + 2a_n\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &\quad - c_n\|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2. \end{aligned} \quad (3.4)$$



For each  $n \in \mathbb{N}$ , set  $S_n := \|v_{2n-1} - \bar{v}\|^2$ . Then  $S_{n+1} = \|v_{2n+1} - \bar{v}\|^2$  and (3.4) become

$$\begin{aligned} S_{n+1} &\leq (1 - a_n)(1 - f_n)S_n + 2a_n \langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle + 2f_n(1 - a_n) \langle u - \bar{v}, v_{2n} - \bar{v} \rangle \\ &\quad - c_n \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2 \\ &\quad - h_n(1 - a_n) \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2 \\ &\leq (1 - a_n)(1 - f_n)S_n + 2a_n \langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle + 2f_n(1 - a_n) \langle u - \bar{v}, v_{2n} - \bar{v} \rangle. \end{aligned} \quad (3.5)$$

Case 1:  $\{S_n\}$  is eventually decreasing, i.e., there exists a natural number  $N$  such that  $\|v_{2n+1} - \bar{v}\| \leq \|v_{2n-1} - \bar{v}\|$  for each  $n \geq N$ . So,  $\{S_n\}$  is convergent and  $\lim_{n \rightarrow \infty} \|v_{2n-1} - \bar{v}\|$  exists. For all  $n \in \mathbb{N}$ ,  $c_n \geq c$ ,  $h_n \geq h$  and (3.5), we have that

$$\begin{aligned} 0 &\leq c \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2 \\ &\quad + h(1 - a_n) \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2 \\ &\leq (1 - a_n)(1 - f_n)S_n - S_{n+1} + 2a_n \langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle + 2f_n(1 - a_n) \langle u - \bar{v}, v_{2n} - \bar{v} \rangle. \end{aligned} \quad (3.6)$$

Noting via condition (i) and the fact that  $\{v_n\}$  is bounded that

$$\begin{aligned} \lim_{n \rightarrow \infty} [(1 - a_n)(1 - f_n)S_n - S_{n+1}] &= 0, \\ \lim_{n \rightarrow \infty} 2a_n \langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle &= \lim_{n \rightarrow \infty} 2f_n(1 - a_n) \langle u - \bar{v}, v_{2n} - \bar{v} \rangle = 0, \end{aligned}$$

we conclude from (3.6) that

$$\lim_{n \rightarrow \infty} (c \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\|^2 + h(1 - a_n) \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|^2) = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\| = \lim_{n \rightarrow \infty} \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\| = 0. \quad (3.7)$$

Since  $\{v_{2n}\}$  is a bounded sequence in  $\mathcal{H}$ , there is a subsequence  $\{v_{(2n)_k}\}$  of  $\{v_{2n}\}$  such that  $v_{(2n)_k} \rightharpoonup \bar{x} \in \mathcal{H}$  and

$$\limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2n} - \bar{v} \rangle = \lim_{k \rightarrow \infty} \langle u - \bar{v}, v_{(2n)_k} - \bar{v} \rangle = \langle u - \bar{v}, \bar{x} - \bar{v} \rangle.$$

On the other hand,  $0 < a \leq \delta_n \leq b < 2\kappa_1$ , there exists a subsequence  $\{\delta_{n_{k_j}}\}$  of  $\{\delta_n\}$  such that  $\{\delta_{n_{k_j}}\}$  converges to a number  $\bar{\delta} \in [a, b]$ . By Lemma 2.5, we have

$$\begin{aligned} &\|v_{(2n)_{k_j}} - J_{\bar{\delta}}^{G_1}(I - \bar{\delta} B_1)v_{(2n)_{k_j}}\| \\ &\leq \|v_{(2n)_{k_j}} - J_{\delta_{n_{k_j}}}^{G_1}(I - \delta_{n_{k_j}} B_1)v_{(2n)_{k_j}}\| \\ &\quad + \|J_{\delta_{n_{k_j}}}^{G_1}(I - \bar{\delta} B_1)v_{(2n)_{k_j}} - J_{\bar{\delta}}^{G_1}(I - \bar{\delta} B_1)v_{(2n)_{k_j}}\| \\ &\quad + \|J_{\delta_{n_{k_j}}}^{G_1}(I - \delta_{n_{k_j}} B_1)v_{(2n)_{k_j}} - J_{\delta_{n_{k_j}}}^{G_1}(I - \bar{\delta} B_1)v_{(2n)_{k_j}}\| \end{aligned}$$

$$\begin{aligned} &\leq \|v_{(2n)k_j} - J_{\delta_{n_{k_j}}}^{G_1}(I - \delta_{n_{k_j}}B_1)v_{(2n)k_j}\| + |\delta_{n_{k_j}} - \bar{\delta}| \|B_1 u_{n_{k_j}}\| \\ &\quad + \frac{|\delta_{n_{k_j}} - \bar{\delta}|}{\bar{\delta}} \|J_{\bar{\delta}}^{G_1}(I - \bar{\delta}B_1)v_{(2n)k_j} - (I - \bar{\delta}B_1)v_{(2n)k_j}\| \rightarrow 0. \end{aligned} \quad (3.8)$$

By (3.8) and Lemma 2.9,  $\bar{x} \in \text{Fix}(J_{\bar{\delta}}^{G_1}(I - \bar{\delta}B_1)) = (B_1 + G_1)^{-1}(0)$ . Since  $0 < c \leq \gamma_n \leq d < 2\kappa_2$ , there exists a subsequence  $\{\gamma_{n_{k_j}}\}$  of  $\{\gamma_n\}$  such that  $\{\gamma_{n_{k_j}}\}$  converges to a number  $\bar{\gamma} \in [c, d]$ . We have that

$$\begin{aligned} \|v_{2n+1} - v_{2n}\| &= \|a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\| \\ &\leq a_n \|u - v_{2n}\| + c_n \|J_{\delta_n}^{G_1}(I - \delta_n B_1)v_{2n} - v_{2n}\| \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \|v_{2n} - v_{2n-1}\| &= \|f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\| \\ &\leq f_n \|u - v_{2n-1}\| + h_n \|J_{\gamma_n}^{G_2}(I - \gamma_n B_2)v_{2n-1} - v_{2n-1}\|. \end{aligned} \quad (3.10)$$

Since  $\{v_n\}$  is bounded, we conclude from (3.7), (3.9), (3.10), and conditions (i), (ii) that

$$\lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0. \quad (3.11)$$

By Lemma 2.5, we have

$$\begin{aligned} &\|v_{(2n)k_j+1} - J_{\bar{\gamma}}(I - \bar{\gamma}B_2)v_{(2n)k_j+1}\| \\ &\leq \|v_{(2n)k_j+1} - J_{\gamma_{n_{k_j}}}(I - \gamma_{n_{k_j}}B_2)v_{(2n)k_j+1}\| \\ &\quad + \|J_{\gamma_{n_{k_j}}}(I - \gamma_{n_{k_j}}B_2)v_{(2n)k_j+1} - J_{\gamma_{n_{k_j}}}(I - \bar{\gamma}B_2)v_{(2n)k_j+1}\| \\ &\quad + \|J_{\gamma_{n_{k_j}}}(I - \bar{\gamma}B_2)v_{(2n)k_j+1} - J_{\bar{\gamma}}(I - \bar{\gamma}B_2)v_{(2n)k_j+1}\| \\ &\leq \|v_{(2n)k_j+1} - J_{\gamma_{n_{k_j}}}(I - \gamma_{n_{k_j}}B_2)v_{(2n)k_j+1}\| + |\gamma_{n_{k_j}} - \bar{\gamma}| \|B_2 v_{(2n)k_j+1}\| \\ &\quad + \frac{|\gamma_{n_{k_j}} - \bar{\gamma}|}{\bar{\gamma}} \|J_{\bar{\gamma}}(I - \bar{\gamma}B_2)v_{(2n)k_j+1} - (I - \bar{\gamma}B_2)v_{(2n)k_j+1}\| \rightarrow 0. \end{aligned} \quad (3.12)$$

Since  $\lim_{n \rightarrow \infty} \|v_{2n+1} - v_{2n}\| = 0$ , we know that  $v_{(2n)k_j+1} \rightharpoonup \bar{x}$ . By (3.12) and Lemma 2.9, we know that  $J_{\bar{\gamma}}^{G_2}(I - \bar{\gamma}B_2)\bar{x} = \bar{x}$ . So,  $\bar{x} \in (B_2 + G_2)^{-1}(0)$ . This shows that  $\bar{x} \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$ . It follows from  $\bar{v} := P_{(B_1+G_1)^{-1}(0) \cap (B_2+G_2)^{-1}(0)} u$  and Lemma 2.3 that

$$\limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2n} - \bar{v} \rangle = \langle u - \bar{v}, \bar{x} - \bar{v} \rangle \leq 0. \quad (3.13)$$

By (3.11) and (3.13),

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle \\ &= \limsup_{n \rightarrow \infty} (\langle u - \bar{v}, v_{2n+1} - v_{2n} \rangle + \langle u - \bar{v}, v_{2n} - \bar{v} \rangle) \\ &\leq \limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2n+1} - v_{2n} \rangle + \limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2n} - \bar{v} \rangle \leq 0. \end{aligned} \quad (3.14)$$

Applying Lemma 2.7 to inequality (3.5) with  $t_n = 2\langle u - \bar{v}, v_{2n+1} - \bar{v} \rangle$  and  $k_n = 2(1 - a_n)\langle u - \bar{v}, v_{2n} - \bar{v} \rangle$ , we obtain from (3.13) and (3.14) and conditions (i), (ii) that  $\lim_{n \rightarrow \infty} S_n = 0$ . That is,  $\lim_{n \rightarrow \infty} v_{2n-1} = \bar{v}$ . And then it follows from (3.11) that  $\lim_{n \rightarrow \infty} v_{2n} = \bar{v}$ . Thus,  $\lim_{n \rightarrow \infty} v_n = \bar{v}$ .

Case 2: Suppose that  $\{S_n\}$  is not an eventually decreasing sequence. Let  $\{S_{n_i}\}$  be a subsequence of  $\{S_n\}$  such that  $S_{n_i} \leq S_{n_{i+1}}$  for all  $i \geq 0$ , also consider the sequence of integers  $\{\tau(n)\}_{n \geq n_0}$ , defined by  $\tau(n) = \max\{k \leq n, S_k < S_{k+1}\}$ , for some  $n_0$  ( $n_0$  is a sufficiently large number). Then  $\{\tau(n)\}_{n \geq n_0}$  is a nondecreasing sequence as  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ , and for all  $n \geq n_0$ , one has that

$$S_{\tau(n)} \leq S_{\tau(n)+1} \quad \text{and} \quad S_n \leq S_{\tau(n)+1}. \quad (3.15)$$

That is,  $\max\{S_{\tau(n)}, S_n\} \leq S_{\tau(n)+1}$ . For such  $n \geq n_0$ , it follows from (3.15) that

$$\|v_{2\tau(n)-1} - \bar{v}\| \leq \|v_{2\tau(n)+1} - \bar{v}\| \quad (3.16)$$

and

$$\|v_{2n-1} - \bar{v}\| \leq \|v_{2\tau(n)+1} - \bar{v}\|. \quad (3.17)$$

From (3.15) and (3.5), we obtain

$$\begin{aligned} S_{\tau(n)} &\leq S_{\tau(n)+1} \\ &\leq (1 - a_{\tau(n)})(1 - f_{\tau(n)})S_{\tau(n)} + 2a_{\tau(n)}\langle u - \bar{v}, v_{2\tau(n)+1} - \bar{v} \rangle \\ &\quad + 2f_{\tau(n)}(1 - a_{\tau(n)})\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle \\ &\quad - c_{\tau(n)} \|J_{\delta_{\tau(n)}}^{G_1} (I - \delta_{\tau(n)} B_1) v_{2\tau(n)} - v_{2\tau(n)}\|^2 \\ &\quad - h_{\tau(n)} \|J_{\gamma_{\tau(n)}}^{G_2} (I - \gamma_{\tau(n)} B_2) v_{2\tau(n)-1} - v_{2\tau(n)-1}\|^2 \\ &\leq (1 - a_{\tau(n)})(1 - f_{\tau(n)})S_{\tau(n)} + 2a_{\tau(n)}\langle u - \bar{v}, v_{2\tau(n)+1} - \bar{v} \rangle \\ &\quad + 2f_{\tau(n)}(1 - a_{\tau(n)})\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle. \end{aligned} \quad (3.18)$$

Just as the argument of Case 1, we have

$$\begin{cases} \lim_{n \rightarrow \infty} \|v_{2\tau(n)} - J_{\delta_{\tau(n)}}^{G_1} (I - \delta_{\tau(n)} B_1) v_{2\tau(n)}\| = 0, \\ \lim_{n \rightarrow \infty} \|v_{2\tau(n)-1} - J_{\gamma_{\tau(n)}}^{G_2} (I - \gamma_{\tau(n)} B_2) v_{2\tau(n)-1}\| = 0, \\ \limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2\tau(n)+1} - \bar{v} \rangle \leq 0, \\ \limsup_{n \rightarrow \infty} \langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle \leq 0, \\ \lim_{n \rightarrow \infty} \|v_{2\tau(n)+1} - v_{2\tau(n)}\| = 0. \end{cases} \quad (3.19)$$

By (3.15) and (3.18), we have

$$\begin{aligned} S_{\tau(n)+1} &\leq (1 - a_{\tau(n)})(1 - f_{\tau(n)})S_{\tau(n)+1} + 2a_{\tau(n)}\langle u - \bar{v}, v_{2\tau(n)+1} - \bar{v} \rangle \\ &\quad + 2f_{\tau(n)}(1 - a_{\tau(n)})\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle \end{aligned}$$

for all  $n \geq n_0$ . This implies that

$$\begin{aligned} S_{\tau(n)+1} &\leq (1 - a_{\tau(n)})(1 - f_{\tau(n)})S_{\tau(n)+1} + a_{\tau(n)}K\|v_{2\tau(n)+1} - v_{2\tau(n)}\| \\ &\quad + 2[f_{\tau(n)}(1 - a_{\tau(n)}) + a_{\tau(n)}]\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle \end{aligned}$$

for all  $n \geq n_0$ , where  $K = 2\|u - \bar{v}\|$ . Furthermore, we have

$$\begin{aligned} S_{\tau(n)+1} &\leq \frac{a_{\tau(n)}K\|v_{2\tau(n)+1} - v_{2\tau(n)}\|}{a_{\tau(n)} + f_{\tau(n)}(1 - a_{\tau(n)})} \\ &\quad + 2\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle \\ &\leq K\|v_{2\tau(n)+1} - v_{2\tau(n)}\| + 2\langle u - \bar{v}, v_{2\tau(n)} - \bar{v} \rangle. \end{aligned} \quad (3.20)$$

Hence, it follows from (3.19) and (3.20) that

$$\lim_{n \rightarrow \infty} S_{\tau(n)+1} = 0. \quad (3.21)$$

By (3.15) and (3.21), we know that  $\lim_{n \rightarrow \infty} S_n = 0$ . Then, just as the argument in the proof of Case 1, we obtain  $\lim_{n \rightarrow \infty} v_n = \bar{v}$ . Therefore, the proof is completed.  $\square$

### Remark 3.1

- (i) If we put  $B_1 = B_2 = 0$  in Theorem 3.1, then Theorem 3.1 is reduced to Theorem 7 in [20].
- (ii) Boikanyo *et al.* [20] showed that strong convergence theorem of a proximal point algorithm with error can be obtained from strong convergence of a proximal point algorithm without errors. Therefore, in Theorem 3.1, we study strong convergence of variational inclusion problems without error.

As a simple consequence of Theorem 3.1, we have the following theorem.

**Theorem 3.2** Suppose that  $(B_1 + G_1)^{-1}(0)$  is nonempty, and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$  are sequences in  $[0, 1]$  such that  $a_n + b_n + c_n = 1$ ,  $0 < a_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ , define a sequence  $\{x_n\}$  by

$$\begin{cases} x_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ x_{n+1} := a_n u + b_n x_n + c_n J_{\delta_n}^{G_1}(I - \delta_n B_1)x_n, \quad n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n = P_{(B_1 + G_1)^{-1}(0)} u$  if the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\sum_{n=1}^{\infty} a_n = \infty$ ;
- (ii)  $0 < a \leq \delta_n \leq b < 2\kappa_1$  for each  $n \in \mathbb{N}$  and for some  $a, b \in \mathbb{R}^+$ ;
- (iii)  $\liminf_{n \rightarrow \infty} c_n > 0$ .

*Proof* Set  $f_n = 0$ ,  $B_2 = 0$ ,  $g_n + h_n = 1$ ,  $\{g_n\}$  and  $\{h_n\}$  are sequences in  $[0, 1]$ , and  $G_2 = \partial t_{\mathcal{H}_1}$ . Define a sequence  $v_n$  by

$$v_{2n+1} = a_n u + b_n x_n + c_n J_{\delta_n}^{G_1}(I - \delta_n B_1)x_n, \quad n \in \mathbb{N} \cup \{0\}$$

and

$$v_{2n} = v_{2n-1} = x_{n-1}.$$

Then

$$v_{2n+1} = a_n u + b_n v_{2n} + c_n J_{\delta_n}^{G_1} (I - \delta_n B_1) v_{2n}, \quad n \in \mathbb{N} \cup \{0\},$$

and

$$v_{2n} = f_n u + g_n v_{2n-1} + h_n v_{2n-1} = f_n u + g_n v_{2n-1} + h_n J_{\gamma_n}^{\partial \iota \mathcal{H}_1} v_{2n-1}, \quad n \in \mathbb{N} \cup \{0\}.$$

Since

$$G_2^{-1}(0) = (\partial \iota \mathcal{H}_1)^{-1} 0 = \text{Fix}(J_{\gamma_n}^{\partial \iota \mathcal{H}_1}) = \text{Fix}(P_{\mathcal{H}_1}) = \mathcal{H}_1,$$

it is easy to see that

$$(B_1 + G_1)^{-1}(0) = ((B_1 + G_1)^{-1}(0) \cap \mathcal{H}_1) = ((B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)) \neq \emptyset.$$

Then Theorem 3.2 follows from Theorem 3.1.  $\square$

**Remark 3.2** Following the same argument as in Remark 12 in [20], we see that Theorems 3.1 and 3.2 contain [23, Theorem 3.3], [24, Theorem 1], [32, Theorems 1-4] and many other recent results as special cases.

## 4 Applications

Now, we recall the following multiple sets split feasibility problem (MSSFP-A1):

$$\text{Find } \bar{x} \in \mathcal{H}_1 \text{ such that } \bar{x} \in G_1^{-1}(0) \cap G_2^{-1}(0), A_1 \bar{x} \in \text{Fix}(F_1) \text{ and } A_2 \bar{x} \in \text{Fix}(F_2).$$

**Theorem 4.1** [33] *Given any  $\bar{x} \in \mathcal{H}_1$ ,  $\{\lambda_n\}$  and  $\{\gamma_n\}$  are sequences in  $(0, \infty)$ .*

- (i) *If  $\bar{x}$  is a solution of (MSSFP-A1), then*  

$$J_{\lambda_n}^{G_1} (I - \rho_n A_1^* (I - F_1) A_1) J_{\gamma_n}^{G_2} (I - \sigma_n A_2^* (I - F_2) A_2) \bar{x} = \bar{x} \text{ for each } n \in \mathbb{N}.$$
- (ii) *Suppose that  $J_{\lambda_n}^{G_1} (I - \rho_n A_1^* (I - F_1) A_1) J_{\gamma_n}^{G_2} (I - \sigma_n A_2^* (I - F_2) A_2) \bar{x} = \bar{x}$  with*  

$$0 < \rho_n < \frac{2}{\|A_1\|^2 + 2}, \quad 0 < \sigma_n < \frac{2}{\|A_2\|^2 + 2} \text{ for each } n \in \mathbb{N} \text{ and the solution set of (MSSFP-A1) is nonempty. Then } \bar{x} \text{ is a solution of (MSSFP-A1).}$$

In order to study the convergence theorems for the solution set of multiple split feasibility problem (MSSFP-A1), we need the following problems and the following essential tool which is a special case of Theorem 3.2 in [33]:

$$(\text{SFP-1}) \quad \text{Find } \bar{x} \in \mathcal{H}_1 \text{ such that } \bar{x} \in \text{Fix}(J_{\rho_n}^{G_1}) \text{ and } A_1 \bar{x} \in \text{Fix}(F_1).$$

**Lemma 4.1** *Given any  $\bar{x} \in \mathcal{H}_1$ .*

- (i) *If  $\bar{x}$  is a solution of (SFP-1), then  $J_{\rho_n}^{G_1} (I - \rho_n A_1^* (I - F_1) A_1) \bar{x} = \bar{x}$  for each  $n \in \mathbb{N}$ .*

- (ii) Suppose that  $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}$  with  $0 < \rho_n < \frac{2}{\|A_1\|^2 + 2}$  for each  $n \in \mathbb{N}$ , and the solution set of (SFP-1) is nonempty. Then  $(I - \rho_n A_1^*(I - F_1)A_1)$  and  $J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)$  are averaged and  $\bar{x}$  is a solution of (SFP-1).

*Proof* (i) Suppose that  $\bar{x} \in \mathcal{H}_1$  is a solution of (SFP-1). Then  $\bar{x} \in \text{Fix}(J_{\rho_n}^{G_1})$ ,  $A_1\bar{x} \in \text{Fix}(F_1)$  for each  $n \in \mathbb{N}$ . It is easy to see that

$$J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}, \quad n \in \mathbb{N}.$$

(ii) Since the solution set of (SFP-1) is nonempty, there exists  $\bar{w} \in \mathcal{H}_1$  such that  $\bar{w} \in \text{Fix}(J_{\rho_n}^{G_1})$ ,  $A_1\bar{w} \in \text{Fix}(F_1)$ . Then  $\bar{w} \in G_1^{-1}(0)$ . If we put  $G_2 = G_1$  and  $F_2 = F_1$ , we get that the solution set of (MSSFP-A1) is nonempty. By Lemma 2.1 we have that

$$A_1^*(I - F_1)A_1 \text{ is } \frac{1}{\|A_1\|^2} \text{-ism.} \quad (4.1)$$

By (4.1),  $0 < \rho_n < \frac{2}{\|A_1\|^2 + 2}$ , and Lemma 2.8(ii), (iii), we know that

$$I - \rho_n A_1^*(I - F_1)A_1 \text{ is averaged for each } n \in \mathbb{N}. \quad (4.2)$$

On the other hand, for each  $n \in \mathbb{N}$ ,  $J_{\rho_n}^{G_1}$  is a firmly nonexpansive mappings, it is easy to see that

$$J_{\rho_n}^{G_1} \text{ is } \frac{1}{2} \text{-averaged.} \quad (4.3)$$

Hence, by (4.2), (4.3) and Lemma 2.8(iv) and (v), we see that

$$J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1) \text{ is averaged.}$$

Since

$$J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x},$$

so

$$J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)\bar{x} = \bar{x}.$$

Then Lemma 4.1 follows from Theorem 4.1 by taking  $G_1 = G_2$ ,  $F_1 = F_2$  and  $\rho_n = r_n$ .  $\square$

**Remark 4.1** From the following result, we know that Lemma 4.1 is more useful than Theorem 4.1.

**Theorem 4.2** Suppose that the solution set  $\Omega_{A_1}$  of (MSSFP-A1) is nonempty and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{f_n\}$ ,  $\{g_n\}$ ,  $\{h_n\}$  are sequences in  $[0, 1]$  such that  $a_n + b_n + c_n = 1$ ,  $f_n + g_n + h_n = 1$ ,  $0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrarily fixed  $u \in \mathcal{H}$ , a sequence  $\{v_n\}$  is defined by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1)v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - F_2)A_2)v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A1}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \rho_n \leq b < \frac{2}{\|A_1\|^2 + 2}$ ,  $0 < c \leq \sigma_n \leq d < \frac{2}{\|A_2\|^2 + 2}$  for each  $n \in \mathbb{N}$ , and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Since  $F_i$  is firmly nonexpansive, it follows from Lemma 4.1 that  $A_i^*(I - F_i)A_i$  is  $\frac{1}{\|A_i\|^2}$ -ism for each  $i = 1, 2$ . For each  $i = 1, 2$ , put  $B_i = A_i^*(I - F_i)A_i$  in Lemma 4.1. Then algorithm in Theorem 3.1 follows immediately from algorithm in Theorem 4.2. Since  $\Omega_{A1}$  is nonempty, by Lemma 4.1, we have that

$$\bar{w} \in (\text{Fix}(J_{\rho_n}^{G_1}((I - \rho_n A_1^*(I - F_1)A_1))) \cap \text{Fix}(J_{\sigma_n}^{G_2}((I - \sigma_n A_2^*(I - F_2)A_2)))) \neq \emptyset \quad (4.4)$$

for each  $n \in \mathbb{N}$ . This implies that

$$\bar{w} \in (\text{Fix}(J_{\rho_n}^{G_1}(I - \rho_n B_1)) \cap (\text{Fix}(J_{\sigma_n}^{G_2}(I - \sigma_n B_2)))) \neq \emptyset \quad (4.5)$$

for each  $n \in \mathbb{N}$ . Hence,

$$\bar{w} \in ((B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)) \neq \emptyset.$$

By Theorem 3.1,  $\lim_{n \rightarrow \infty} v_n = \bar{x}$ , where  $\bar{x} = P_{(B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)} u$ . That is,

$$\bar{x} \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$$

and

$$\|\bar{x} - u\| \leq \|q - u\|$$

for all  $q \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$ . Since

$$\bar{x} \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0),$$

we know that

$$\bar{x} \in \text{Fix}(J_{\rho_n}^{G_1}(I - \rho_n B_1)) \cap \text{Fix}(J_{\sigma_n}^{G_2}(I - \sigma_n B_2)).$$

That is,

$$\bar{x} \in \text{Fix}(J_{\rho_n}^{G_1}(I - \rho_n A_1^*(I - F_1)A_1))$$

and

$$\bar{x} \in \text{Fix}(J_{\sigma_n}^{G_2}(I - \sigma_n A_2^*(I - F_2)A_2)).$$

By Lemma 4.1, we get that  $\bar{x} \in \Omega_{A_1}$ . Similarly, if  $q \in (B_1 + G_1)^{-1}(0) \cap (B_2 + G_2)^{-1}(0)$ , then  $q \in \Omega_{A_1}$ . Therefore  $\bar{x} = P_{\Omega_{A_1}}u$ . This shows that  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the optimization problem

$$\min_{q \in \Omega_{A_1}} \|q - u\|.$$

Therefore, the proof is completed.  $\square$

In the following theorem, we study the following multiple sets split feasibility problem (MSSMVIP-A2):

$$\text{Find } \bar{x} \in \mathcal{H}_1 \text{ such that } \bar{x} \in C, A_1 \bar{x} \in Q \text{ and } A_2 \bar{x} \in Q'.$$

Let  $\Omega_{A_2}$  denote the solution set of (MSSMVIP-A2). The following theorem is a special case of Theorem 4.3. Hence, it is also a special case of Theorem 4.2.

**Theorem 4.3** Suppose that  $\Omega_{A_2}$  is nonempty, and that  $\{a_n\}, \{b_n\}, \{c_n\}, \{f_n\}, \{g_n\}, \{h_n\}$  are sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1, f_n + g_n + h_n = 1, 0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ , a sequence  $\{v_n\}$  is defined by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n P_C(I - \rho_n A_1^*(I - P_Q)A_1)v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n P_C(I - \sigma_n A_2^*(I - P_{Q'})A_2)v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A_2}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \rho_n \leq b < \frac{2}{\|A_1\|^2 + 2}, 0 < c \leq \sigma_n \leq d < \frac{2}{\|A_2\|^2 + 2}$  for each  $n \in \mathbb{N}$  and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0, \liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Put  $G_1 = G_2 = \partial_{\iota_C}, F_1 = P_Q$ , and  $F_2 = P_{Q'}$ . Then  $G_1, G_2$  are two set-valued maximum monotone mappings,  $F_1$  and  $F_2$  are firmly nonexpansive mappings. Since  $J_{\rho_n}^{\partial_{\iota_C}} = P_C$  and  $J_{\sigma_n}^{\partial_{\iota_C}} = P_C$ , we have  $\text{Fix}(F_1) = \text{Fix}(P_Q) = Q, \text{Fix}(F_2) = \text{Fix}(P_{Q'}) = Q', \text{Fix}(J_{\rho_n}^{\partial_{\iota_C}}) = \text{Fix}(P_C) = C$  and  $\text{Fix}(J_{\sigma_n}^{\partial_{\iota_C}}) = \text{Fix}(P_C) = C$ . Then Theorem 4.3 follows from Theorem 4.2.  $\square$



In the following theorem, we study the following split feasibility problem (MSSMVIP-A3):

Find  $\bar{x} \in \mathcal{H}_1$  such that  $\bar{x} \in C \cap Q'$ ,  $A_1 \bar{x} \in Q$  where  $Q'$  is a nonempty closed subset of  $\mathcal{H}_1$ .

Let  $\Omega_{A3}$  denote the solution set of problem (MSSMVIP-A3). The following is also a special case of Theorem 4.3.

**Theorem 4.4** *Suppose that  $Q'$  is a nonempty closed convex subset of  $\mathcal{H}_1$ ,  $\Omega_{A3}$  is nonempty, and  $\{a_n\}, \{b_n\}, \{c_n\}, \{f_n\}, \{g_n\}, \{h_n\}$  are sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1, f_n + g_n + h_n = 1, 0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ , a sequence  $\{v_n\}$  is defined by*

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n P_C(I - \rho_n A_1^*(I - P_Q)A_1)v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n P_C(I - \sigma_n(I - P_{Q'}))v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A3}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \rho_n \leq b < \frac{2}{\|A_1\|^2 + 2}, 0 < c < \sigma_n < d < \frac{2}{3}$  for each  $n \in \mathbb{N}$  and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0, \liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Put  $A_2 = I$  and  $\mathcal{H}_1 = \mathcal{H}_3$  in Theorem 4.3. Then Theorem 4.4 follows from Theorem 4.3.  $\square$

In the following theorem, we study the following convex feasibility problem (MSSMVIP-A4):

Find  $\bar{x} \in \mathcal{H}_1$  such that  $\bar{x} \in C \cap Q \cap Q'$ , where  $Q, Q'$  are nonempty closed subsets of  $\mathcal{H}_1$ .

Let  $\Omega_{A4}$  denote the solution set of (MSSMVIP-A4). The following is a special case of Theorem 4.3.

**Theorem 4.5** *Suppose that  $Q$  and  $Q'$  are nonempty closed convex subsets of  $\mathcal{H}_1$ ,  $\Omega_{A4}$  is nonempty, and  $\{a_n\}, \{b_n\}, \{c_n\}, \{f_n\}, \{g_n\}, \{h_n\}$  are sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1, f_n + g_n + h_n = 1, 0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ ,*

a sequence  $\{v_n\}$  is defined by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n P_C(I - \rho_n(I - P_Q))v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n P_C(I - \sigma_n(I - P_{Q'}))v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A4}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a < \rho_n < b < \frac{2}{3}$ ,  $0 < c < \sigma_n < d < \frac{2}{3}$  for each  $n \in \mathbb{N}$  and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Put  $A_1 = A_2 = I$  and  $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$  in Theorem 4.3. Then Theorem 4.5 follows from Theorem 4.3.  $\square$

In the following theorem, we study the following convex feasibility problem (MSSMVIP-A5):

Find  $\bar{x} \in \mathcal{H}_1$  such that  $\bar{x} \in Q \cap Q'$ , where  $Q$  and  $Q'$  are nonempty closed convex subsets of  $\mathcal{H}_1$ .

Let  $\Omega_{A5}$  denote the solution set of (MSSMVIP-A5).

The following existent theorem of a convex feasibility problem follows immediately from Theorem 4.5.

**Theorem 4.6** Suppose that  $Q$  and  $Q'$  are nonempty closed convex subsets of  $\mathcal{H}_1$ ,  $\Omega_{A5}$  is nonempty, and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{f_n\}$ ,  $\{g_n\}$ ,  $\{h_n\}$  are sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1$ ,  $f_n + g_n + h_n = 1$ ,  $0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ . Define a sequence  $\{v_n\}$  by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n (I - \rho_n(I - P_Q))v_{2n}, & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n (I - \sigma_n(I - P_{Q'}))v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A5}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;

- (iii)  $0 < a < \rho_n < b < \frac{2}{3}$ ,  $0 < c < \sigma_n < d < \frac{2}{3}$  for each  $n \in \mathbb{N}$  and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Put  $C = \mathcal{H}_1$ , then  $P_C = P_{\mathcal{H}_1}$ . Then Theorem 4.6 follows from Theorem 4.5.  $\square$

In the following theorem, we study the following system of convexly constrained linear inverse problem (SCCLIP):

Find  $\bar{x} \in \mathcal{H}_1$  such that  $\bar{x} \in C$ ,  $A_1 \bar{x} = b$  and  $A_2 \bar{x} = b'$ , where  $b \in \mathcal{H}_2$  and  $b' \in \mathcal{H}_3$ .

Let  $\Omega_{A_6}$  denote the solution set of (SCCLIP).

**Theorem 4.7** Suppose that  $\Omega_{A_6}$  is nonempty, and  $b \in \mathcal{H}_2$ ,  $b' \in \mathcal{H}_3$ . Let  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{f_n\}$ ,  $\{g_n\}$ , and  $\{h_n\}$  be sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1$ ,  $f_n + g_n + h_n = 1$ ,  $0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ , a sequence  $\{v_n\}$  is defined by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n P_C(v_{2n} - \rho_n A_1^*(A_1 v_{2n} - b)), & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n P_C(v_{2n-1} - \sigma_n A_2^*(A_2 v_{2n-1} - b')), & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A_6}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \rho_n \leq b < \frac{2}{\|A_1\|^2 + 2}$ ,  $0 < c \leq \sigma_n \leq d < \frac{2}{\|A_2\|^2 + 2}$  for each  $n \in \mathbb{N}$  and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

*Proof* Put  $Q = \{b\}$  and  $Q' = \{b'\}$ . Then Theorem 4.7 follows from Theorem 4.2.  $\square$

In the following theorem, we study the following convexly constrained linear inverse problem (CCLIP):

Find  $\bar{x} \in \mathcal{H}_1$  such that  $\bar{x} \in C \cap Q'$  and  $A_1 \bar{x} = b$ , where  $b \in \mathcal{H}_2$  and  $Q'$  is a nonempty closed convex subset of  $\mathcal{H}_1$ .

Let  $\Omega_{A_7}$  denote the solution set of (CCLIP).

**Theorem 4.8** Suppose that  $Q'$  is a nonempty closed convex subset of  $\mathcal{H}_1$ .  $\Omega_{A_7}$  is nonempty,  $b \in \mathcal{H}_2$ , and  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{f_n\}$ ,  $\{g_n\}$ ,  $\{h_n\}$  are sequences in  $[0, 1]$  with  $a_n + b_n + c_n = 1$ ,  $f_n + g_n + h_n = 1$ ,  $0 < a_n < 1$ , and  $0 < f_n < 1$  for each  $n \in \mathbb{N}$ . For an arbitrary fixed  $u \in \mathcal{H}$ , a sequence  $\{v_n\}$  is defined by

$$\begin{cases} v_0 \in \mathcal{H} \text{ is chosen arbitrarily,} \\ v_{2n+1} := a_n u + b_n v_{2n} + c_n P_C(v_{2n} - \rho_n (A_1 v_{2n} - b)), & n \in \mathbb{N} \cup \{0\}, \\ v_{2n} := f_n u + g_n v_{2n-1} + h_n P_C(I - \sigma_n (I - P_{Q'}))v_{2n-1}, & n \in \mathbb{N}. \end{cases}$$

Then  $\lim_{n \rightarrow \infty} v_n$  is a unique solution of the following optimization problem:

$$\min_{q \in \Omega_{A7}} \|q - u\|$$

provided the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f_n = 0$ ;
- (ii) either  $\sum_{n=1}^{\infty} a_n = \infty$  or  $\sum_{n=1}^{\infty} f_n = \infty$ ;
- (iii)  $0 < a \leq \rho_n \leq b < \frac{2}{\|A_1\|^2 + 2}$ ,  $0 < c < \sigma_n < d < \frac{2}{3}$  for each  $n \in \mathbb{N}$ , and for some  $a, b, c, d \in \mathbb{R}^+$ ;
- (iv)  $\liminf_{n \rightarrow \infty} c_n > 0$ ,  $\liminf_{n \rightarrow \infty} h_n > 0$ .

**Remark 4.2** The iteration in Theorem 4.8 is different from the Landweber iterative method [19]:

$$x_{n+1} := x_n + \gamma A^T(b - Ax_n), \quad n \in \mathbb{N}.$$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

L-JL designed and coordinated this research project and revised the paper. Y-DC carried out the project, drafted and revised the manuscript. C-SC coordinated the project and revised the paper.

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