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Existence of solutions for weighted $p(r)$-Laplacian impulsive system mixed type boundary value problems

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Abstract

This paper investigates the existence of solutions for weighted $p(r)$-Laplacian impulsive system mixed type boundary value problems. The proof of our main result is based upon Gaines and Mawhin's coincidence degree theory. Moreover, we obtain the existence of nonnegative solutions.

Keywords: Weighted $p(r)$-Laplacian, impulsive system, coincidence degree

1 Introduction

In this paper, we mainly consider the existence of solutions for the weighted $p(r)$-Laplacian system

$$-(w(r)|u'|^{p(r)-2}u'(r))' + f(r, u(r), (w(r))^{\frac{1}{p(r)-1}}u'(r)) = 0, \quad r \in (0, T), \quad r \neq r_i, \tag{1}$$

where u: $[0, T] \to \mathbb{R}^N$, with the following impulsive boundary conditions

$$\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = A_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}}u'(r)), \quad i = 1, \ldots, k, \tag{2}$$

$$\begin{aligned}&\lim_{r\to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim_{r\to r_i^-} w(r)|u'|^{p(r)-2}u'(r)\\ &= B_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}}u'(r)), \quad i = 1, \ldots, k,\end{aligned} \tag{3}$$

$$au(0) - b\lim_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}}u'(r) = 0, \text{ and } cu(T) + d\lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0, \tag{4}$$

where $p \in C([0, T], \mathbb{R})$ and $p(r) > 1$, $-\Delta_{p(r)}\, u := -(w(r)\,|u'|^{p(r)-2}\, u'(r))'$ is called weighted $p(r)$-Laplacian; $0 < r_1 < r_2 < \ldots < r_k < T$; $A_i, B_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$; $a, b, c, d \in [0, +\infty)$, $ad + bc > 0$.

Throughout the paper, $o(1)$ means functions which uniformly convergent to 0 (as $n \to +\infty$); for any $v \in \mathbb{R}^N$, v^j will denote the j-th component of v; the inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$; $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Denote $J = [0, T]$, $J' = [0, T]\backslash\{r_0, r_1,\ldots, r_{k+1}\}$, $J_0 = [r_0, r_1]$, $J_i = (r_i, r_{i+1}]$, $i = 1, \ldots, k$, where $r_0 = 0$, $r_{k+1} = T$. Denote J_i^o the interior of J_i, $i = 0, 1,\ldots, k$. Let $PC(J, \mathbb{R}^N) = \{x$: J

$\to \mathbb{R}^N \mid x \in C(J_i, \mathbb{R}^N)$, $i = 0, 1,..., k$, and $\lim_{r\to r_i^+} x(r)$ exists for $i = 1,..., k\}$; $w \in PC(J, \mathbb{R})$ satisfies $0 < w(r)$, $\forall r \in J'$, and $(w(\cdot))^{\frac{-1}{p(\cdot)-1}} \in L^1(0,T)$; $\lim_{r\to r_i^+} w(r)|x'|^{p(r)-2}x'(r)$, $\lim_{r\to r_i^+} w(r)|x'|^{p(r)-2}x'(r)$ and $\lim_{r\to r_{i+1}^-} w(r)|x'|^{p(r)-2}x'(r)$ exists for $i = 0, 1,..., k\}$. For any $x = (x^1,..., x^N) \in PC(J, \mathbb{R}^N)$, denote $|x^i|_0 = \sup_{r\in J'} |x^i(r)|$. Obviously, $PC(J, \mathbb{R}^N)$ is a Banach space with the norm $||x||_0 = (\sum_{i=1}^N |x^i|_0^2)^{\frac{1}{2}}$, $PC^1(J, \mathbb{R}^N)$ is a Banach space with the norm $||x||_1 = ||x||_0 + ||(w(r))^{\frac{1}{p(r)-1}} x'||_0$. In the following, $PC(J, \mathbb{R}^N)$ and $PC^1(J, \mathbb{R}^N)$ will be simply denoted by PC and PC^1, respectively. Let $L^1 = L^1(J, \mathbb{R}^N)$ with the norm $||x||_{L^1} = (\sum_{i=1}^N |x^i|_{L^1}^2)^{\frac{1}{2}}$, $\forall x \in L^1$, where $|x^i|_{L^1} = \int_0^T |x^i(r)|dr$. We will denote

$$u(r_i^+) = \lim_{r\to r_i^+} u(r), \quad u(r_i^-) = \lim_{r\to r_i^-} u(r),$$

$$w(0)|u'|^{p(0)-2}u'(0) = \lim_{r\to 0^+} w(r)|u'|^{p(r)-2}u'(r),$$
$$w(T)|u'|^{p(T)-2}u'(T) = \lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r).$$

The study of differential equations and variational problems with nonstandard $p(r)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electro-rheological fluids, image processing, etc. (see [1-4]). Many results have been obtained on this problems, for example [1-25]. If $p(r) \equiv p$ (a constant), (1) is the well-known p-Laplacian system. If $p(r)$ is a general function, $-\Delta_{p(r)}$ represents a nonhomogeneity and possesses more nonlinearity, thus $-\Delta_{p(r)}$ is more complicated than $-\Delta_p$; for example, if $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(\cdot)} = \inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x}$$

is zero in general, and only under some special conditions $\lambda_{p(\cdot)} > 0$ (see [8,17-19]), but the property of $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

Impulsive differential equations have been studied extensively in recent years. Such equations arise in many applications such as spacecraft control, impact mechanics, chemical engineering and inspection process in operations research (see [26-28] and the references therein). It is interesting to note that $p(r)$-Laplacian impulsive boundary problems are about comparatively new applications like ecological competition, respiratory dynamics and vaccination strategies. On the Laplacian impulsive differential equation boundary value problems, there are many results (see [29-37]). There are many methods to deal with this problem, e.g., subsupersolution method, fixed point theorem, monotone iterative method and coincidence degree. Because of the nonlinearity of $-\Delta_p$, results on the existence of solutions for p-Laplacian impulsive differential equation boundary value problems are rare (see [38,39]). On the Laplacian ($p(x) \equiv 2$) impulsive differential equations mixed type boundary value problems, we refer to [30,32,34].

In [39], Tian and Ge have studied nonlinear IBVP

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x(t)), \quad t \neq t_i, a.e. t \in [a, b], \\ \lim_{t\to t_i^+} \rho(t)\Phi_p(x'(t)) - \lim_{t\to t_i^-} \rho(t)\Phi_p(x'(t)) = I_i(x(t_i)), \quad i = 1, \ldots, l, \\ \alpha x'(a) - \beta x(a) = \sigma_1, \quad \gamma x'(b) + \sigma x(b) = \sigma_2, \end{cases} \tag{5}$$

where $\Phi_p(x) = |x|^{p-2} x$, $p > 1$, $\rho, s \in L^\infty [a, b]$ with *essin* $f_{[a,\ b]}\ \rho > 0$, and *essin* $f_{[a,b]}\ s > 0$, $0 < \rho(a)$, $p(b) < \infty$, $\sigma_1 \leq 0$, $\sigma_2 \geq 0$, $\alpha, \beta, \gamma, \sigma > 0$, $a = t_0 < t_1 < \ldots < t_l < t_{l+1} = b$, $I_i \in C([0, +\infty), [0, \infty))$, $i = 1, \ldots, l$, $f \in C([a, b] \times [0, +\infty), [0, \infty))$, $f(\cdot, 0)$ is nontrivial. By using variational methods, the existence of at least two positive solutions was obtained.

In [24,25], the present author investigates the existence of solutions of $p(r)$-Laplacian impulsive differential equation (1-3) with periodic-like or multi-point boundary value conditions.

In this paper, we consider the existence of solutions for the weighted $p(r)$-Laplacian impulsive differential system mixed type boundary value condition problems, when $p(r)$ is a general function. The proof of our main result is based upon Gaines and Mawhin's coincidence degree theory. Since the nonlinear term f in (5) is independent on the first-order derivative, and the impulsive conditions are simpler than (2), our main results partly generalized the results of [30,32,34,39]. Since the mixed type boundary value problems are different from periodic-like or multi-point boundary value conditions, and this paper gives two kinds of mixed type boundary value conditions (linear and nonlinear), our discussions are different from [24,25] and have more difficulties. Moreover, we obtain the existence of nonnegative solutions. This paper was motivated by [24-26,38,40].

Let $N \geq 1$, the function f: $J \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory; by this, we mean:

(i) for almost every $t \in J$, the function $f(t, \cdot, \cdot)$ is continuous;
(ii) for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, the function $f(\cdot, x, y)$ is measurable on J;
(iii) for each $R > 0$, there is a $\alpha_R \in L^1(J, \mathbb{R})$, such that, for almost every $t \in J$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, one has

$$|f(t, x, y)| \leq \alpha_R(t).$$

We say a function u: $J \to \mathbb{R}^N$ is a solution of (1) if $u \in PC^1$ with $w(\cdot)\ |u'|^{p(\cdot)-2}\ u'(\cdot)$ absolutely continuous on every J_i^o, $i = 0, 1, \ldots, k$, which satisfies (1) *a.e.* on J.

This paper is divided into three sections; in the second section, we present some preliminary. Finally, in the third section, we give the existence of solutions and nonnegative solutions of system (1)-(4).

2 Preliminary

Let X and Y be two Banach spaces and L: $D(L) \subset X \to Y$ be a linear operator, where $D(L)$ denotes the domain of L. L will be a Fredholm operator of index 0, *i.e.*, ImL is closed in Y and the linear spaces $KerL$ and $coImL$ have the same dimension which is finite. We define $X_1 = KerL$ and $Y_1 = coImL$, so we have the decompositions $X = X_1 \oplus coKerL$ and $Y = Y_1 \oplus ImL$. Now, we have the linear isomorphism Λ: $X_1 \to Y_1$ and the continuous linear projectors P: $X \to X_1$ and Q: $Y \to Y_1$ with $KerQ = ImL$ and $ImP = X_1$.

Let Ω be an open bounded subset of X with $\Omega \cap D(L) \neq \emptyset$. Operator $S : \overline{\Omega} \to Y$ be a continuous operator. In order to define the coincidence degree of (L, S) in Ω, as in [40,41], denoted by $d(L - S, \Omega)$, we assume that

$$Lx \neq Sx \quad \text{for all } x \in \partial\Omega.$$

It is easy to see that the operator $M : \overline{\Omega} \to X$, $M = (L + \Lambda P)^{-1} (S + \Lambda P)$ is well defined, and

$$Lx^* = Sx^* \text{ if and only if } x^* = Mx^*.$$

If M is continuous and compact, then S is called L-compact, and the Leray-Schauder degree of $I_X - M$ (where I_X is the identity mapping of X) is well defined in Ω, and we will denote it by $d_{LS} (I_X - M, \Omega, 0)$. This number is independent of the choice of P, Q and Λ (up to a sign) and we can define

$$d(L - S, \Omega) := d_{LS}(I_X - M, \Omega, 0).$$

Definition 2.1. (see [40,41]) The coincidence degree of (L, S) in Ω, denoted by $d(L - S, \Omega)$, is defined as $d(L - S, \Omega) = d_{LS} (I_X - M, \Omega, 0)$.

There are many papers about coincidence degree and its applications (see [40-43]).

Proposition 2.2. (see [40]) (i) (Existence property). If $d(L - S, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $Lx = Sx$.

(ii) (Homotopy invariant property). If $H : \overline{\Omega} \times [0, 1] \to Y$ is continuous, L-compact and $Lx \neq H(x, \lambda)$ for all $x \in \partial\Omega$ and $\lambda \in [0, 1]$, then $d(L - H (\cdot, \lambda), \Omega)$ is independent of λ.

The effect of small perturbations is negligible, as is proved in the next Proposition (see [41] Theorem III.3, page 24).

Proposition 2.3. Assume that $Lx \neq Sx$ for each $x \in \partial\Omega$. If S_ε is such that $\sup_{x\in \partial\Omega}||S_\varepsilon x||_Y$ is sufficiently small, then $Lx \neq Sx + S_\varepsilon x$ for all $x \in \partial\Omega$ and $d(L - S - S_\varepsilon, \Omega) = d(L - S, \Omega)$.

For any $(r, x) \in (J \times \mathbb{R}^N)$, denote $\phi_{p(r)}(x) = |x|^{p(r)-2}x$. Obviously, ϕ has the following properties

Proposition 2.4 (see [41]) ϕ is a continuous function and satisfies

(i) For any $r \in [0, T]$, $\phi_{p(r)}(\cdot)$ is strictly monotone, i.e.,

$$\langle \varphi_{p(r)}(x_1) - \varphi_{p(r)}(x_2), x_1 - x_2 \rangle > 0, \quad \forall x_1, x_2 \in \mathbb{R}^N, \quad x_1 \neq x_2;$$

(ii) There exists a function η: $[0, +\infty) \to [0, +\infty)$, $\eta(s) \to +\infty$ as $s \to +\infty$, such that

$$\langle \varphi_{p(r)}(x), x \rangle \geq \eta(|x|)|x|, \quad \text{for all } x \in \mathbb{R}^N.$$

It is well known that $\phi_{p(r)}(\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $r \in J$. Denote

$$\varphi_{p(r)}^{-1}(x) = |x|^{\frac{2-p(r)}{p(r)-1}}x, \quad \text{for } x \in \mathbb{R}^N\backslash\{0\}, \quad \varphi_{p(r)}^{-1}(0) = 0.$$

It is clear that $\varphi_{p(r)}^{-1}(\cdot)$ is continuous and sends bounded sets to bounded sets, and $\varphi_{p(r)}^{-1}(\cdot) = \varphi_{q(r)}(\cdot)$ where $\frac{1}{p(r)} + \frac{1}{q(r)} \equiv 1$. Let $X = \{(x_1, x_2) \mid x_1 \in PC, x_2 \in PC\}$ with the norm $||(x_1, x_2)||_X = || x_1||_0 + ||x_2||_0$, $Y = L^1 \times L^1 \times \mathbb{R}^{2(k+1)N}$, and we define the norm on Y as

$$||(y_1, y_2, z_1, \dots, z_{2(k+1)})||_Y = ||y_1||_{L^1} + ||y_2||_{L^1} + \sum_{m=1}^{2(k+1)} |z_m|, \forall(y_1, y_2, z_1, \dots, z_{2(k+1)}) \in Y,$$

where $y_1, y_2 \in L^1$, $z_m \in \mathbb{R}^N$, $m = 1,\dots, 2(k + 1)$, then X and Y are Banach spaces.
Define L: $D(L) \subset X \to Y$ and S: $X \to Y$ as the following

$$Lx = (x_1', x_2', \Delta x_1(r_i), \Delta x_2(r_i), 0, 0),$$

$$\begin{aligned} Sx = \big(&\varphi_{q(r)}\left(x_2/w(r)\right), f(r, x_1, \varphi_{q(r)}(x_2)), A_i, B_i, \\ &ax_1(0) - b\varphi_{q(0)}(x_2(0)), cx_1(T) + dx_2(T)\big), \end{aligned}$$

where

$$\begin{gathered} \Delta x_j(r_i) = x_j(r_i^+) - x_j(r_i^-), \quad j = 1, 2, \quad i = 1, \dots, k; \\ A_i = A_i(x_1(r_i^-), \varphi_{q(r_i)}(x_2(r_i^-))), \quad B_i = B_i(x_1(r_i^-), \varphi_{q(r_i)}(x_2(r_i^-))), \quad i = 1, \dots, k. \end{gathered} \tag{6}$$

Obviously, the problem (1)-(4) can be written as $Lx = Sx$, where L: $X \to Y$ is a linear operator, S: $X \to Y$ is a nonlinear operator, and X and Y are Banach spaces.

Since

$$ImL = \{(y_1, y_2, a_i, b_i, 0, 0) | \forall y_1, y_2 \in L^1, \forall a_i, b_i \in \mathbb{R}^N, i = 1, \dots, k\},$$

we have dim$KerL$ = dim(Y/ImL) = $2N < +\infty$ is even and ImL is closed in Y, then L is a Fredholm operator of index zero. Define

$$\begin{gathered} P : X \to X, (x_1, x_2) \to (x_1(0), x_2(0)), \\ Q : Y \to Y, (y_1, y_2, a_i, b_i, h_1, h_2) \to (0, 0, 0, 0, h_1, h_2), \forall y_1, y_2 \in L^1, \forall a_i, b_i, h_1, h_2 \in \mathbb{R}^N, i = 1, \dots, k, \end{gathered}$$

at the same time the projectors P: $X \to X$ and Q: $Y \to Y$ satisfy

$$dim(ImP) = dim(KerL) = dim(Y/ImL) = dim(ImQ).$$

Since ImQ is isomorphic to $KerL$, there exists an isomorphism Λ: $KerL \to ImQ$. It is easy to see that $L\mid_{D(L)\cap KerP} : D(L) \cap KerP \to ImL$ is invertible. We denote the inverse of that mapping by K_p, then $K_p : ImL \to D(L) \cap KerP$ as

$$K_p z = \left(\int_0^t y_1(r)\mathrm{d}r + \sum_{r_i<t} a_i, \int_0^t y_2(r)\mathrm{d}r + \sum_{r_i<t} b_i\right), \quad \forall z = (y_1, y_2, a_i, b_i, 0, 0) \in ImL,$$

then

$$K_p(I-Q)Sx=\left(\int_0^t \varphi_{q(r)}((w(r))^{-1}x_2)\mathrm{d}r+\sum_{r_i<t}A_i(x_1(r_i^-),\varphi_{q(r_i)}(x_2(r_i^-))),\right.$$
$$\left.\int_0^t f(r,x_1,\varphi_{q(r)}(x_2))\mathrm{d}r+\sum_{r_i<t}B_i(x_1(r_i^-),\varphi_{q(r_i)}(x_2(r_i^-)))\right).$$

Proposition 2.5 (i) $K_p(\cdot)$ is continuous;

(ii) $K_p\ (I - Q)S$ is continuous and compact.

Proof. (i) It is easy to see that $K_p(\cdot)$ is continuous. Moreover, the operator $\Psi(y)=\int_0^t y(r)\mathrm{d}r$ sends equi-integrable set of L^1 to relatively compact set of PC.

(ii) It is easy to see that $K_p(I - Q)Sx \in X$, $\forall x \in X$. Since $(w(r))^{\frac{-1}{p(r)-1}} \in L^1$ and f is Caratheodory, it is easy to check that S is a continuous operator from X to Y, and the operators $(x_1, x_2) \to \phi_{q(r)}\ ((w(r))^{-1}\ x_2)$ and $(x_1, x_2) \to f\ (r, x_1, \phi_{q(r)}\ ((w(r))^{-1}\ x_2))$ both send bounded sets of X to equi-integrable set of L^1. Obviously, A_i, B_i and QS are compact continuous. Since f is Caratheodory, by using the Ascoli-Arzela theorem, we can show that the operator $K_p(I-Q)S:\overline{\Omega}\to X$ is continuous and compact. This completes the proof.

Denote

$$S(x,\lambda)=\Big(\lambda\varphi_{q(r)}\left(x_2/w(r)\right),\lambda^{p(r)}f(r,x_1,\varphi_{q(r)}(x_2)),\lambda^2A_i,\lambda^{p(r_i)}B_i,$$
$$ax_1(0)-b\varphi_{q(0)}(x_2(0)),cx_1(T)+\mathrm{d}x_2(T)\Big),$$

where A_i, B_i are defined in (6), $i = 1,..., k$.

Consider

$$Lx=S(x,\lambda).$$

Define $M(\cdot,\cdot):\overline{\Omega}\times[0,1]\to X$ as $M(\cdot, \cdot) = (L + \Lambda P)^{-1}\ (S(\cdot, \cdot) + \Lambda P)$, then

$$\begin{aligned}M(\cdot,\lambda)&=(L+\Lambda P)^{-1}(S(\cdot,\lambda)+\Lambda P)\\&=(K_p+\Lambda^{-1})((I-Q)S(\cdot,\lambda)+QS(\cdot,\lambda)+\Lambda P)\\&=K_p(I-Q)S(\cdot,\lambda)+\Lambda^{-1}(QS(\cdot,\lambda)+\Lambda P)\\&=K_p(I-Q)S(\cdot,\lambda)+\Lambda^{-1}QS(\cdot,\lambda)+P.\end{aligned}$$

Since $(I - Q)S(\cdot, 0) = 0$ and $K_p\ (0) = 0$, we have

$$d(L-S(\cdot,0),\Omega)=d_{LS}(I_X-M(\cdot,0),\Omega,0)=d_{LS}(I_X-\Lambda^{-1}QS(\cdot,0)-P,\Omega,0).$$

It is easy to see that all the solutions of $Lx = S(x, 0)$ belong to $KerL$, then

$$d_{LS}(I_X-\Lambda^{-1}QS(\cdot,0)-P,\Omega,0)=d_B(I_{KerL}-\Lambda^{-1}QS(\cdot,0)-P|_{KerL},\Omega\cap KerL,0).$$

Notice that $P\ |_{KerL} = I_{KerL}$, then

$$d(L-S(\cdot,0),\Omega)=d_{LS}(I_X-M(\cdot,0),\Omega,0)=d_B(\Lambda^{-1}QS(\cdot,0),\Omega\cap KerL,0).$$

Proposition 2.6 (continuation theorem) (see [40]). Suppose that L is a Fredholm operator of index zero and S is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. If the following conditions are satisfied,

(i) for each $\lambda \in (0, 1)$, every solution x of

$$Lx = S(x, \lambda)$$

is such that $x \notin \partial\Omega$;

(ii) $QS(x, 0) \neq 0$ for $x \in \partial\Omega \cap KerL$ and $d_B(\Lambda^{-1} QS(\cdot,0), \Omega \cap KerL, 0) \neq 0$, then the operator equation $Lx = S(x, 1)$ has one solution lying in $\overline{\Omega}$.

The importance of the above result is that it gives sufficient conditions for being able to calculate the coincidence degree as the Brouwer degree (denoted with d_B) of a related finite dimensional mapping. It is known that the degree of finite dimensional mappings is easier to calculate. The idea of the proof is the use of the homotopy of the problem $Lx = S(x, 1)$ with the finite dimensional one $Lx = S(x, 0)$.

Let us now consider the following simple impulsive problem

$$\left.\begin{array}{l} (w(r)\varphi_{p(r)}(u'(r)))' = g(r), \quad r \in J', \\ \lim\limits_{r\to r_i^+} u(r) - \lim\limits_{r\to r_i^-} u(r) = a_i, \quad i = 1, \dots, k, \\ \lim\limits_{r\to r_i^+} w(r)\varphi_{p(r)}(u'(r)) - \lim\limits_{r\to r_i^-} w(r)\varphi_{p(r)}(u'(r)) = b_i, \quad i = 1, \dots, k, \\ au(0) - b\lim\limits_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}} u'(r) = 0, \text{ and } cu(T) + d\lim\limits_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0, \end{array}\right\} \tag{7}$$

where $J' = [0, T]\backslash\{r_0, r_1, \dots, r_{k+1}\}$, $a_i, b_i \in \mathbb{R}^N$; $g \in L^1$.

If u is a solution of (7), then we have

$$w(r)\varphi_{p(r)}(u'(r)) = w(0)\varphi_{p(0)}(u'(0)) + \sum_{r_i<r} b_i + \int_0^r g(t)\mathrm{d}t, \quad \forall r \in J'. \tag{8}$$

Denote $\rho_0 = w(0)_{\phi p(0)}(u'(0))$. Obviously, ρ_0 is dependent on g, a_i, b_i. Define F: $L^1 \to PC$ as

$$F(g)(r) = \int_0^r g(t)dt, \ \forall r \in J, \quad \forall g \in L^1.$$

By (8), we have

$$u(r) = u(0) + \sum_{r_i<r} a_i + F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho_0 + \sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(r), \quad \forall r \in J. \tag{9}$$

If $a \neq 0$, then the boundary condition $au(0) - b\lim\limits_{r\to 0^+}(w(r))^{\frac{1}{p(r)-1}}u'(r) = 0$ implies that

$$u(r) = \frac{b}{a}\varphi_{q(0)}(\rho_0) + \sum_{r_i<r} a_i + F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho_0 + \sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(r), \quad \forall r \in J.$$

The boundary condition $cu(T) + d \lim_{r \to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0$ implies that

$$c\frac{b}{a}\varphi_{q(0)}(\rho_0) + c\sum_{i=1}^{k} a_i + cF\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho_0 + \sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(T)$$
$$+d\left(\rho_0 + \sum_{i=1}^{k} b_i + F(g)(T)\right) = 0.$$

Denote $H = L^1 \times \mathbb{R}^{2kN}$ with the norm

$$||h||_H = ||g||_{L^1} + \sum_{i=1}^{k}|a_i| + \sum_{i=1}^{k}|b_i|, \forall h = (g, a_i, b_i) \in H,$$

then H is a Banach space. For fixed $h \in H$, we denote

$$\Theta_h(\rho) = c\frac{b}{a}\varphi_{q(0)}(\rho) + c\sum_{i=1}^{k} a_i + cF\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(T)$$
$$+d\left(\rho + \sum_{i=1}^{k} b_i + F(g)(T)\right).$$

Lemma 2.7 The mapping $\Theta_h(\cdot)$ has the following properties

(i) For any fixed $h \in H$, the equation

$$\Theta_h(\rho) = 0 \tag{10}$$

has a unique solution $\rho(h) \in \mathbb{R}^N$.

(ii) The mapping ρ: $H \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover, $|\rho(h)| \leq 3N[(2N\frac{E+1}{E}\sum_{i=1}^{k}|a_i|)^{p\#-1} + \sum_{i=1}^{k}|b_i| + ||g||_{L^1}]$, where $h = (g, a_i, b_i) \in H$, $E = \int_0^T (w(r))^{\frac{-1}{p(r)-1}}dr$, the notation $p^{\#}$ means $C^{p\#-1} = \begin{cases} C^{p^+-1}, & C > 1 \\ C^{p^--1}, & C \leq 1 \end{cases}$.

Proof. (i) From Proposition 2.4, it is immediate that

$$\langle \Theta_h(\rho_1) - \Theta_h(\rho_2), \rho_1 - \rho_2 \rangle > 0, \quad \mathit{for} \quad \rho_1 \neq \rho_2,$$

and hence, if (10) has a solution, then it is unique.

Let $R_0 = 3N[(2N\frac{E+1}{E}\sum_{i=1}^{k}|a_i|)^{p\#-1} + \sum_{i=1}^{k}|b_i| + \| g\|_{L^1}]$. Since $(w(r))^{\frac{-1}{p(r)-1}} \in L^1(0, T)$ and $F(g) \in PC$, if $|\rho| > R_0$, it is easy to see that there exists a j_0 such that, the j_0-th component ρ^{j_0} of ρ satisfies

$$|\rho^{j_0}| \geq \frac{1}{N}|\rho|. \tag{11}$$

Obviously,

$$\left|\sum_{r_i<r} b_i + F(g)(r)\right| \leq \sum_{r_i<r}|b_i| + |F(g)(r)| \leq \sum_{i=1}^{k}|b_i| + ||g||_{L^1}, \quad \forall r \in 0, T],$$

then

$$\left|\sum_{r_i<r} b_i + F(g)(r)\right| \le \sum_{i=1}^{k} |b_i| + ||g||_{L^1} \le \frac{R_0}{3N} < \frac{|\rho|}{3N}, \quad \forall r \in [0,T], \tag{12}$$

and

$$\left|\rho + \sum_{r_i<r} b_i + F(g)(r)\right| \le |\rho| + \left|\sum_{r_i<r} b_i + F(g)(r)\right| < \frac{4\,|\rho|}{3}, \quad \forall r \in [0,T]. \tag{13}$$

By (11) and (12), the j_0-th component of $\rho + \sum_{r_i<r} b_i + F(g)(r)$ keeps the same sign of ρ^{j_0} on J and

$$\left|\rho^{j_0} + \sum_{r_i<r} b_i^{j_0} + F(g)^{j_0}(r)\right| \ge \left|\rho^{j_0}\right| - \left|\sum_{r_i<r} b_i^{j_0} + F(g)^{j_0}(r)\right| > \frac{2\,|\rho|}{3N}, \quad \forall r \in J. \tag{14}$$

Combining (13) and (14), the j_0-th component $\varphi_{q(r)}^{j_0}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} b_i + F(g)(r)\right)\right]$ of $\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} b_i + F(g)(r)\right)\right]$ satisfies

$$\begin{aligned}\left|\varphi_{q(r)}^{j_0}(\rho + \sum_{r_i<r} b_i + F(g)(r))\right| &= \left|\rho + \sum_{r_i<r} b_i + F(g)(r)\right|^{q(r)-2} \left|\rho^{j_0} + \sum_{r_i<r} b_i^{j_0} + F(g)^{j_0}(r)\right| \\ &> \frac{2}{3N}\left|\rho + \sum_{r_i<r} b_i + F(g)(r)\right|^{q(r)-2} |\rho| \\ &> \frac{1}{2N}\left|\rho + \sum_{r_i<r} b_i + F(g)(r)\right|^{q(r)-1}.\end{aligned}$$

From the definition $\varphi_{q(r)}(\cdot) = \varphi_{p(r)}^{-1}(\cdot)$, we have $\frac{1}{p(r)} + \frac{1}{q(r)} \equiv 1$, then $q(r) - 1 = \frac{1}{p(r)-1}$, and

$$\begin{aligned}\left|\varphi_{q(r)}^{j_0}\left(\rho + \sum_{r_i<r} b_i + F(g)(r)\right)\right| &> \frac{1}{2N}\left|\rho + \sum_{r_i<r} b_i + F(g)(r)\right|^{\frac{1}{p(r)-1}} \\ &\ge \frac{1}{2N}\left||\rho| - \left|\sum_{r_i<r} b_i + F(g)(r)\right|\right|^{\frac{1}{p(r)-1}} \\ &\ge \frac{1}{2N}\left|\left(2N\frac{E+1}{E}\sum_{i=1}^{k}|a_i|\right)^{p^\#-1}\right|^{\frac{1}{p(r)-1}} \\ &\ge \frac{1}{2N}2N\frac{E+1}{E}\sum_{i=1}^{k}|a_i| = \frac{E+1}{E}\sum_{i=1}^{k}|a_i|.\end{aligned}$$

Without loss of generality, we may assume that $\rho^{j_0} > 0$, then we have

$$F^{j_0}\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(T) > (E+1)\sum_{i=1}^{k}|a_i| \ge \sum_{i=1}^{k} a_i^{j_0}.$$

Therefore, the j_0-th component of $\sum_{i=1}^{k} a_i + F\{\varphi_{q(r)}[(w(r))^{-1}(\rho + \sum_{r_i<r} b_i + F(g)(r))]\}(T)$ keeps the same sign of ρ^{j_0}. Since the j_0-th component of $\rho + \sum_{i=1}^{k} b_i + F(g)(T)$ keeps the same sign of ρ^{j_0}, $a, b, c, d \in [0, +\infty)$ and $ad + bc > 0$, we can easily see that the j_0-th component of $\Theta_h(\rho)$ keeps the same sign of ρ^{j_0}, and thus

$$\Theta_h(\rho) \neq 0.$$

Let us consider the equation

$$\lambda\Theta_h(\rho) + (1-\lambda)\rho = 0, \lambda \in [0,1]. \tag{15}$$

According to the above discussion, all the solutions of (15) belong to $b(R_0 + 1) = \{x \in \mathbb{R}^N | \ |x| < R_0 + 1\}$. So, we have

$$d_B[\Theta_h(\rho), b(R_0+1), 0] = d_B[I, b(R_0+1), 0] \neq 0.$$

It means the existence of solutions of $\Theta_h(\rho) = 0$.

In this way, we define a mapping $\rho(h)$: $H \to \mathbb{R}^N$, which satisfies

$$\Theta_h(\rho(h)) = 0.$$

(ii) By the proof of (i), we also obtain ρ sends bounded set to bounded set, and

$$\left|\rho(h)\right| \leq 3N\left[\left(2N\frac{E+1}{E}\sum_{i=1}^{k}|a_i|\right)^{p\#-1} + \sum_{i=1}^{k}|b_i| + ||g||_{L^1}\right].$$

It only remains to prove the continuity of ρ. Let $\{u_n\}$ is a convergent sequence in H and $u_n \to u$, as $n \to +\infty$. Since $\{\rho(u_n)\}$ is a bounded sequence, it contains a convergent subsequence $\{\rho(u_{n_j})\}$ satisfies $\rho(u_{n_j}) \to \rho_*$ as $j \to +\infty$. Since $\Theta_h(\rho)$ consists of continuous functions, and

$$\Theta_{u_{n_j}}(\rho(u_{n_j})) = 0,$$

Letting $j \to +\infty$, we have

$$\Theta_u(\rho_*) = 0,$$

from (i) we get $\rho_* = \rho(u)$, it means that ρ is continuous.

This completes the proof.

If $a = 0$, the boundary condition $au(0) - b\lim_{r\to 0^+}(w(r))^{\frac{1}{p(r)-1}}u'(r) = 0$ implies that

$$\lim_{r\to 0^+}(w(r))^{\frac{1}{p(r)-1}}u'(r) = 0.$$

Since $ad + bc > 0$, we have $c > 0$. Thus,

$$u(r) = u(0) + \sum_{r_i<r} a_i + F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(r), \quad \forall r \in J,$$

the boundary condition $cu(T) + d\lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0$ implies that

$$u(0) + \sum_{i=1}^{k} a_i + F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(T) + \frac{d}{c}\left(\sum_{i=1}^{k} b_i + F(g)(T)\right) = 0.$$

Denote G: $H \to \mathbb{R}^N$ as

$$G(h) = -\sum_{i=1}^{k} a_i - F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\sum_{r_i<r} b_i + F(g)(r)\right)\right]\right\}(T) - \frac{d}{c}\left(\sum_{i=1}^{k} b_i + F(g)(T)\right).$$

It is easy to see that

Lemma 2.8 The function $G(\cdot)$ is continuous and sends bounded sets to bounded sets. Moreover, $\left|G(h)\right| \leq \frac{3N(c+d)}{c}[\sum_{i=1}^{k}|a_i| + E(\sum_{i=1}^{k}|b_i| + ||g||_{L^1})^{\frac{1}{p^*-1}} + \sum_{i=1}^{k}|b_i| + ||g||_{L^1}]$, where $E = \int_0^T (w(r))^{\frac{-1}{p(r)-1}} dr$, the notation p^* means $C^{\frac{1}{p^*-1}} = \begin{cases} C^{\frac{1}{p^+-1}}, & C \leq 1 \\ C^{\frac{1}{p^--1}}, & C > 1 \end{cases}$.

3 Main results and proofs

In this section, we will apply coincidence degree to deal with the existence of solutions for (1)-(4). In the following, we always use C and C_i to denote positive constants, if it cannot lead to confusion.

Theorem 3.1 Assume that Ω is an open bounded set in X such that the following conditions hold.

(1^0) For each $\lambda \in (0, 1)$ the problem

$$Lx = S(x, \lambda) \tag{16}$$

has no solution on $\partial\Omega$.

(2^0) $(0, 0) \in \Omega$.

Then, problem (1)-(4) has a solution u satisfies $(u, v) \in \overline{\Omega}$, where $v = w(r)\phi_{p(r)}(u'(r))$, $\forall r \in J'$.

Proof. Let us consider the following operator equation

$$Lx = S(x, \lambda). \tag{17}$$

It is easy to see that $x = (x_1, x_2)$ is a solution of $Lx = S(x, 1)$ if and only if $x_1(r)$ is a solution of (1)-(4) and $x_2(r) = w(r)\varphi_{p(r)}(x_1'(r))$, $\forall r \in J'$.

According to Proposition 2.5, we can conclude that $S(\cdot, \cdot)$ is L-compact from $X \times [0, 1]$ to Y. We assume that for $\lambda = 1$, (16) does not have a solution on $\partial\Omega$, otherwise we complete the proof. Now from hypothesis (1^0), it follows that (16) has no solutions for $(x, \lambda) \in \partial\Omega \times (0, 1]$. For $\lambda = 0$, (17) is equivalent to $Lx = S(x, 0)$, namely the following usual problem

$$\left.\begin{array}{ll} x'_1 = 0, & r \in (0, T), \\ x'_2 = 0, & r \in (0, T), \\ ax_1(0) - b\varphi_{q(0)}(x_2(0)) = 0, & cx_1(T) + \mathrm{d}x_2(T) = 0. \end{array}\right\}$$

The problem (??) is a usual differential equation. Hence,

$$x_1 \equiv c_1, \quad x_2 \equiv c_2,$$

where $c_1, c_2 \in \mathbb{R}^N$ are constants. The boundary value condition of (??) holds,

$$ac_1 - b\varphi_{q(0)}(c_2) = 0, \quad cc_1 + dc_2 = 0.$$

Since $(ad + bc) > 0$, we have

$$c_1 = 0, \quad c_2 = 0,$$

which together with hypothesis (2^0), implies that $(0, 0) \in \Omega$. Thus, we have proved that (16) has no solution on $\partial\Omega \times [0, 1]$. It means that the coincidence degree $d[L - S(\cdot, \lambda), \Omega]$ is well defined for each $\lambda \in [0, 1]$. From the homotopy invariant property of that degree, we have

$$d[L - S(\cdot, 1), \Omega] = d[L - S(\cdot, 0), \Omega]. \tag{18}$$

Now, it is clear that the following problem

$$Lx = S(x, 1) \tag{19}$$

is equivalent to problem (1)-(4), and (18) tells us that problem (19) will have a solution if we can show that

$$d[L - S(\cdot, 0), \Omega] \neq 0.$$

Since by hypothesis (2^0), this last degree

$$d[L - S(\cdot, 0), \Omega] = d_B[\omega_*, \Omega \cap \mathbb{R}^{2N}, 0] \neq 0,$$

where $\omega_*(c_1, c_2) = (ac_1 - b\phi_{q(0)}(c_2), cc_1 + dc_2)$. This completes the proof.

Our next theorem is a consequence of Theorem 3.1. Denote

$$z^- = \min_{r\in J} z(r), \ z^+ = \max_{r\in J} z(r), \text{ for } z \in C(J, \mathbb{R}).$$

Theorem 3.2 Assume that the following conditions hold

(1^0) $a > 0$;

(2^0) $\lim_{|u| + |v| \to +\infty} (f(r, u, v)/(|u| + |v|)^{\beta(r) - 1}) = 0$, for $r \in J$ uniformly, where $\beta(r) \in C(J, \mathbb{R})$, and $1<\beta^- \le \beta^+ <p^-$;

(3^0) $\sum_{i=1}^{k} |A_i(u, v)| \le C_1(|u| + |v|)^{\theta}$ when $|u| + |v|$ is large enough, where $0 < \theta < \frac{p^- - 1}{p^+ - 1}$;

(4^0) $\sum_{i=1}^{k} |B_i(u, v)| \le C_2(|u| + |v|)^{\varepsilon}$ when $|u| + |v|$ is large enough, where $0 \le \varepsilon <\beta^+ - 1$.

Then, problem (1)-(4) has at least one solution.

Proof. Now, we consider the following operator equation

$$Lx = S(x, \lambda). \tag{20}$$

For any $\lambda \in (0, 1]$, $x = (x_1, x_2) = (u, v)$ is a solution of (20) if and only if $v(r) = \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)) (\forall r \in J')$ and $u(r)$ is a solution of the following

$$\left.\begin{aligned}
&\frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)))' = \lambda^{p(r)} f(r, u, \tfrac{1}{\lambda}(w(r))^{\frac{1}{p(r)-1}} u') \quad r \in (0, T), \quad r \neq r_i,\\
&\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = \lambda^2 A_i(\lim_{r\to r_i^-} u(r), \tfrac{1}{\lambda} \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), \quad i = 1, \ldots, k,\\
&\lim_{r\to r_i^+} \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)) - \lim_{r\to r_i^-} \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r))\\
&= \lambda^{p(r_i)} B_i(\lim_{r\to r_i^-} u(r), \tfrac{1}{\lambda} \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), \quad i = 1, \ldots, k,\\
&au(0) - b\tfrac{1}{\lambda}(w(0))^{\frac{1}{p(0)-1}} u'(0) = 0, \text{ and } cu(T) + d\frac{1}{\lambda^{p(T)-1}} w(T)|u'|^{p(T)-2} u'(T) = 0.
\end{aligned}\right\} \tag{21}$$

We claim that all the solutions of (21) are uniformly bounded for $\lambda \in (0, 1]$. In fact, if it is false, we can find a sequence (u_n, λ_n) of solutions for (21), such that $||u_n||_1 > 1$ and $||u_n||_1 \to +\infty$ when $n \to +\infty$, $\lambda_n \in (0, 1]$. Since (u_n, λ_n) are solutions of (21), we have

$$w(r)|u_n'|^{p(r)-2}u_n'(r) = \lambda_n^{p(r)-1}\left[\frac{1}{\lambda_n^{p(0)-1}}\rho_n + \sum_{r_i<r}\lambda_n^{p(r_i)}B_i + \int_0^r \lambda_n^{p(t)} f\left(t, u_n, \frac{1}{\lambda_n}(w(t))^{\frac{1}{p(t)-1}}u'_n\right)dt\right],$$

for any $r \in J'$, where $\rho_n = w(0)\varphi_{p(0)}(u_n'(0))$ and

$$A_i = A_i\left(\lim_{r\to r_i^-} u_n(r), \frac{1}{\lambda_n}\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'_n(r)\right), B_i = B_i\left(\lim_{r\to r_i^-} u_n(r), \frac{1}{\lambda_n}\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'_n(r)\right).$$

By computation, we have

$$\begin{aligned}
\left|\sum_{r_i<r}\lambda_n^2 A_i\right| &\le C_1\lambda_n^2\left(||u_n||_0 + \frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\theta} \le \lambda_n C_3||u_n||_1^{\theta},\\
\left|\sum_{r_i<r}\lambda_n^{p(r_i)}B_i\right| &\le C_2\lambda_n^{p(r_i)}\left(||u_n||_0 + \frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\beta^+-1} \le C_4||u_n||_1^{\beta^+-1},\\
\left|\int_0^T \lambda_n^{p(r)} f\left(r, u_n, \frac{1}{\lambda_n}(w(r))^{\frac{1}{p(r)-1}}u'_n\right)dr\right| &\le C_5\lambda_n^{p^-}\left(||u_n||_0 + \frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\beta^+-1}\\
&\le C_5||u_n||_1^{\beta^+-1}.
\end{aligned} \tag{22}$$

Denote

$$\Gamma_n(r) = \frac{1}{\lambda_n^{p(0)-1}}\rho_n + \sum_{r_i<r}\lambda_n^{p(r_i)}B_i + \int_0^r \lambda_n^{p(t)} f\left(t, u_n, \frac{1}{\lambda_n}(w(t))^{\frac{1}{p(t)-1}}u'_n\right)dt, \quad \forall r \in J.$$

We claim that

$$\frac{1}{\lambda_n^{p(0)-1}}|\rho_n| \le 3NC_6\left[||u_n||_1^{\theta_*(p^+-1)} + ||u_n||_1^{\beta^+-1}\right], \quad n = 1, 2, \ldots, \text{ where } \theta_* \in \left(\theta, \frac{p^- - 1}{p^+ - 1}\right). \tag{23}$$

If it is false, without loss of generality, we may assume that

$$\frac{1}{\lambda_n^{p(0)-1}}|\rho_n| > 3N(C_4 + C_5)\left[||u_n||_1^{\theta_*(p^+-1)} + ||u_n||_1^{\beta^+-1}\right], \quad n = 1, 2, \ldots,$$

then for any $n = 1, 2, \ldots$, there is a $j_n \in \{1, \ldots, N\}$ such that the j_n-th component $\rho_n^{j_n}$ of ρ_n satisfies

$$\frac{1}{\lambda_n^{p(0)-1}}|\rho_n^{j_n}| > 3(C_4 + C_5)\left[||u_n||_1^{\theta_*(p^+-1)} + ||u_n||_1^{\beta^+-1}\right], \quad n = 1, 2, \ldots$$

Thus, when n is large enough, the j_n-th component $\Gamma_n^{j_n}(r)$ of $\Gamma_n(r)$ keeps the same sign as $\rho_n^{j_n}$ and satisfies

$$\left|\Gamma_n^{j_n}(r)\right| > (C_4 + C_5)\left[||u_n||_1^{\theta_*(p^+-1)} + ||u_n||_1^{\beta^+-1}\right], \quad \forall r \in J', \quad n = 1, 2, \ldots$$

When n is large enough, we can conclude that the j_n-th component $F^{j_n}\{\varphi_{q(r)}[\Gamma_n(r)]\}(T)$ of $F\{\phi_{q(r)}[\Gamma_n(r)]\}(T)$ keeps the same sign as $\rho_n^{j_n}$ and satisfies

$$\left|F^{j_n}\{\varphi_{q(r)}[\Gamma_n(r)]\}\right| > C_7||u_n||_1^{\theta_*}, \quad \forall r \in J. \tag{24}$$

Since

$$\begin{aligned} u_n(r) &= u_n(0) + \sum_{r_i<r} \lambda_n^2 A_i + \lambda_n F\{\varphi_{q(r)}\left[(w(r))^{-1}\Gamma_n(r)\right]\} \\ &= \frac{b}{a}\varphi_{q(0)}\left(\frac{1}{\lambda_n^{p(0)-1}}\rho_n\right) + \sum_{r_i<r} \lambda_n^2 A_i + \lambda_n F\{\varphi_{q(r)}[(w(r))^{-1}\Gamma_n(r)]\}, \quad \forall r \in J, \quad n = 1, 2, \dots, \end{aligned}$$

from (22) and (24), we can see that $u_n^{j_n}(r)(\forall r \in J)$ keeps the same sign as $\rho_n^{j_n}$, when n is large enough.

But the boundary value conditions (4) mean that

$$cu_n(T) + d\frac{1}{\lambda_n^{p(T)-1}} \lim_{r\to T^-} w(r)|u_n'|^{p(r)-2}u_n'(r) = cu_n(T) + d\Gamma_n(T) = 0, \quad n = 1, 2, \dots$$

It is a contradiction. Thus (23) is valid. Therefore,

$$w(r)\left|u'_n(r)\right|^{p(r)-1} \le C_7\left(||u_n||_1^{\theta_*(p^+-1)} + ||u_n||_1^{\beta^+-1}\right), \quad \forall r \in J'.$$

It means that

$$||(w(r))^{\frac{1}{p(r)-1}}u_n'||_0 \le o(1)||u_n||_1, \text{ where } o(1) \text{ tends to 0 uniformly as } n \to \infty. \tag{25}$$

From (22), (23) and (25), for any $r \in J$, we have

$$\begin{aligned} |u_n(r)| &= \left|u_n(0) + \int_0^r u'_n(t)dt + \sum_{r_i<r}\lambda_n^2 A_i\right| \le \left|u_n(0)\right| + \left|\int_0^r u'_n(t)dt\right| + \left|\sum_{r_i<r}\lambda_n^2 A_i\right| \\ &\le \left|u_n(0)\right| + \int_0^r (w(t))^{\frac{-1}{p(t)-1}}\left|(w(t))^{\frac{1}{p(t)-1}}u'_n(t)\right|dt + C_3||u_n||_1^{\theta} \\ &\le \frac{b}{a}\left|\frac{1}{\lambda_n}(w(0))^{\frac{1}{p(0)-1}}u'_n(0)\right| + E\,\|\,(w(r))^{\frac{1}{p(r)-1}}u'_n(r)||_0 + C_3||u_n||_1^{\theta} \\ &\le o(1)||u_n||_1, \text{ where } o(1) \text{ tends to 0 uniformly as } n \to \infty, \end{aligned}$$

then

$$||u_n||_0 \le o(1)||u_n||_1, \text{ where } o(1) \text{ tends to 0 uniformly as } n \to \infty. \tag{26}$$

From (25) and (26), we get that all the solutions of (20) are uniformly bounded for any $\lambda \in (0, 1]$.

When $\lambda = 0$, if (x_1, x_2) is a solution of (20), then (x_1, x_2) is a solution of the following usual equation

$$\begin{cases} x'_1 = 0, & r \in (0, T), \\ x'_2 = 0, & r \in (0, T), \\ ax_1(0) - b\varphi_{q(0)}(x_2(0)) = 0, & cx_1(T) + dx_2(T) = 0, \end{cases}$$

we have

$$(x_1, x_2) = (0, 0).$$

Thus, there exists a large enough $R_0 > 0$ such that all the solutions of (20) belong to $B(R_0) = \{x \in X \mid \| x \|_X < R_0\}$. Thus, (20) has no solution on $\partial B(R_0)$. From theorem 3.1, we obtain that (1)-(4) has at least one solution. This completes the proof.

Theorem 3.3 Assume that the following conditions hold

(1^0) $a = 0$;

(2^0) $\lim_{|u|+|v|\to+\infty} f(r,u,v)/(|u|+|v|)^{\varepsilon} = 0$ for $r \in J$ uniformly, where $0 \le \varepsilon$ min(1, $p^- - 1$);

(3^0) $\sum_{i=1}^{k} |A_i(u,v)| \le C_1(|u|+|v|)^{\theta}$when $|u| + |v|$ is large enough, where $0 < \theta < 1$;

(4^0) $\sum_{i=1}^{k} |B_i(u,v)| \le C_2(|u|+|v|)^{\varepsilon}$when $|u| + |v|$ is large enough, where $0 \le \varepsilon < \min(1, p^- - 1)$.

Then, problem (1)-(4) has at least one solution.

Proof Now, we consider the following operator equation

$$Lx = S(x,\lambda). \tag{27}$$

If (x_1, x_2) is a solution of (27) when $\lambda = 0$, then (x_1, x_2) is a solution of the following usual equation

$$\begin{cases} x'_1 = 0, & r \in (0,T), \\ x'_2 = 0, & r \in (0,T), \\ ax_1(0) - b\varphi_{q(0)}(x_2(0)) = 0, \ cx_1(T) + dx_2(T) = 0. \end{cases}$$

Then, we have

$$(x_1, x_2) = (0,0).$$

For any $\lambda \in (0, 1]$, $x = (x_1, x_2) = (u, v)$ is a solution of (27) if and only if $v(r) = \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)) (\forall r \in J')$ and $u(r)$ is a solution of the following

$$\left.\begin{array}{l} \left(\frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r))\right)' = \lambda^{p(r)} f(r, u, \frac{1}{\lambda}(w(r))^{\frac{1}{p(r)-1}} u'), r \in (0,T), r \ne r_i, \\ \lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = \lambda^2 A_i(\lim_{r\to r_i^-} u(r), \frac{1}{\lambda} \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), i = 1,\dots,k, \\ \lim_{r\to r_i^+} \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)) - \lim_{r\to r_i^-} \frac{1}{\lambda^{p(r)-1}} w(r)\varphi_{p(r)}(u'(r)) \\ = \lambda^{p(r_i)} B_i(\lim_{r\to r_i^-} u(r), \dfrac{1}{\lambda} \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), i = 1,\dots,k, \\ \lim_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}} u'(r) = 0, \text{ and } cu(T) + d\frac{1}{\lambda^{p(T)-1}} \lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0. \end{array}\right\} \tag{28}$$

We only need to prove that all the solutions of (28) are uniformly bounded for $\lambda \in (0, 1]$.

In fact, if it is false, we can find a sequence (u_n, λ_n) of solutions for (28), such that $\|u_n\|_1 > 1$ and $\|u_n\|_1 \to +\infty$ when $n \to +\infty$. Since (u_n, λ_n) are solutions of (28), we have

$$w(r)|u_n'|^{p(r)-2}u_n'(r) = \lambda_n^{p(r)-1}\left[\sum_{r_i<r} \lambda_n^{p(r_i)} B_i + \int_0^r \lambda_n^{p(t)} f\left(t, u_n, \frac{1}{\lambda_n}(w(t))^{\frac{1}{p(t)-1}} u'_n\right) dt\right], \quad \forall r \in J',$$

where $B_i = B_i(\lim_{r\to r_i^-} u_n(r), \frac{1}{\lambda_n} \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u_n'(r))$.

From conditions (2^0) and (4^0), we have

$$\left|\sum_{r_i<r}\lambda_n^{p(r_i)}B_i\right| \le C_2\lambda_n^{p^-}\left(||u_n||_0+\frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\varepsilon} \le C_4||u_n||_1^{\varepsilon},$$

$$\left|\int_0^T \lambda_n^{p(r)}f(r,u_n(r),\frac{1}{\lambda_n}(w(r))^{\frac{1}{p(r)-1}}u'_n(r))dr\right| \le C_5\lambda_n^{p^-}\left(||u_n||_0+\frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\varepsilon} \le C_5||u_n||_1^{\varepsilon}.$$

Thus,

$$w(r)|u'_n(r)|^{p(r)-1} \le C_6||u_n||_1^{\varepsilon}, \quad \forall r\in J', \quad n=1,2,\dots,$$
$$||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0 \le o(1)||u_n||_1, \quad n=1,2,\dots \tag{29}$$

Denote

$$\Upsilon_n(r)=\sum_{r_i<r}\lambda_n^{p(r_i)}B_i+F\left(\lambda_n^{p(r)}f(r,u_n(r),\frac{1}{\lambda_n}(w(r))^{\frac{1}{p(r)-1}}u'_n(r))\right)(r), \quad \forall r\in J, \quad n=1,2,\dots$$

By solving $u_n(r)$, we have

$$u_n(r)=u_n(0)+\sum_{r_i<r}\lambda_n^2A_i+\lambda_nF\left\{\varphi_{q(r)}\left[(w(r))^{-1}\Upsilon_n(r)\right]\right\}, \quad \forall r\in J, x \tag{30}$$

where $A_i=A_i(\lim_{r\to r_i^-}u_n(r),\frac{1}{\lambda_n}\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'_n(r))$.

From condition (3^0), we have

$$\left|\sum_{r_i<r}\lambda_n^2A_i\right| \le C_1\lambda_n^2\left(||u_n||_0+\frac{1}{\lambda_n}||(w(r))^{\frac{1}{p(r)-1}}u'_n||_0\right)^{\theta} \le \lambda_nC_3||u_n||_1^{\theta}.$$

The boundary value condition implies

$$u_n(0)+\sum_{i=1}^{k}\lambda_n^2A_i+\lambda_nF\left\{\varphi_{q(r)}\left[(w(r))^{-1}\Upsilon_n(r)\right]\right\}(T)=-\frac{d}{c}\Upsilon_n(T). \tag{31}$$

From (31) and conditions (2^0), (3^0) and (4^0), we have

$$\left|u_n(0)\right| \le o(1)||u_n||_1, \text{ where } o(1) \text{ tends to 0 uniformly as } n\to\infty. \tag{32}$$

From (30) and (32), we have

$$||u_n||_0 \le o(1)||u_n||_1, \quad n=1,2,\dots \tag{33}$$

From (29) and (33), we can conclude that $\{||u_n||_1\}$ is uniformly bounded for $\lambda \in (0, 1]$. This completes the proof.

Now, let us consider the following mixed type boundary value condition

$$au(0)-b\lim_{r\to0^+}(w(r))^{\frac{1}{p(r)-1}}u'(r)=0, \text{ and } cu(T)+d\lim_{r\to T^-}(w(r))^{\frac{1}{p(r)-1}}u'(r)=0. \tag{34}$$

Theorem 3.4 Assume that the following conditions hold

(1^0) $\lim_{|u|+|v|\to+\infty}(f(r,u,v)/(|u|+|v|)^{\beta(r)-1})=0$, for $r\in J$ uniformly, where $\beta(r)\in C(J,\mathbb{R})$, and $1<\beta^-\le\beta^+<p^-$;

(2^0) $\sum_{i=1}^{k} |A_i(u, v)| \le C_1(|u| + |v|)^{\theta}$when $|u| + |v|$ is large enough, where $0 < \theta < \frac{p^- - 1}{p^+ - 1}$;

(3^0) $\sum_{i=1}^{k} |B_i(u, v)| \le C_2(|u| + |v|)^{\varepsilon}$when $|u| + |v|$ is large enough, where $0 \le \varepsilon < \beta^+ - 1$.

Then, problem (1) with (2), (3) and (34) has at least one solution.

Proof. It is similar to the proof of Theorem 3.2 and Theorem 3.3, we omit it.

Denote

$$f_\delta(r, u, v) = f(r, u, v) + \delta f_*(r, u, v),$$

where f_* (r, u, v) is Caratheodory.

Let us consider

$$-(w(r)|u'|^{p(r)-2}u'(r))' + f_\delta(r, u(r), (w(r))^{\frac{1}{p(r)-1}} u'(r)) = 0, \quad \forall r \in J'. \tag{35}$$

Theorem 3.5 Under the conditions of Theorem 3.2, Theorem 3.3 or Theorem 3.4, then (35) with (2), (3) and (4) or (34) has at least a solution when δ is small enough.

Proof We only need to prove the existence of solutions under the conditions of Theorem 3.2, the rest is similar. If $\delta = 0$, the proof of Theorem 3.2 means that all the solutions of (35) with (2), (3) and (4) are bounded and belong to $U(R_0) = \{u \in PC^1 | \ ||u||_1 < R_0\}$. Define $S_\delta : X \to Y$ as

$$S_\delta x = \left(\varphi_{q(r)}\left(\frac{x_2}{w(r)}\right), f_\delta(r, x_1, \varphi_{q(r)}(x_2)), A_i, B_i, ax_1(0) - b\varphi_{q(0)}(x_2(0)), cx_1(T) + dx_2(T)\right),$$

where $A_i = A_i(x_1(r_i^-), \varphi_{q(r_i)}(x_2(r_i^-)))$, $B_i = B_i(x_1(r_i^-), \varphi_{q(r_i)}(x_2(r_i^-)))$.

Since f_* (r, u, v) is a Caratheodory function, we have $||S_\delta x - S_0 x||_Y \to 0$ as $\delta \to 0$, for $x \in \overline{U(R_0)}$ uniformly. According to Proposition 2.3, we get the existence of solutions.

In the following, we will consider the existence of nonnegative solutions. For any $x = (x^1, ..., x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ means $x^i \ge 0$ for any $i = 1, ..., N$.

Theorem 3.6 We assume

(i) $f(r, u, v) \le 0$, $\forall (r, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$;

(ii) for any $i = 1, ..., k$, $B_i(u, v) \le 0, \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$;

(iii) for any $i = 1, ..., k$, $j = 1, ..., N$, $A_i^j(u, v)v^j \ge 0 \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then, the solution u in Theorem 3.2, Theorem 3.3 or Theorem 3.4 is nonnegative.

Proof We only need to prove that the solution u in Theorem 3.2 is nonnegative, and the rest is similar. Denote

$$N_f(u) = f(r, u(r), (w(r))^{\frac{1}{p(r)-1}} u'(r)),$$

$$D = c\frac{b}{a}\varphi_{q(0)}(\rho) + c\sum_{i=1}^{k} A_i + cF\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} B_i + F(N_f(u))(r)\right)\right]\right\}(T)$$
$$+ d\left[\rho + \sum_{i=1}^{k} B_i + F(N_f(u))(T)\right],$$

where

$$A_i = A_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), i = 1, \quad \dots, k,$$

$$B_i = B_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), \quad i = 1, \dots, k.$$

Similar to (8) and (9), we have

$$\begin{gathered} w(r)\varphi_{p(r)}(u'(r)) = \rho + \sum_{r_i<r} B_i + F(N_f(u))(r), \quad \forall r \in J', \\ u(r) = u(0) + \sum_{r_i<r} A_i + F\left\{\varphi_{q(r)}\left[(w(r))^{-1}\left(\rho + \sum_{r_i<r} B_i + F(N_f(u))(r)\right)\right]\right\}(r), \quad \forall r \in J, \end{gathered} \tag{36}$$

where

$$u(0) = \frac{b}{a}\varphi_{q(0)}(\rho),$$

and ρ is the solution (unique) of

$$D = 0. \tag{37}$$

Denote

$$\Phi(r) = w(r)\varphi_{p(r)}(u'(r)), \quad \forall r \in J'.$$

From (i), (ii) and (36), we can see that $\Phi(r)$ is decreasing, namely

$$\Phi(t_2) - \Phi(t_1) \le 0, \ \forall t_2, \ t_1 \in J', \quad t_2 > t_1. \tag{38}$$

We claim that

$$\rho \ge 0. \tag{39}$$

If it is false, then there exists some $j_0 \in \{1, ..., N\}$, such that the j_0-th component ρ^{j_0} of ρ satisfies

$$\rho^{j_0} < 0. \tag{40}$$

Combining (i), (ii), (iii) and (40), we can see that the j_0-th component D^{j_0} of D is negative. It is a contradiction to (37). Thus, (39) is valid. So, we have

$$u(0) = \frac{b}{a}\varphi_{q(0)}(\rho) \ge 0.$$

We claim that

$$\Phi(T) \le 0. \tag{41}$$

If it is false. Then, there exists some $j_1 \in \{1,..., N\}$, such that the j_1-th component $\Phi^{j_1}(T)$ of $\Phi(T)$ satisfies

$$\Phi^{j_1}(T) > 0. \tag{42}$$

From (38) and (42), we have

$$\Phi^{j_1}(t) > 0, \forall t \in J'.$$

Combining (i), (ii), (iii) and (42), we can see that the j_1-th component D^{j_1} of D is positive. It is a contradiction to (37). Thus, (41) is valid.

If $c > 0$. We have

$$u(T) = -\frac{d}{c}\Phi(T) \geq 0.$$

Since $\Phi(r)$ is decreasing, $\Phi(0) = \rho \geq 0$ and $\Phi(T) \leq 0$, for any $j = 1, ..., N$, there exists $\xi_j \in J$ such that

$$\Phi^j(r) \geq 0, \forall r \in (0, \xi_j) \text{ and } \Phi^j(r) \leq 0, \quad \forall r \in (\xi_j, T).$$

Combining condition (iii), we can conclude that $u^j(r)$ is increasing on $[0, \xi_j]$, and $u^j(r)$ is decreasing on $(\xi_j, T]$. Notice that $u(0) \geq 0$ and $u(T) \geq 0$, then we have $u(r) \geq 0, \forall r \in [0, T]$.

If $c = 0$, boundary condition (4) means that $\Phi(T) = 0$. Since $\Phi(r)$ is decreasing, we get that $\Phi(r) \geq 0$. Combining condition (iii), we can conclude that $u(r)$ is increasing on J, namely $u(t_2) - u(t_1) \geq 0, \forall t_2, t_1 \in J, t_2 > t_1$. Notice that $u(0) \geq 0$, then we have $u(r) \geq 0, \forall t \in J$. This completes the proof.

Corollary 3.7 We assume

(i) $f(r, u, v) \leq 0, \forall (r, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$ with $u \geq 0$;
(ii) for any $i = 1, ..., k$, $B_i(u, v) \leq 0, \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ with $u \geq 0$;
(iii) for any $i = 1, ..., k$, $j = 1, ..., N$, $A_i^j(u, v)v^j \geq 0, \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ with $u \geq 0$.

Then, we have

(1_0) Under the conditions of Theorem 3.2 or Theorem 3.3, (1)-(4) has a nonnegative solution.

(2_0) Under the conditions of Theorem 3.4, (1) with (2), (3) and (34) has a nonnegative solution.

Proof We only need to prove that (1)-(4) has a nonnegative solution under the conditions of Theorem 3.2, and the rest is similar. Define

$$\phi(u) = (\phi_*(u^1), \ldots, \phi_*(u^N)),$$

where

$$\phi_*(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Denote

$$\widetilde{f}(r, u, v) = f(r, \phi(u), v), \quad \forall (r, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N,$$

then $\widetilde{f}(r, u, v)$ satisfies Caratheodory condition, and $\widetilde{f}(r, u, v) \leq 0$ for any $(r, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $i = 1, ..., k$, we denote

$$\widetilde{A}_i(u, v) = A_i(\phi(u), v), \widetilde{B}_i(u, v) = B_i(\phi(u), v), \forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

then $\widetilde{A}_i$ and $\widetilde{B}_i$ are continuous and satisfy

$$\widetilde{B}_i(u,v) \le 0, \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N \quad \text{for any } i = 1,\dots,k,$$
$$\widetilde{A}_i^j(u,v)v^j \ge 0, \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \quad \text{for any } i = 1,\dots,k, \quad j = 1,\dots,N.$$

Obviously, we have

$(2^0)'$ $\lim\limits_{|u|+|v|\to+\infty} (\widetilde{f}(r,u,v)/(|u|+|v|)^{\beta(r)-1}) = 0$, for $r \in J$ uniformly, where $\beta(r) \in C(J, \mathbb{R})$, and$1 < \beta^- \le \beta^+ < p^-$;

$(3^0)'$ $\sum_{i=1}^{k} \left|\widetilde{A}_i(u,v)\right| \le C_1(|u|+|v|)^{\theta}$ when $|u| + |v|$ is large enough, where $0 < \theta < \frac{p^- - 1}{p^+ - 1}$;

$(4^0)'$ $\sum_{i=1}^{k} \left|\widetilde{B}_i(u,v)\right| \le C_2(|u|+|v|)^{\varepsilon}$ when $|u| + |v|$ is large enough, where $0 \le \varepsilon < \beta^+ - 1$.

Let us consider

$$\left.\begin{array}{l}
(w(r)\varphi_{p(r)}(u'(r)))' = \widetilde{f}(r, u(r), (w(r))^{\frac{1}{p(r)-1}} u'(r)), r \in J', \\
\lim\limits_{r\to r_i^+} u(r) - \lim\limits_{r\to r_i^-} u(r_i) = \widetilde{A}_i(\lim\limits_{r\to r_i^-} u(r), \lim\limits_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), \quad i = 1,\dots,k, \\
\lim\limits_{r\to r_i^+} w(r)\varphi_{p(r)}(u'(r)) - \lim\limits_{r\to r_i^-} w(r)\varphi_{p(r)}(u'(r)) \\
= \widetilde{B}_i(\lim\limits_{r\to r_i^-} u(r), \lim\limits_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)), \quad i = 1,\dots,k, \\
au(0) - b \lim\limits_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}} u'(r) = 0, \text{ and } cu(T) + d \lim\limits_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0.
\end{array}\right\} \quad (43)$$

From Theorem 3.2 and Theorem 3.6, we can see that (43) has a nonnegative solution u. Since $u \ge 0$, we have $\varphi(u) = u$, and then

$$\widetilde{f}(r, u(r), (w(r))^{\frac{1}{p(r)-1}} u'(r)) = f(r, u(r), (w(r))^{\frac{1}{p(r)-1}} u'(r)),$$
$$\widetilde{A}_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)) = A_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)),$$
$$\widetilde{B}_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)) = B_i(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)).$$

Thus, u is a nonnegative solution of (1)-(4). This completes the proof.

4 Examples

Example 4.1. Consider the following problem

$$(P_1)\left\{\begin{array}{l}
(w(r)\varphi_{p(r)}(u'(r)))' = -g(r) - |u|^{q(r)-2}u - \sigma w(r)|u'|^{q(r)-2}u' - \delta u e^{|u| + \left|(w(r))^{\frac{1}{p(r)-1}} u'\right|}, r \in J', \\
\lim\limits_{r\to r_i^+} u(r) - \lim\limits_{r\to r_i^-} u(r) = \sigma \lim\limits_{r\to r_i^-} |u(r)|^{\frac{-1}{2}} u(r) + \lim\limits_{r\to r_i^-} \left|(w(r))^{\frac{1}{p(r)-1}} u'(r)\right|^{\frac{-1}{2}} (w(r))^{\frac{1}{p(r)-1}} u'(r), \\
\lim\limits_{r\to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim\limits_{r\to r_i^-} w(r)|u'|^{p(r)-2}u'(r) \\
= \sigma \lim\limits_{r\to r_i^-} (w(r))^{\frac{3}{p(r)-1}} |u'(r)|^2 u'(r) - \lim\limits_{r\to r_i^-} |u(r)|^2 u(r), \\
u(0) - \lim\limits_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}} u'(r) = 0, \text{ and } u(T) + \lim\limits_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0,
\end{array}\right.$$

where $p(r) = 5 + \cos 3r$, $q(r) = 3 + \frac{1}{2}\sin 2r$, $0 \le g(r) \in L^1$, $e_0, e_1 \in \mathbb{R}^N$, $w(r) = 3 + \sin r$, σ is a nonnegative parameter.

Obviously, $g(r) + |u|^{q(r)-2}u + \sigma w(r)|u'|^{q(r)-2}u' + \delta u e^{|u|+\left|(w(r))^{\frac{1}{p(r)-1}}u'\right|}$ is Caratheodory, $q(r) \leq 3.5 < 4 \leq p(r) \leq 6$, then the conditions of Theorem 3.5 are satisfied, then (P_1) has a solution when $\delta > 0$ is small enough. Moreover, when $\sigma = 0$, the conditions of Corollary 3.7 are satisfied, then (P_1) has a nonnegative solution.

Example 4.2. Consider the following problem

$$(P_2)\begin{cases} (w(r)\varphi_{p(r)}(u'(r)))' = -g(r) - |u|^{q(r)-2}u - \sigma w(r)|u'|^{q(r)-2}u' - \delta u e^{|u|+\left|(w(r))^{\frac{1}{p(r)-1}}u'\right|}, \ r \in J', \\ \lim\limits_{r\to r_i^+} u(r) - \lim\limits_{r\to r_i^-} u(r) = \sigma \lim\limits_{r\to r_i^-} \left|u(r)\right|^{\frac{-1}{2}} u(r) + \lim\limits_{r\to r_i^-} \left|(w(r))^{\frac{1}{p(r)-1}}u'(r)\right|^{\frac{-1}{2}} (w(r))^{\frac{1}{p(r)-1}}u'(r), \\ \lim\limits_{r\to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim\limits_{r\to r_i^-} w(r)|u'|^{p(r)-2}u'(r) \\ = \sigma \lim\limits_{r\to r_i^-} (w(r))^{\frac{3}{p(r)-1}} \left|u'(r)\right|^{\frac{-1}{3}} u'(r) - \lim\limits_{r\to r_i^-} \left|u(r)\right|^{\frac{-1}{3}} u(r), \\ \lim\limits_{r\to 0^+} (w(r))^{\frac{1}{p(r)-1}}u'(r) = 0, \text{ and } u(T) + \lim\limits_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r) = 0, \end{cases}$$

where $p(r) = 5 + \cos 3r$, $q(r) = \frac{3}{2} + \frac{1}{4}\sin 2r$, $0 \leq g(r) \in L^1$, $e_0, e_1 \in \mathbb{R}^N$, $w(r) = 3+\sin r$, σ is a nonnegative parameter.

Obviously, $g(r) + |u|^{q(r)-2}u + \sigma w(r)|u'|^{q(r)-2}u' + \delta u e^{|u|+\left|(w(r))^{\frac{1}{p(r)-1}}u'\right|}$ is Caratheodory, $1 < q(r) < 2 < 4 \leq p(r) \leq 6$, then conditions of Theorem 3.5 are satisfied, then (P_2) has a solution when $\delta > 0$ is small enough. Moreover, when $\sigma = 0$, the conditions of Corollary 3.7 are satisfied, and (P_2) has a nonnegative solution.

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