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On hyperbolic equations with double characteristics in the presence of transition

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Abstract
The paper deals with the study of the Cauchy problem for a class of hyperbolic second order operators with double characteristics in the presence of a transition. In particular, we obtain some *a priori* local estimates and, by means of these estimates, we prove local and global existence theorems.

MSC: Primary 35L20; secondary 35B45

Keywords: Sobolev spaces; Cauchy problem; hyperbolic equations; pseudodifferential operators

1 Introduction

Let $\Omega = [0,+\infty[\,\times\Omega_0$, Ω_0 be an open subset of $\mathbb{R}^n$ ($n \geq 2$). Let $x = (x_0, x_1, \dots, x_m, x_{m+1}, \dots, x_n) = (x_0, x', x'') \in \Omega$, where we set $x' = (x_1, \dots, x_m) \in \Omega'$ and $x'' = (x_{m+1}, \dots, x_n) \in \Omega''$, Ω' is the projection of Ω_0 on the hyperplane $x'' = 0$ and Ω'' is the projection of Ω_0 on the hyperplane $x' = 0$. Let us consider the following class of hyperbolic second order operators with double characteristics in the presence of a transition:

$$P = D^2_{x_0} - \operatorname{Div}_{x'}\big(A(x',x'')D_{x'}\big) - \big(x_0 + \lambda - \alpha(x')\big)^2 \operatorname{Div}_{x''}\big(B(x'')D_{x''}\big) + \gamma(x), \quad \text{in } \Omega, \tag{1}$$

with C^∞ coefficients, where $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 0, 1, \dots, n$, $D_{x'} = \frac{1}{i}\nabla_{x'} = (D_{x_1}, \dots, D_{x_m})$, $D_{x''} = \frac{1}{i}\nabla_{x''} = (D_{x_{m+1}}, \dots, D_{x_n})$, $\operatorname{Div}_{x'} = \frac{1}{i}\operatorname{div}_{x'}$, $\operatorname{Div}_{x''} = \frac{1}{i}\operatorname{div}_{x''}$ and λ is a positive parameter.

For $\xi = (\xi_0, \xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_n) = (\xi_0, \xi', \xi'')$, where we set $\xi' = (\xi_1, \dots, \xi_m)$, $\xi'' = (\xi_{m+1}, \dots, \xi_n)$ and have fixed λ, let us denote by

$$\begin{aligned} p(x,\xi) = {} & -\xi_0^2 + \sum_{i,j=1}^m a_{ij}\big(x',x''\big)\xi'_i\xi'_j + \sum_{h=1}^m \partial_{x_h} a_{ij}\big(x',x''\big)\xi'_h \\ & + \big(x_0 + \lambda - \alpha(x')\big)^2 \sum_{i,j=m+1}^n b_j\big(x''\big)\xi''_i\xi''_j \\ & + \big(x_0 + \lambda - \alpha(x')\big)^2 \sum_{h=m+1}^n \partial_{x_h} b_j\big(x''\big)\xi''_h + \gamma(x) \end{aligned}$$

the *symbol* of P, by Σ the characteristic set

$$\Sigma = \{\rho = (x,\xi) \in T^*\Omega : p(\rho) = 0, \nabla p(\rho) = 0\},$$

where $T^*\Omega = \Omega \times (\mathbb{R}^n \setminus \{0\})$ is the cotangent bundle related to Ω, and by F_p the *fundamental matrix* of P at ρ, namely

$$F_p(\rho) = \frac{1}{2}\begin{pmatrix} p''_{x\xi}(\rho) & p''_{\xi\xi}(\rho) \\ -p''_{xx}(\rho) & -p''_{\xi x}(\rho)y \end{pmatrix}, \quad \forall \rho \in \Sigma.$$

The spectrum of $F(\rho)$, which we denote by $\mathrm{Spec}(F(\rho))$, has a remarkable importance for the study of the well-posedness of the Cauchy-Dirichlet problem for P.

Let us note that (see [1])

$$z \in \mathrm{Spec}(F(\rho)) \quad \Leftrightarrow \quad -z, \bar{z} \in \mathrm{Spec}(F(\rho)).$$

It is well known that $F(\rho)$ has only pure imaginary eigenvalues with a possible exception of a pair of non-zero real eigenvalues $\pm\lambda$ (see [1, 2]). If $F(\rho)$ has a pair of non-zero real eigenvalues, we say that P is effectively hyperbolic at ρ. If $F(\rho)$ has only pure imaginary eigenvalues and, moreover, if in the Jordan normal form of $F(\rho)$ corresponding to the eigenvalue 0, there are only Jordan blocks of dimension 2, *i.e.*, $\mathrm{Ker}\, F(\rho)^2 \cap \mathrm{Im}\, F(\rho)^2 = \{0\}$, we say that P is non-effectively hyperbolic of type 1 at ρ. Instead, if $F(\rho)$ has only pure imaginary eigenvalues and, moreover, if in the Jordan normal form of $F(\rho)$ corresponding to the eigenvalue 0, there is only a Jordan block of dimension 4 and no block of dimension 3, *i.e.*, $\mathrm{Ker}\, F(\rho)^2 \cap \mathrm{Im}\, F(\rho)^2$ is 2-dimensional, we say that P is non-effectively hyperbolic of type 2 at ρ. Besides let us set

$$\Sigma_+ = \{\rho \in \Sigma : P \text{ is effectively hyperbolic at } \rho\},$$

$$\Sigma_- = \{\rho \in \Sigma : P \text{ is non-effectively hyperbolic of type 1 at } \rho\},$$

$$\Sigma_0 = \{\rho \in \Sigma : P \text{ is non-effectively hyperbolic of type 2 at } \rho\}.$$

It is easy to verify

$$\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$$

Finally we say that we have a *transition* exactly when at least two among the above sets are nonempty.

The Cauchy problem for hyperbolic operators with double characteristics has been widely studied by many authors either in the case in which $F_p(\rho)$ has two real nonzero eigenvalues $\forall\rho \in \Sigma$ or in the case in which all the nonzero eigenvalues of $F_p(\rho)$ are purely imaginary numbers, $\forall\rho \in \Sigma$ (see for instance [1, 3–9]). Recently, another class of hyperbolic second order operators with double characteristics has been considered in [2]. For this class the C^∞ well-posedness of the Cauchy problem is studied. Moreover, Carleman estimates are obtained for non-effectively hyperbolic operators. In [10], for a different class of hyperbolic second order operators some energy estimates are established and the C^∞

well-posedness of the Cauchy problem for non-effectively hyperbolic operators is studied. We emphasize that in [2] and [10] the authors obtain *a priori* estimates when $\Sigma = \Sigma_- \sqcup \Sigma_0$. Instead we get *a priori* estimates when $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$ or $\Sigma = \Sigma_- \sqcup \Sigma_0$ or $\Sigma = \Sigma_0 \sqcup \Sigma_+$ or $\Sigma = \Sigma_-$ or $\Sigma = \Sigma_+$. In fact, in the class of operators (1), studied also in [11, 12] and [13], both in the case in which $F_p(\rho)$ has two distinct real eigenvalues and in the case in which all the eigenvalues are purely imaginary numbers can occur. Namely, on the variety characteristic a transition from a case to another one can be considered. More precisely, if $p(x,\xi) = \xi_0^2 - \sum_{j=1}^m \xi_j^2 + (x_0 + \lambda - \alpha(x'))^2 \sum_{j=m+1}^n \xi_j^2$, setting $\beta(x) = x_0 + \lambda - \alpha(x')$, if $|\nabla_{x'}\alpha(x')| < 1$ and $\beta(x) = 0$ ($\xi_0 = \xi_1 = \cdots = \xi_m = 0$, $\sum_{j=m+1}^n \xi_j^2 = 1$), then $F_p(\rho)$ has two distinct nonzero real eigenvalues. As a consequence, P is effectively hyperbolic. Instead if $|\nabla_{x'}\alpha(x')| > 1$ and $\beta(x) = 0$ ($\xi_0 = \xi_1 = \cdots = \xi_m = 0$, $\sum_{j=m+1}^n \xi_j^2 = 1$), $F_p(\rho)$ has two nonzero imaginary eigenvalues, then P is non-effectively hyperbolic. Therefore Σ_+ is the set of points of Σ for which $|\nabla_{x'}\alpha(x)| < 1$, Σ_- is the set of points of Σ for which $|\nabla_{x'}\alpha(x)| > 1$ and Σ_0 is the set of points of Σ in which $|\nabla_{x'}\alpha(x)| = 1$. Hence, even if we consider the particular class of operators (1), we have a transition from effectively hyperbolic to non-effectively hyperbolic.

In [11], an *a priori* estimate for solutions of a class of hyperbolic equations depending on a parameter $(-\partial_{x_0}^2 + \partial_{x_1}^2 + (x_0 + \lambda - \alpha(x_1))^2 \partial_{x_2}^2)u = f$ related to a Cauchy-Dirichlet problem is proved. Then in [14] energy estimates and existence and uniqueness results are established. For the Cauchy problem related to the same class of hyperbolic operators, a global existence and uniqueness theorem is obtained in [12, 13] and energy estimates for solutions are established in [15]. In this paper, we study the general class of hyperbolic second order operators with double characteristics in the presence of a transition (1). Under suitable assumptions on the coefficients that allow the transition on the variety characteristic, we obtain, first of all, *a priori* local estimate near the boundary and, then, distant from it. Such estimates allow us to prove existence theorems for the following Cauchy problem in the set Ω:

$$\begin{cases} Pu = f, \quad \text{in } \Omega, \\ u(0,x',x'') = 0, \qquad \partial_{x_0} u(0,x',x'') = 0, \end{cases} \tag{2}$$

see Section 6.

Let us assume that:

(i) all the coefficients $a_{ij}(x',x'')$, $i = 1,\dots,m$, and $b_j(x'')$, $j = m+1,\dots,n$ of the operator (1) belong to $C^\infty(\Omega_0) \cap L^\infty(\Omega_0)$ and $C_0^\infty(\Omega'') \cap L^\infty(\Omega'')$, respectively, for every $k > 0$;

(ii) setting $g(x') = \dfrac{\alpha(x')}{\operatorname{div}_{x'} \overline{\alpha}(x')}$, where $\overline{\alpha}(x')$ is a vector with m components equal to $\alpha(x')$, and $h(x') = 1 - \operatorname{div}_{x'} \overline{g}(x_1)$, $g, h \in C^\infty$, $h(x') \in [h_1, h_2]$, $\forall x' \in \Omega'$, with $0 < h_1 < h_2 < 4$;

(iii) there exists $\lambda > 0$ such that $|g(x')| \le \lambda$, $\forall x' \in \Omega'$;

(iv) setting $C(x',x'') = \operatorname{div}_{x'} \overline{A}(x',x'')g(x') + 2[A(x',x'')\operatorname{div}_{x'} \overline{g}(x') - \Lambda(x',x'')]$, for every $(x',x'') \in \Omega_0$, where $\Lambda(x',x'')$ is a matrix with m columns equal to $A(x',x'')\nabla_{x'} g(x')$, the matrices A and B are positive definite and C is positive semidefinite, namely

$$\exists L_1 \ge m : A(x',x'')\xi'\xi' \ge L_1 \|\xi'\|^2, \quad \forall \xi' \in \mathbb{R}^m,$$

$$\exists L_2 \ge 0 : B(x'')\xi''\xi'' \ge L_2 \|\xi''\|^2, \quad \forall \xi'' \in \mathbb{R}^{n-m},$$

$$C(x',x'')\xi'\xi' \ge 0, \quad \forall \xi' \in \mathbb{R}^m.$$

It is worth remarking that in the study of hyperbolic operators considered in this note, the major difficulties in order to establish *a priori* estimates regard to the case in which the function $\beta(x,\lambda) = x_0 + \lambda - \alpha(x')$ assumes positive and negative values in $\overline{\Omega}$. Let us observe that if $m = 1$, setting $A(x',x'') = (a(x',x''))$, as a result $C(x',x'') = \operatorname{div}_{x'} a(x',x'')g(x')$. Moreover, if $a(x',x'')$ is a constant function, then $C(x',x'') = 0$. Therefore, if $m = 1$ and $A(x',x'')$ is a constant function, assumption (iv) naturally occurs.

Example 1.1 Let $\alpha(x_1) = e^{\frac{x_1^3}{3}+x_1}$ be a function defined in $\mathbb{R}$ and let $P = D^2_{x_0} - D^2_{x_1} - (x_0 + \lambda - \alpha(x_1))^2 D^2_{x_2}$. It is easy to verify that $g(x_1) = \dfrac{1}{x_1^2+1}$ and $h(x_1) = \dfrac{2x_1}{(x_1^2+1)^2} + 1$ in $\mathbb{R}$. Let us remark that $1 - \frac{3\sqrt{3}}{8} \leq h(x_1) \leq 1 + \frac{3\sqrt{3}}{8}$, $\forall x_1 \in \mathbb{R}$. Moreover, the assumption (iii) is satisfied if we choose $\lambda \geq 1$. Therefore, we have $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$, namely we have a transition from effectively hyperbolic to non-effectively hyperbolic.

Example 1.2 Let us consider the function $\alpha(x_1,x_2) = e^{ax_1+bx_2}$ in $\mathbb{R}^2$, where $a + b \neq 0$, and the operator

$$P = D^2_{x_0} - \left(3D^2_{x_1} + D^2_{x_1x_2} + 4D^2_{x_2}\right) - \left(x_0 + \lambda - \alpha(x_1,x_2)\right)^2 D^2_{x_3} - \left(x_0 + \lambda - \alpha(x_1,x_2)\right)^2 D^2_{x_4}$$

in $[0,+\infty[\times\mathbb{R}^4$. We observe that $g(x_1,x_2) = \frac{1}{a+b}$ and $h(x_1,x_2) = 1$ in $\mathbb{R}^2$. Let us remark that

$$A = \begin{pmatrix} 3 & \frac{1}{2} \\ \frac{1}{2} & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As a consequence, $A + C = A$. The matrices A and B are defined positive with constants $L_1 = \frac{5}{2}$ and $L_2 = 1$, respectively. Moreover, assumption (iii) holds if $\lambda \geq \frac{1}{|a+b|}$. Therefore, $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$, then we have transition.

Example 1.3 Let us consider the function $\alpha(x_1,x_2,x_3) = (x_1 + x_2 + x_3)^2$ in $]-k,k[^3$, with $k > 0$ and the operator

$$\begin{aligned} P = D^2_{x_0} &- \left(4D^2_{x_1} + 4D^2_{x_2} + 4D^2_{x_3} + \frac{1}{2}D^2_{x_1x_2} + \frac{1}{2}D^2_{x_1x_3} + \frac{1}{2}D^2_{x_2x_3}\right) \\ &- \left(x_0 + \lambda - \alpha(x_1,x_2,x_3)\right)^2 D^2_{x_4} \end{aligned}$$

in $[0,+\infty[\times]-k,k[^3\times\mathbb{R}$. It is easy to verify that $g(x_1,x_2,x_3) = \frac{1}{6}(x_1 + x_2 + x_3)$ and $h(x_1,x_2,x_3) = \frac{1}{2}$ in $]-k,k[^3$. Moreover, as a result

$$A = \begin{pmatrix} 4 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 4 \end{pmatrix}, \qquad B = (1), \qquad C = \begin{pmatrix} \frac{5}{2} & -\frac{5}{4} & -\frac{5}{4} \\ -\frac{5}{4} & \frac{5}{2} & -\frac{5}{4} \\ -\frac{5}{4} & -\frac{5}{4} & \frac{5}{2} \end{pmatrix}.$$

The matrices A and B are positive definite with constants $L_1 = \frac{7}{2}$ and $L_2 = 1$, respectively, and C is positive semidefinite. Finally, assumption (iii) is ensured when $\lambda \geq \frac{1}{2}k$. For k large enough, $\Sigma = \Sigma_- \sqcup \Sigma_0 \sqcup \Sigma_+$ results, namely we have transition from effectively hyperbolic to non-effectively hyperbolic.

The paper is organized as follows. In Section 2 some preliminary notations are given. In Section 3 *a priori* estimates are proved. In Section 4, estimates in Sobolev spaces with $s < 0$ by means of the pseudodifferential operator theory are obtained. Section 5 deals with a local existence theorem near the boundary. Then a regularity result for the solution u to the Cauchy problem (2) is shown. At last a global existence result is proved in Section 6.

2 Notations and preliminaries

Let $\alpha = (\alpha_0, \alpha', \alpha'') \in \mathbb{N}_0^{n+1}$. We denote by ∂^α the derivative of order $|\alpha|$, while $\partial^h_{x_j}$ means, as usually, the derivative of order h with respect to x_j and $\partial^h_{x_j,x_p}$ denotes the derivative of order h with respect to x_j and x_p.

Let us denote by $(\cdot,\cdot)$, $\|\cdot\|$, $\|\cdot\|_{H^r}$ $(r \in \mathbb{N}_0)$ the L^2-scalar product, the L^2-norm and the H^r-norm, respectively.

$C_0^\infty(\overline{\Omega})$ is the space of the restrictions to $\overline{\Omega}$ of functions φ belonging to $C_0^\infty(\mathbb{R}^{n+1})$ such that φ vanishes with all the derivatives in $[0,+\infty[\,\times \partial\Omega_0$.

Let $s \in \mathbb{R}$, let us denote by $\|\cdot\|_{H^{0,0,s}}$ the norm given by

$$\|u\|^2_{H^{0,0,s}(\overline{\Omega})} = \frac{1}{(2\pi)^{n-m}} \int_0^{+\infty} dx_0 \int_{\mathbb{R}^m} dx' \int_{\mathbb{R}^{n-m}} \bigl(1 + |\xi''|^2\bigr)^s \bigl|\widehat{u}(x_0, x', \xi'')\bigr|^2 \, d\xi'',$$
$$\forall u \in C_0^\infty(\overline{\Omega}),$$

where the Fourier transform is done only with respect to the variable x_2. Moreover, let us denote by A_s the pseudodifferential operator, given by

$$A_s u = \frac{1}{(2\pi)^{n-m}} \int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''} \bigl(1 + |\xi''|^2\bigr)^{\frac{s}{2}} \widehat{u}(x_0, x_1, \xi'') \, d\xi'', \quad \forall u \in C_0^\infty(\overline{\Omega}). \tag{3}$$

Let us recall that $A_s : C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$. For every $\varphi(x'') \in C_0^\infty(\mathbb{R}^{n-m})$, the operator $\varphi A_s u$ extends as a linear continuous operator from $H^{0,0,r}_{\text{comp.}}(\overline{\Omega})$ to $H^{0,0,r-s}_{\text{loc}}(\overline{\Omega})$, where $r, s \in \mathbb{R}$ (see [16]). Moreover, denoted by $\mathcal{U}_{x''}$ the projection of $\operatorname{supp} u$ on the hyperplane $x'' = 0$, if $\operatorname{supp} u \subseteq \mathbb{R}^{m-n} \backslash \mathcal{U}_{x''}$, then $\varphi A_s u$ is regularizing with respect to the variable x'', namely as a result

$$\|\varphi A_s u\|_{H^{0,0,r}} \le c \|u\|_{H^{0,0,r'}},$$
$$\forall r, r' \in \mathbb{R}, u \in C^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq [0,+\infty[\times \Omega' \times \Omega'' \setminus \operatorname{supp} \varphi.$$

Let us remark that the norms $\|u\|_{H^{0,0,s}(\Omega)}$ and $\|A_s u\|_{L^2(\Omega)}$ are equivalent.

Finally, let $s, p \in \mathbb{R}$, let us denote by $\|\cdot\|_{H^{s,p}}$ the norm given by

$$\|u\|^2_{H^{p,s}(\overline{\Omega})} = \sum_{|h| \le p} \int_0^{+\infty} dx_0 \int_{\mathbb{R}^m} dx' \int_{\mathbb{R}^{n-m}} \frac{1}{(2\pi)^{n-m}} \bigl(1 + |\xi''|^2\bigr)^p \bigl|\partial^h_{x_0,x'} \widehat{u}(x_0, x', \xi'')\bigr|^2 \, d\xi'',$$
$$\forall u \in C_0^\infty(\overline{\Omega}).$$

3 *A priori* estimates

Lemma 3.1 *Let $\Omega_k = [0,k[\,\times \Omega_0$, for every $k > 0$, let $u \in C_0^\infty(\overline{\Omega}_k)$, as a result*

$$\|u\|_{L^2(\Omega_k)} \le 2k \|\partial_{x_0} u\|, \tag{4}$$

$$\|u(0,x',x'')\|^2_{L^2(\Omega_0)} \le 4k\|\partial_{x_0}u\|^2, \tag{5}$$

$$\|u\|^2_{L^2(\Omega_k)} + \|u(0,x',x'')\|^2_{L^2(\Omega_0)} \le 4(k^2+k)\|\partial_{x_0}u\|^2_{L^2(\Omega_k)}. \tag{6}$$

Proof Let $u \in C_0^\infty(\overline{\Omega})$. We have

$$\begin{aligned} 0 &= \int_\Omega \partial_{x_0}\bigl(x_0 u^2(x)\bigr)\,dx \\ &= \int_\Omega u^2(x)\,dx + 2\int_\Omega x_0 u(x)\partial_{x_0}u(x)\,dx, \end{aligned}$$

and therefore

$$\int_\Omega u^2(x)\,dx = -2\int_\Omega x_0 u(x)\partial_{x_0}u(x)\,dx.$$

In particular, in Ω_k we obtain

$$\|u\|^2 \le 2k\|u\|\|\partial_{x_0}u\|,$$

which implies (4). Analogously, by using the following equality:

$$\int_\Omega \partial_{x_0}u^2(x)\,dx = -\int_{\Omega_0} u^2(0,x',x'')\,dx'\,dx'',$$

we obtain (5). Finally, collecting (4) and (5), we have (6). □

Now, we are able to prove the following *a priori* estimate.

Theorem 3.1 *Let* $\Omega_k = [0,k[\times\Omega_0$ *be a subset of* Ω, *where* $k > 0$. *Let us suppose that* g, h *satisfy* (i), (ii), *and* (iii). *Then there exists a constant* $c > 0$ *such that*

$$\|\partial_{x_0}u\| + \sum_{j=1}^m \|\partial_{x_j}u\| + \sum_{j=m+1}^n \|(x_0+\lambda-\alpha(x'))\partial_{x_j}u\| \le c(\|Pu\|+\|u\|),$$
$$\forall u \in C_0^\infty(\overline{\Omega}_k). \tag{7}$$

Proof Let us integrate by parts in the inner product

$$\begin{aligned} 2\bigl((x_0+\lambda)\partial_{x_0}u(x),Pu\bigr) = {}& \|\partial_{x_0}u\|^2 + \lambda\int_{\Omega_0}\bigl(\partial_{x_0}u(0,x',x'')\bigr)^2\,dx'\,dx'' \\ & - 2\int_\Omega A(x',x'')\nabla_{x'}u(x)\cdot(x_0+\lambda)\nabla_{x'}\partial_{x_0}u(x)\,dx \\ & - 2\int_\Omega \bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot(x_0+\lambda)\nabla_{x''}\partial_{x_0}u(x)\,dx \\ & - \int_\Omega \gamma(x)u^2(x)\,dx \\ & - \int_\Omega \partial_{x_0}\gamma(x)(x_0+\lambda)u^2(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{\Omega_0} \gamma\bigl(0,x',x''\bigr)u^2\bigl(0,x',x''\bigr)\,dx'\,dx'' \\
&= \|\partial_{x_0}u\|^2 + \lambda \int_{\Omega_0} \bigl(\partial_{x_0}u\bigl(0,x',x''\bigr)\bigr)^2\,dx'\,dx'' \\
&\quad + \int_\Omega A\bigl(x',x''\bigr)\nabla_{x'}u(x)\cdot\nabla_{x'}u(x)\,dx \\
&\quad + \lambda \int_{\Omega_0} A\bigl(x',x''\bigr)\nabla_{x'}u\bigl(0,x',x''\bigr)\cdot\nabla_{x'}u\bigl(0,x',x''\bigr)\,dx'\,dx'' \\
&\quad + 2\int_\Omega \bigl(x_0+\lambda-\alpha\bigl(x'\bigr)\bigr)B\bigl(x''\bigr)\nabla_{x''}u(x)\cdot(x_0+\lambda)\nabla_{x''}u(x)\,dx \\
&\quad + \int_\Omega \bigl(x_0+\lambda-\alpha\bigl(x'\bigr)\bigr)^2 B\bigl(x''\bigr)\nabla_{x''}u(x)\cdot\nabla_{x''}u(x)\,dx \\
&\quad + \lambda \int_{\Omega_0} \bigl(\lambda-\alpha\bigl(x'\bigr)\bigr)^2 B\bigl(x''\bigr)\nabla_{x''}u\bigl(0,x',x''\bigr) \\
&\quad \cdot \nabla_{x''}u\bigl(0,x',x''\bigr)\,dx'\,dx'' \\
&\quad - \int_\Omega \gamma(x)u^2(x)\,dx - \int_\Omega (x_0+\lambda)u^2(x)\partial_{x_0}\gamma(x)\,dx \\
&\quad - \lambda \int_{\Omega_0} \gamma\bigl(0,x',x''\bigr)u^2\bigl(0,x',x''\bigr)\,dx'\,dx''.
\end{aligned} \tag{8}
$$

On the other hand, by integrating by parts in the inner product, we obtain

$$
\begin{aligned}
2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),Pu\bigr) &= -2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\partial_{x_0}^2u(x)\bigr) \\
&\quad + 2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\operatorname{div}_{x'}A\bigl(x',x''\bigr)\nabla_{x'}u(x)\bigr) \\
&\quad + 2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\bigl(x_0+\lambda-\alpha\bigl(x'\bigr)\bigr)^2\operatorname{div}_{x''}B\bigl(x''\bigr)\nabla_{x''}u(x)\bigr) \\
&\quad + 2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\gamma(x)u(x)\bigr).
\end{aligned} \tag{9}
$$

Let us compute separately every inner product:

$$
\begin{aligned}
-2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\partial_{x_0}^2u(x)\bigr) &= 2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\partial_{x_0}\overline{u}(x),\partial_{x_0}u(x)\bigr) \\
&\quad + 2\int_{\Omega_0}\partial_{x_0}u\bigl(0,x',x''\bigr)g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}\bigl(0,x',x''\bigr)\,dx'\,dx'' \\
&= -\int_\Omega \operatorname{div}_{x'}\overline{g}\bigl(x'\bigr)\bigl(\partial_{x_0}u(x)\bigr)^2\,dx \\
&\quad + 2\int_{\Omega_0}\partial_{x_0}u\bigl(0,x',x''\bigr)g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}\bigl(0,x',x''\bigr)\,dx'\,dx''.
\end{aligned} \tag{10}
$$

For the second one, we have

$$
\begin{aligned}
&2\bigl(g\bigl(x'\bigr)\operatorname{div}_{x'}\overline{u}(x),\operatorname{div}_{x'}A\bigl(x',x''\bigr)\nabla_{x'}u(x)\bigr) \\
&\quad = -2\sum_{h=1}^m\int_\Omega A\bigl(x',x''\bigr)\nabla_{x'}u(x)\cdot g\bigl(x'\bigr)\nabla_{x'}\partial_{x_h}u(x)\,dx \\
&\qquad - 2\sum_{h=1}^m\int_\Omega A\bigl(x',x''\bigr)\nabla_{x'}u(x)\cdot\partial_{x_h}u(x)\nabla_{x'}g\bigl(x'\bigr)\,dx.
\end{aligned} \tag{11}
$$

Moreover, as a result

$$\begin{aligned}
&-2\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}\partial_{x_h}u(x)\,dx\\
&\quad=2\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}\partial_{x_h}u(x)\cdot g(x')\nabla_{x'}u(x)\,dx\\
&\qquad+2\sum_{h=1}^{m}\int_{\Omega}\partial_{x_h}A(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}u(x)\,dx\\
&\qquad+2\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u(x)\cdot \partial_{x_h}g(x')\nabla_{x'}u(x)\,dx,
\end{aligned}$$

and that implies

$$\begin{aligned}
&-2\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u\cdot g(x')\nabla_{x'}\partial_{x_h}u(x)\,dx\\
&\quad=\sum_{h=1}^{m}\int_{\Omega}\partial_{x_h}A(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}u(x)\,dx\\
&\qquad+\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u(x)\cdot \partial_{x_h}g(x')\nabla_{x'}u(x)\,dx.
\end{aligned}\tag{12}$$

Substituting (12) in (11), we obtain

$$\begin{aligned}
&2\bigl(g(x')\operatorname{div}_{x'}\overline{u}(x),\operatorname{div}_{x'}\bigl(A(x',x'')\nabla_{x'}u(x)\bigr)\bigr)\\
&\quad=\sum_{h=1}^{m}\int_{\Omega}\partial_{x_h}A(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}u(x)\,dx\\
&\qquad+\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u(x)\cdot \partial_{x_h}g(x')\nabla_{x'}u(x)\,dx\\
&\qquad-2\sum_{h=1}^{m}\int_{\Omega} A(x',x'')\nabla_{x'}u(x)\cdot \partial_{x_h}u(x)\nabla_{x'}g(x')\,dx.
\end{aligned}\tag{13}$$

We compute

$$\begin{aligned}
&\sum_{h=1}^{m}\partial_{x_h}A(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}u(x)+\sum_{h=1}^{m}A(x',x'')\nabla_{x'}u(x)\cdot\partial_{x_h}g(x')\nabla_{x'}u(x)\\
&\quad=\operatorname{div}_{x'}\overline{A}(x',x'')\nabla_{x'}u(x)\cdot g(x')\nabla_{x'}u(x)+A(x',x'')\nabla_{x'}u(x)\cdot\operatorname{div}_{x'}\overline{g}(x')\nabla_{x'}u(x)\\
&\quad=\bigl(\operatorname{div}_{x'}\overline{A}(x',x'')g(x')+A(x',x'')\operatorname{div}_{x'}\overline{g}(x')\bigr)\nabla_{x'}u(x)\cdot\nabla_{x'}u(x).
\end{aligned}\tag{14}$$

Moreover, as a result

$$\sum_{h=1}^{m}A(x',x'')\nabla_{x'}u\cdot\partial_{x_h}u\nabla_{x'}g(x')=\Lambda(x',x'')\nabla_{x'}u(x)\cdot\nabla_{x'}u(x),\tag{15}$$

where $\Lambda(x',x'')$ is a matrix with m columns equal to $A(x',x'')\nabla_{x'}g(x')$. Taking into account (13), (14), and (15), we obtain

$$\begin{aligned}
&2\big(g(x')\operatorname{div}_{x'}\overline{u}(x),\operatorname{div}_{x'}A(x',x'')\nabla_{x'}u(x)\big)\\
&\quad=\int_\Omega\big[\operatorname{div}_{x'}\overline{A}(x',x'')g(x')+2A(x',x'')\operatorname{div}_{x'}\overline{g}(x')-2\Lambda(x',x'')\big]\nabla_{x'}u(x)\cdot\nabla_{x'}u(x)\,dx\\
&\qquad-\int_\Omega A(x',x'')\operatorname{div}_{x'}\overline{g}(x')\nabla_{x'}u\cdot\nabla_{x'}u\,dx\\
&\quad=\int_\Omega C(x',x'')\nabla_{x'}u(x)\cdot\nabla_{x'}u(x)\,dx-\int_\Omega A(x',x'')\nabla_{x'}u\cdot\nabla_{x'}u\,dx\\
&\qquad+\int_\Omega h(x')A(x',x'')\nabla_{x'}u\cdot\nabla_{x'}u\,dx, \qquad (16)
\end{aligned}$$

where we have set $C(x',x'')=\operatorname{div}_{x'}\overline{A}(x',x'')g(x')+2[A(x',x'')\operatorname{div}_{x'}\overline{g}(x')-\Lambda(x',x'')]$, for every $(x',x'')\in\Omega_0$.

Let us consider

$$\begin{aligned}
&2\big(g(x')\operatorname{div}_{x'}\overline{u}(x),\big(x_0+\lambda-\alpha(x')\big)^2\operatorname{div}_{x''}\big(\overline{B}(x'')\nabla_{x''}u(x)\big)\big)\\
&\quad=-2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)^2B(x'')\nabla_{x''}u(x)\cdot g(x')\nabla_{x''}\partial_{x_h}u(x)\,dx\\
&\quad=2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)^2B(x'')\nabla_{x''}\partial_{x_h}u(x)\cdot g(x')\nabla_{x''}u(x)\,dx\\
&\qquad-4\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)\partial_{x_h}\alpha(x')B(x'')\nabla_{x''}u(x)\cdot g(x')\nabla_{x''}u(x)\,dx\\
&\qquad+2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)^2B(x'')\nabla_{x''}u(x)\cdot\partial_{x_h}g(x')\nabla_{x''}u(x)\,dx, \qquad (17)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&-2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)^2B(x'')\nabla_{x''}u\cdot g(x')\nabla_{x''}\partial_{x_h}u\,dx\\
&\quad=-2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)\partial_{x_h}\alpha(x')B(x'')\nabla_{x''}u\cdot g(x')\nabla_{x''}\partial_{x_h}u\,dx\\
&\qquad+\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)^2B(x'')\nabla_{x''}u(x)\cdot\partial_{x_h}g(x')\nabla_{x''}u(x)\,dx. \qquad (18)
\end{aligned}$$

Substituting (18) in (17) and using assumption (ii), we have

$$\begin{aligned}
&-2\big(g(x')\operatorname{div}_{x'}\overline{u}(x),\big(x_0+\lambda-\alpha(x')\big)^2\operatorname{div}_{x''}\overline{B}(x'')\nabla_{x''}u(x)\big)\\
&\quad=-2\sum_{h=1}^m\int_\Omega\big(x_0+\lambda-\alpha(x')\big)\partial_{x_h}\alpha(x')B(x'')\nabla_{x''}u(x)\cdot g(x)\nabla_{x''}u(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{h=1}^{m}\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot\partial_{x_h}g(x')\nabla_{x''}u(x)\,dx\\
&=-2\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)B(x'')\nabla_{x''}u(x)\cdot\alpha(x)\nabla_{x''}u(x)\,dx\\
&\quad+\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot\operatorname{div}_{x'}\overline{g}(x')\nabla_{x''}u(x)\,dx\\
&=-2\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)B(x'')\nabla_{x''}u(x)\cdot\alpha(x)\nabla_{x''}u(x)\,dx\\
&\quad+\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot\nabla_{x''}u(x)\,dx\\
&\quad-\int_{\Omega}\bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot h(x')\nabla_{x''}u(x)\,dx.
\end{aligned}
\tag{19}
$$

Finally, we have

$$
\begin{aligned}
2\bigl(g(x')\operatorname{div}_{x'}\overline{u}(x),\gamma(x)u(x)\bigr)&=-\int_{\Omega}\operatorname{div}_{x'}\overline{\gamma}(x)g(x')u^2(x)\,dx\\
&\quad-\int_{\Omega}\gamma(x)\operatorname{div}_{x'}\overline{g}(x')u^2(x)\,dx\\
&=-\int_{\Omega}\operatorname{div}_{x'}\overline{\gamma}(x)g(x')u^2(x)\,dx\\
&\quad-\int_{\Omega}\gamma(x)u^2(x)\,dx\\
&\quad+\int_{\Omega}\gamma(x)h(x')u^2(x)\,dx.
\end{aligned}
\tag{20}
$$

By adding (10), (16), (19), and (20), we have

$$
\begin{aligned}
&2\bigl((x_0+\lambda)\partial_{x_0}u+g(x')\operatorname{div}_{x''}\overline{u},Pu\bigr)\\
&=\|\partial_{x_0}u\|^2+\int_{\Omega}\bigl(4-h(x')\bigr)\bigl(x_0+\lambda-\alpha(x')\bigr)^2 B(x'')\nabla_{x''}u(x)\cdot\nabla_{x''}u(x)\,dx\\
&\quad+\int_{\Omega_0}\bigl\{\lambda\bigl[\bigl(\partial_{x_0}u(0,x',x'')\bigr)^2+A(x',x'')\nabla_{x'}u(0,x',x'')\cdot\nabla_{x'}u(0,x',x'')\bigr]\\
&\quad+2\partial_{x_0}u(0,x',x'')g(x')\operatorname{div}_{x'}\overline{u}(0,x',x'')\bigr\}\,dx'\,dx''\\
&\quad+\int_{\Omega}\bigl(h(x')A(x',x'')+C(x',x'')\bigr)\nabla_{x'}u\cdot\nabla_{x'}u\,dx\\
&\quad+\int_{\Omega}\bigl(2h(x')-3\bigr)\gamma(x)u^2(x)\,dx\\
&\quad-\int_{\Omega}(x_0+\lambda)\partial_{x_0}\gamma(x)u^2(x)\,dx\\
&\quad-\lambda\int_{\partial\Omega}\gamma(0,x',x'')u^2(0,x',x'')\,dx'\,dx''\\
&\quad-\int_{\Omega}\operatorname{div}_{x'}\overline{g}(x')u^2(x)\,dx.
\end{aligned}
$$

Making use of assumptions (i), (ii), (iii), and (iv), we obtain

$$\begin{aligned}
&\big((x_0+\lambda)\partial_{x_0}u+g(x')\operatorname{div}_{x'}\overline{u}(x),Pu\big)\\
&\quad\geq h_1\|\partial_{x_0}u\|^2+L_1h_1\sum_{j=1}^{m}\|\partial_{x_j}u\|^2\\
&\qquad+L_2(4-h_2)\sum_{j=1}^{n-m}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_{j+m}}u\big\|^2+4\big(k^2+k\big)c\|u\|^2,\quad \forall u\in C_0^\infty(\overline{\Omega}_k),
\end{aligned}$$

where $\Omega_k=[0,k[\times\Omega_0$, with $k>0$, from which (7) follows. □

As a consequence, we have the following corollary.

Corollary 3.1 *Under the same assumptions of Theorem* 3.1 *and for k small enough, there exists a constant* $c>0$ *such that*

$$\begin{aligned}
&\|\partial_{x_0}u\|+\sum_{j=1}^{m}\|\partial_{x_j}u\|+\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|+\|u\|\leq c\|Pu\|,\\
&\forall u\in C_0^\infty(\overline{\Omega}_k).
\end{aligned}\tag{21}$$

Proof Taking into account (4) and (7) and choosing a positive number k small enough, we obtain (21). □

4 Estimates in Sobolev spaces with $s<0$ by means of pseudodifferential operator theory

Let us, first, prove some preliminary results.

Lemma 4.1 *Let* $u\in C_0^\infty(\overline{\Omega})$, *with* $\overline{\Omega}=[0,+\infty[\times\Omega'\times\mathbb{R}^{n-m}$, *and let* $\varphi\in C_0^\infty(\mathbb{R}^{n-m})$, *with* $\operatorname{supp}\varphi\subseteq\mathbb{R}^{n-m}\backslash\mathcal{U}_{x''}$. *As a result*

$$\|\varphi A_s u\|_{L^2(\overline{\Omega})}\leq\frac{c_{q,r,s}}{L^q}\|u\|_{H^{0,0,r}(\overline{\Omega})},\quad \forall s\in\mathbb{R},r\in\mathbb{Z}^-,q\geq s+r,$$

where L is the distance between $\operatorname{supp}\varphi$ *and* $\mathcal{U}_{x''}$, *supposed to be greater than* 1.

Proof Let us consider

$$\begin{aligned}
\varphi A_s u&=\frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{n-m}}e^{ix''\cdot\xi''}\varphi\big(x''\big)\big(1+\big|\xi''\big|^2\big)^{\frac{1}{2}}\widehat{u}\big(x_0,x',\xi''\big)\,d\xi''\\
&=\frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{2(n-m)}}e^{i(x''-y'')\cdot\xi''}\varphi\big(x''\big)\big(1+\big|\xi''\big|^2\big)^{\frac{1}{2}}u\big(x_0,x',y''\big)\,dy''\,d\xi''\\
&=\frac{i^{2p}}{(2\pi)^{m-n}}\int_{\mathbb{R}^{2(n-m)}}e^{i(x''-y'')\cdot\xi''}\frac{\varphi(x'')u(x_0,x',y'')}{|x''-y''|^{2p}}\big(\triangle_{\xi''}\big(1+\big|\xi''\big|^2\big)^{\frac{s}{2}}\big)^{[p]}\,dy''\,d\xi''\\
&=\frac{i^{2p}}{(2\pi)^{m-n}}\int_{\mathbb{R}^{n-m}}\big(\triangle_{\xi''}\big(1+\big|\xi''\big|^2\big)^{\frac{s}{2}}\big)^{[p]}\,d\xi''\\
&\quad\times\int_{\mathbb{R}^{n-m}}e^{i(x''-y'')\cdot\xi''}u\big(x_0,x',y''\big)\frac{\psi(\frac{|x''-y''|}{L})\varphi(x'')}{|x''-y''|^{2p}}\,dy'',
\end{aligned}$$

where $m\in\mathbb{N}$ and $\psi\in C^\infty(\mathbb{R})$ such that $\psi(\tau)=1$ if $|\tau|\geq1$, $\psi(\tau)=0$ if $|\tau|\leq\frac{1}{2}$.

This implies

$$\varphi A_s u = \frac{i^{2p}\varphi(x'')}{(2\pi)^{n-m}} \int_{\mathbb{R}^{n-m}} \left(\triangle_{\xi''}\left(1+\left|\xi''\right|^2\right)^{\frac{s}{2}}\right)^{[p]} u\left(x_0,x',x''\right) * \psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\,d\xi'',$$

where the convolution is done with respect to x'', and also

$$\begin{aligned}\mathcal{F}_{x''}(\varphi A_s u) = {}& \frac{i^{2p}\widehat{\varphi}(\eta'')}{2\pi} \\ & * \int_{\mathbb{R}^{n-m}} \left(\triangle_{\xi''}\left(1+\left|\xi''\right|^2\right)^{\frac{s}{2}}\right)^{[p]} \widehat{u}\left(x_0,x',\eta''\right)\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right)d\xi'', \quad (22)\end{aligned}$$

where

$$\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right) = \int_{\mathbb{R}^{n-m}} e^{ix''\cdot(\xi''-\eta'')}\psi\left(\frac{|x''|}{L}\right)\frac{1}{|x''|^{2p}}\,dx''.$$

It results

$$\begin{aligned}&\left(1+\left|\xi''-\eta''\right|^{2r'}\right)\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right) \\ &\quad = \int_{\mathbb{R}^{n-m}} e^{ix''\cdot(\xi''-\eta'')}\psi\left(\frac{|x''|}{L}\right)\frac{1}{|x''|^{2p}}\,dx'' \\ &\qquad + (-1)^r \int_{\mathbb{R}^{n-m}} e^{ix''\cdot(\xi''-\eta'')}\left(\triangle_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{1}{|x''|^{2p}}\right)\right)^{[r]} dx''\end{aligned}$$

and then

$$\left|\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right)\right| \le \frac{c_{r,p}}{(1+|\xi''-\eta''|^{2r'})}\left(\frac{1}{L}\right)^{2p-n+m+1}. \quad (23)$$

Making use of (22) and (23), we obtain

$$\begin{aligned}&\left\|\mathcal{F}_{x''}(\varphi A_s u)\right\| \\ &\quad = \|\varphi A_s u\| \\ &\quad \le \frac{1}{(2\pi)^{n-m}}\left\|\widehat{\varphi}\left(\eta''\right)\right\|_{L^1(\mathbb{R}^{n-m})} \\ &\qquad \cdot \left\|\int_{\mathbb{R}^{n-m}} \left(\triangle_{\xi''}\left(1+\left|\xi''\right|^2\right)^{\frac{s}{2}}\right)^{[p]}\widehat{u}\left(x_0,x',\eta''\right)\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right)d\xi''\right\|_{L^2(\Omega)} \\ &\quad \le c\int_{\mathbb{R}^{n-m}}\left\|\left(\triangle_{\xi''}\left(1+\left|\xi''\right|^2\right)^{\frac{s}{2}}\right)^{[p]}\widehat{u}\left(x_0,x',\eta''\right)\mathcal{F}_{x''}\left(\psi\left(\frac{|x''|}{L}\right)\frac{e^{ix''\cdot\xi''}}{|x''|^{2p}}\right)\right\|_{L^2(\Omega)} d\xi'' \\ &\quad \le \frac{c_{r,p}}{L^{2p-n+m+1}}\int_{\mathbb{R}^{n-m}}\left\|\frac{(\triangle_{\xi''}(1+|\xi''|^2)^{\frac{s}{2}})^{[p]}\widehat{u}(x_0,x',\eta'')}{(1+|\xi''-\eta''|^{2r'})}\right\|_{L^2(\Omega)} d\xi''.\end{aligned}$$

Taking into account the previous inequality and the Peetre inequality (see [16], p. 17), it follows that

$$\|\varphi A_s u\|_{L^2(\overline{\Omega})} \le \frac{c_{r,p,s}}{L^{2p-n+m}}\int_{\mathbb{R}^{n-m}}\frac{\|(1+|\xi''|^2)^{\frac{s}{2}-\frac{2p+n-m}{2}}\widehat{u}(x_0,x',\eta'')\|_{L^2(\Omega)}}{(1+|\xi''-\eta''|^{2r})}\,d\xi''. \quad (24)$$

If $p \geq \frac{s+r+n-m+1}{2}$, setting $q = 2p - n + m + 1$ in (24), $r = 2r'$, results in

$$\|\varphi A_s u\|_{L^2(\Omega)} \leq \frac{c_{q,r,s}}{L^q} \|u\|_{H^{0,0,r}(\Omega)},$$

where the constant $c_{q,r,s}$ is independent on L. □

Taking into account Lemma 4.1, it is easy to show the following.

Lemma 4.2 *Let $\varphi \in C_0^\infty(\mathbb{R}^{n-m})$ such that $\varphi(|\tau''|) = 0$ if $|\tau''| \leq 1$. For every $\varepsilon > 0$, $r \in \mathbb{Z}^-$, $s \in \mathbb{R}$ and for every $u \in C_0^\infty(\overline{\Omega})$, with $\overline{\Omega} = [0, +\infty[\times \Omega' \times \mathbb{R}^{n-m}$, there exists $L > 0$ such that*

$$\left\| \varphi\left(\frac{|x''|}{L}\right) A_s u \right\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{H^{0,0,r}(\Omega)}.$$

Furthermore, we are able to prove the following.

Lemma 4.3 *Let $\varphi \in C_0^\infty(\mathbb{R}^{n-m})$ such that $\varphi(|\tau''|) = 1$ if $|\tau''| \leq 1$. For every $\varepsilon > 0$, $r \in \mathbb{Z}^-$, $s \in \mathbb{R}$ and for every $u \in C_0^\infty(\overline{\Omega})$, with $\overline{\Omega} = [0, +\infty[\times \Omega' \times \mathbb{R}^{n-m}$, there exists $L > 0$ such that*

$$\left\| \left(1 - \varphi\left(\frac{|x''|}{L}\right)\right) A_s u \right\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{H^{0,0,r}(\Omega)}.$$

Proof In order to establish this result we can proceed as Lemma 4.1, but in (22) we need to consider the Fourier transform of the function $\psi(|x''|) = 1 - \varphi(\frac{|x''|}{L})$ instead of $\widehat{\varphi}(\eta'')$ and keep in mind that

$$\widehat{\psi g} = \widehat{\psi} * \widehat{g} = (2\pi\delta - \widehat{\varphi}) * \widehat{g} = 2\pi\widehat{g} - \widehat{\varphi} * \widehat{g}, \quad \forall g \in S'(\mathbb{R}),$$

where $S'(\mathbb{R})$ is the space of tempered distributions defined in $\mathbb{R}$. □

Next, we prove a result concerning estimates near the boundary.

Lemma 4.4 *Let $\Omega =]0, +\infty[\times \Omega_0$, where Ω_0 is an open subset of $\mathbb{R}^n$. For every ε and δ positive, there exists $k > 0$ such that if*

$$I_{k,\delta} = \{x \in \overline{\Omega} : x_0 < k, |x_0 + \lambda - \alpha(x')| > \delta\},$$

as a result

$$\begin{aligned} &\|\partial_{x_0} u\| + \sum_{j=1}^{m} \|\partial_{x_j} u\| + \sum_{j=m+1}^{n} \|(x_0 + \lambda - \alpha(x'))\partial_{x_j} u\| + \|u\| \leq \varepsilon \|Pu\|, \\ &\quad \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq I_{k,\delta}. \end{aligned} \tag{25}$$

Proof Integrating by parts and proceeding as in the first part of Theorem 3.1, we have

$$\begin{aligned} 2\left(e^{\tau x_0}\partial_{x_0} u(x), Pu\right) &= \tau \left\| e^{\frac{1}{2}\tau x_0} \partial_{x_0} u \right\|^2 + \int_{\Omega_0} \left(\partial_{x_0} u(0, x', x'')\right)^2 dx'\, dx'' \\ &\quad + \int_\Omega e^{\tau x_0} A(x', x'') \nabla_{x'} u(x) \cdot \nabla_{x'} u(x)\, dx \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\Omega_0} A(x',x'')\nabla_{x'} u(0,x',x'') \cdot \nabla_{x'} u(0,x',x'')\,dx'\,dx'' \\
&+ 2\int_{\Omega} e^{\tau x_0}\big(x_0+\lambda-\alpha(x')\big)B(x'')\nabla_{x''}u(x)\cdot\nabla_{x''}u(x)\,dx \\
&+ \tau\int_{\Omega} e^{\tau x_0}\big(x_0+\lambda-\alpha(x')\big)^2 B(x'')\nabla_{x''}u(x)\cdot\nabla_{x''}u(x)\,dx \\
&+ \lambda\int_{\Omega_0}\big(\lambda-\alpha(x')\big)^2 B(x'')\nabla_{x''}u(0,x',x'')\cdot\nabla_{x''}u(0,x',x'')\,dx'\,dx'' \\
&- \int_{\Omega}\tau e^{\tau x_0}\gamma(x)u^2(x)\,dx - \int_{\Omega} e^{\tau x_0}u^2(x)\partial_{x_0}\gamma(x)\,dx \\
&- \int_{\Omega_0}\gamma(0,x',x'')u^2(0,x',x'')\,dx'\,dx''. \qquad (26)
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\tau\left(\big\|e^{\frac{1}{2}\tau x_0}\partial_{x_0}u\big\|^2 + L_1\sum_{j=1}^{m}\big\|e^{\frac{1}{2}\tau x_0}\partial_{x_j}u\big\|^2 + L_2\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)e^{\frac{1}{2}\tau x_0}\partial_{x_j}u\big\|^2\right) \\
&\quad\le c\sum_{j=m+1}^{n}\left\|\frac{(x_0+\lambda-\alpha(x'))}{(x_0+\lambda-\alpha(x'))^{\frac{1}{2}}}e^{\frac{1}{2}\tau x_0}\partial_{x_j}u\right\|^2 + \tau\big\|e^{\frac{1}{2}\tau x_0}\big|\gamma(x)\big|^{\frac{1}{2}}u(x)\big\|^2 \\
&\qquad + \big\|e^{\frac{1}{2}\tau x_0}\big|\partial_{x_0}\gamma(x)\big|^{\frac{1}{2}}u(x)\big\|^2 + \int_{\Omega_0}\big|\gamma(0,x',x'')\big|u^2(0,x',x'')\,dx'\,dx'' \\
&\qquad + \big(ce^{\tau x_0}Pu,\partial_{x_0}u\big) \\
&\quad\le \frac{c}{\delta}\sum_{j=m+1}^{n}\big\|e^{\frac{1}{2}\tau x_0}\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|^2 + c\left(\frac{1}{\tau^2}+\frac{1}{\tau}\right)\|\partial_{x_0}u\|^2 + \frac{1}{2}\big\|e^{\frac{1}{2}\tau x_0}Pu\big\|^2 \\
&\qquad + \frac{1}{2}\big\|e^{\frac{1}{2}\tau x_0}\partial_{x_0}u\big\|^2.
\end{aligned}
$$

Choosing $x_0 < \frac{1}{\tau}$, it follows that

$$
\begin{aligned}
&\|\partial_{x_0}u\|^2 + \sum_{j=1}^{m}\|\partial_{x_j}u\|^2 + \sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|^2 \\
&\quad\le \frac{c}{\tau\delta}\sum_{j=m+1}^{n}\big\|e^{\frac{1}{2}\tau x_0}\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|^2 + c\frac{1}{\tau}\|Pu\|^2 + \frac{c}{\tau}\|\partial_{x_0}u\|^2. \qquad (27)
\end{aligned}
$$

For τ large enough, making use of (27) and (4) we obtain the claim. □

As a consequence, we establish the following result.

Lemma 4.5 *For every ε and δ positive, there exists $k > 0$ such that if*

$$
I_{k,\delta} = \big\{x\in\overline{\Omega} : x_0 < k, \big|x_0+\lambda-\alpha(x')\big| > \delta\big\},
$$

for every $s < 0$, as a result

$$\begin{aligned}&\|\partial_{x_0}u\|_{H^{0,0,s}(\Omega)}+\sum_{j=1}^{m}\|\partial_{x_j}u\|_{H^{0,0,s}(\Omega)}+\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|_{H^{0,0,s}(\Omega)}+\|u\|_{H^{0,0,s}(\Omega)}\\&\quad\le\varepsilon\|Pu\|_{H^{0,0,s}(\Omega)},\quad\forall u\in C_0^\infty(\overline{\Omega}):\operatorname{supp}u\subseteq I_{k,\delta}.\end{aligned}\tag{28}$$

Proof Let $\varphi\in C_0^\infty(\mathbb{R})$, set $v_s=\varphi(|x''|)A_su$, as a result $\operatorname{supp}v_s\subseteq I_{k,\delta}$ and $\varphi(|x''|)=1$ in $\mathcal{U}_{x''}$. Therefore we can rewrite (25) with v_s, namely

$$\|\partial_{x_0}v_s\|+\sum_{j=1}^{m}\|\partial_{x_j}v_s\|+\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}v_s\big\|+\|v_s\|\le\varepsilon\|Pv_s\|.\tag{29}$$

Let us compute

$$\begin{aligned}\|\partial_{x_0}v_s\|&\ge\|\partial_{x_0}A_su\|-\big\|(\varphi-1)\partial_{x_0}A_su\big\|\\&=\|\partial_{x_0}A_su\|-\|Ru\|,\end{aligned}\tag{30}$$

where $(\varphi-1)\partial_{x_0}A_s=R$ is a regularizing operator. In the same way,

$$\|\partial_{x_j}v_s\|\ge c\|A_s\partial_{x_j}u\|-\|Ru\|,\quad\forall j=1,\dots,m.\tag{31}$$

Similarly, we obtain

$$\begin{aligned}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}v_s\big\|&=\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}\varphi A_su\big\|\\&=\big\|A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u+(\varphi-1)A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\\&\quad+\big(x_0+\lambda-\alpha(x')\big)[\partial_{x_j},\varphi](A_su)\big\|\\&\ge\big\|A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|-c\|Ru\|,\end{aligned}\tag{32}$$

where we take into account that $(\varphi-1)A_s\partial_{x_j}$ and $[\partial_{x_j},\varphi]A_s$ are regularizing operators. Finally, we have

$$\begin{aligned}\|v_s\|&=\big\|A_su+(\varphi-1)A_su\big\|\\&\ge\|A_su\|-\|Ru\|,\end{aligned}\tag{33}$$

$(\varphi-1)A_s$ being a regularizing operator.

Making use of (30), (31), (32), and (33), it follows that

$$\begin{aligned}&\|\partial_{x_0}v_s\|+\sum_{j=1}^{m}\|\partial_{x_j}v_s\|+\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}v_s\big\|+\|v_s\|\\&\quad\ge\|\partial_{x_0}u\|_{H^{0,0,s}(\Omega_k)}+c\left(\sum_{j=1}^{m}\|\partial_{x_j}u\|_{H^{0,0,s}(\Omega_k)}+\sum_{j=m+1}^{n}\big\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\big\|_{H^{0,0,s}(\Omega_k)}\right)\\&\qquad+\|u\|_{H^{0,0,s}(\Omega_k)}-c\|Ru\|.\end{aligned}$$

Since $\|Ru\| \leq c\|u\|_{H^{0,0,s}(\Omega_k)} \leq ck\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)}$, choosing k small enough, as a result

$$\begin{aligned}
&\|\partial_{x_0} v_s\| + \sum_{j=1}^{m} \|\partial_{x_j} v_s\| + \sum_{j=m+1}^{n} \big\|\big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} v_s\big\| + \|v_s\| \\
&\quad \geq c\Bigg(\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=m+1}^{n} \big\|\big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u\big\|_{H^{0,0,s}(\Omega_k)} \\
&\qquad + \|u\|_{H^{0,0,s}(\Omega_k)}\Bigg).
\end{aligned} \tag{34}$$

On the other hand, we have

$$Pv_s = \varphi\big(\big|x''\big|\big)A_s Pu + \varphi\big(\big|x''\big|\big)[P, A_s](u) + \big[\varphi\big(\big|x''\big|\big), P\big](A_s u).$$

As a consequence, we obtain

$$\|Pv_s\| = \big\|\varphi\big(\big|x''\big|\big)A_s Pu + \varphi\big(\big|x''\big|\big)B_{s+1} u + Ru\big\|, \tag{35}$$

where we set $[P, A_s] = B_{s+1}$, this being a pseudodifferential operator endowed with the symbol with respect to the variable x'' of order $s + 1$. Such a symbol has the following principal part:

$$c\big(x, \xi''\big) = -\frac{1}{i} \sum_{p=m+1}^{n} \big(x_0 + \lambda - \alpha\big(x'\big)\big)^2 \sum_{i,j=m+1}^{n} \partial_{x_p} b_j\big(x''\big)\xi_i''\xi_j''\partial_{\xi_p}\big(1 + \big|\xi''\big|^2\big)^{\frac{s}{2}}.$$

Therefore the symbol $b(x, \xi'')$ can be written as

$$b\big(x, \xi''\big) = c\big(x, \xi''\big) + d\big(x, \xi''\big),$$

where $d(x, \xi'')$ is a symbol of order s and we set

$$B_{s+1} = C_{s+1} + D_s.$$

Moreover, we set $R = [\varphi(|x''|), P]A_s$, which is a regularizing operator.

At last, we remark that

$$C_{s+1} u = C_s'\big(x_0 + \lambda - \alpha\big(x'\big)\big) \sum_{i=m+1}^{n} \partial_{x_j} u$$

and the symbol c_s' of C_s' is given by

$$c_s'\big(x, \xi''\big) = -\frac{1}{i} \sum_{p=m+1}^{n} \big(x_0 + \lambda - \alpha\big(x'\big)\big) \sum_{j=m+1}^{n} \partial_{x_p} b_j\big(x''\big)\xi_j''\partial_{\xi_p}\big(1 + \big|\xi''\big|^2\big)^{\frac{s}{2}}.$$

By such insights and by (35), as a result

$$\begin{aligned}\|Pv_s\| &\le c\left(\|A_sPu\| + \sum_{j=m+1}^{n}\left\|C_s'\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\right\| + \|D_su\| + \|Ru\|\right)\\ &\le c\left(\|Pu\|_{H^{0,0,s}(\Omega)} + \sum_{j=m+1}^{n}\left\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\right\|_{H^{0,0,s}(\Omega)} + \|u\|_{H^{0,0,s}(\Omega)}\right).\end{aligned} \tag{36}$$

By using (29), (34), (36), and for ε small enough, the claim follows. □

Now, we are able to prove the following theorem.

Theorem 4.1 *Let* $\Omega_k = [0,k[\,\times\Omega_0$, *with* k *such that* (21) *holds, let* $\Omega_0 = \Omega'\times\mathbb{R}^{n-m}$, *let* Ω' *be an open set of* $\mathbb{R}^{n-m}$ *and let* $B(x'')$ *be a constant. Under assumptions* (i) *and* (ii), *for every* $s\in\mathbb{Z}^-$ *there exists* $c>0$ *such that*

$$\begin{aligned}&\|\partial_{x_0}u\|_{H^{0,0,s}(\Omega)} + \sum_{j=1}^{m}\|\partial_{x_j}u\|_{H^{0,0,s}(\Omega)} + \sum_{j=m+1}^{n}\left\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\right\|_{H^{0,0,s}(\Omega)} + \|u\|_{H^{0,0,s}(\Omega)}\\ &\quad\le c\|Pu\|_{H^{0,0,s}(\Omega_k)},\quad \forall u\in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u\subseteq\Omega_k.\end{aligned} \tag{37}$$

Proof Let $\varphi\in C_0^\infty(\mathbb{R})$ and $\varphi(\tau')=1$ if $|\tau'|\le 1$ and $\mathcal{U}_{x''}\subseteq[-L,L]^{n-m}$. Setting $v_s=\varphi(\frac{|x''|}{L})A_su$ in (21), as a result

$$\|\partial_{x_0}v_s\| + \sum_{j=1}^{m}\|\partial_{x_j}v_s\| + \sum_{j=m+1}^{n}\left\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}v_s\right\| + \|v_s\| \le c\|Pv_s\|. \tag{38}$$

Furthermore, we have

$$\begin{aligned}\|\partial_{x_0}v_s\| &= \left\|\partial_{x_0}A_su + (\varphi-1)\partial_{x_0}A_su\right\|\\ &\ge \|A_s\partial_{x_0}u\| - \left\|(\varphi-1)A_s\partial_{x_0}u\right\|.\end{aligned}$$

Taking into account Lemma 4.3, for L large enough, it follows that

$$\|\partial_{x_0}v_s\| \ge c\|A_s\partial_{x_0}u\|. \tag{39}$$

Making use of the same technique, we have

$$\|\partial_{x_j}v_s\| \ge c\|A_s\partial_{x_j}u\|,\quad \forall j=1,\dots,m, \tag{40}$$

and similarly

$$\begin{aligned}\left\|\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}v_s\right\| &= \bigg\|A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u + (\varphi-1)A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\\ &\qquad + \left[\partial_{x_j},\varphi\left(\frac{|x''|}{L}\right)\right]A_s\big(x_0+\lambda-\alpha(x')\big)u\bigg\|\\ &\ge c\left\|A_s\big(x_0+\lambda-\alpha(x')\big)\partial_{x_j}u\right\| - \frac{c}{L}\left\|A_s\big(x_0+\lambda-\alpha(x')\big)u\right\|,\end{aligned} \tag{41}$$

where L is large enough. Finally, we have

$$\begin{aligned} \|v_s\| &= \|A_s u\| + \left\| \left(1 - \varphi\left(\frac{|x''|}{L}\right)\right) A_s u \right\| \\ &\geq (1-\varepsilon)\|u\|_{H^{0,0,s}(\Omega)}, \end{aligned} \tag{42}$$

having used Lemma 4.2.

On the other hand we have the result

$$Pv_s = \varphi\left(\frac{|x''|}{L}\right) A_s P u + \left[\varphi\left(\frac{|x''|}{L}\right), P\right](A_s u). \tag{43}$$

Let us observe that

$$\|A_s P u\| = \|Pu\|_{H^{0,0,s}(\Omega)}, \tag{44}$$

from the continuity property of the pseudodifferential operators (see [16], Theorem 2.1) and making use of Lemma 4.2

$$\left\| \left[\varphi\left(\frac{|x''|}{L}\right), P\right](A_s u) \right\| \leq \varepsilon \|u\|_{H^{0,0,s}(\Omega)}. \tag{45}$$

By using (43), (44), and (45), we have

$$\begin{aligned} \|Pv_s\| &\leq \left\| \varphi\left(\frac{|x''|}{L}\right) A_s P u \right\| + \left\| \varphi\left(\frac{|x''|}{L}\right) B_s u \right\| + \left\| \left[\varphi\left(\frac{|x''|}{L}\right), P\right](A_s u) \right\| \\ &\leq c\|Pu\|_{H^{0,0,s}(\Omega)} + \varepsilon \|u\|_{H^{0,0,s}(\Omega)}. \end{aligned} \tag{46}$$

Making use of (38), (39), (40), (41), and (42), choosing ε small enough and taking into account Lemma 3.1, as a result

$$\begin{aligned} &\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega)} + \sum_{j=m+1}^{n} \left\| \left(x_0 + \lambda - \alpha(x')\right) \partial_{x_j} u \right\|_{H^{0,0,s}(\Omega)} + \|u\|_{H^{0,0,s}(\Omega)} \\ &\quad \leq c\|Pu\|_{H^{0,0,s}(\Omega)} + \varepsilon \|u\|_{H^{0,0,s}(\Omega)} \\ &\quad \leq c\|Pu\|_{H^{0,0,s}(\Omega)} + \varepsilon \|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)}, \end{aligned} \tag{47}$$

and for ε small enough the claim is established. □

Now, we prove an estimate in Sobolev spaces with $s < 0$.

Theorem 4.2 *Under assumptions* (i), (ii), (iii), *and* (iv), *for every* $s \in \mathbb{R}_0^-$ *there exists* $c > 0$ *such that*

$$\begin{aligned} &\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega)} + \sum_{j=m+1}^{n} \left\| \left(x_0 + \lambda - \alpha(x')\right) \partial_{x_j} u \right\|_{H^{0,0,s}(\Omega)} + \|u\|_{H^{0,0,s}(\Omega)} \\ &\quad \leq c\|Pu\|_{H^{0,0,s}(\Omega_k)}, \quad \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq \Omega_k = [0, k[\times \Omega_0. \end{aligned} \tag{48}$$

Proof Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(|x''|) = 1$ on $\mathcal{U}_{x''}$ and let k such that (21) holds.

Then, for every $u \in C_0^\infty(\overline{\Omega})$ such that $\operatorname{supp} u \subseteq \Omega_k$, as a result

$$\|\partial_{x_0} v_s\| + \sum_{j=1}^{m} \|\partial_{x_j} v_s\| + \sum_{j=m+1}^{n} \left\|\left(x_0 + \lambda - \alpha\left(x'\right)\right)\partial_{x_j} v_s\right\| + \|v_s\| \le c\|Pv_s\|,$$

where $v_s = \varphi(|x''|)A_s u$.

Proceeding as in the proof of (34) (see from (29) to (34)), we obtain

$$\begin{aligned}
&\|\partial_{x_0} v_s\| + \sum_{j=1}^{m} \|\partial_{x_j} v_s\| + \sum_{j=m+1}^{n} \left\|\left(x_0 + \lambda - \alpha\left(x'\right)\right)\partial_{x_j} v_s\right\| + \|v_s\| \\
&\quad \ge c\left(\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=m+1}^{n} \left\|\left(x_0 + \lambda - \alpha\left(x'\right)\right)\partial_{x_j} u\right\|_{H^{0,0,s}(\Omega_k)} \right. \\
&\qquad \left. + \|u\|_{H^{0,0,s}(\Omega_k)} \right).
\end{aligned}$$

On the other hand, we have

$$Pv_s = \varphi\left(\left|x''\right|\right)A_s Pu + \left[\varphi\left(\left|x'\right|\right), P\right](A_s u) + \varphi\left(\left|x'\right|\right)[P, A_s]u = \varphi\left(\left|x'\right|\right)A_s Pu + Ru + B_{s+1}u,$$

where $R = [\varphi(|x'|), P]A_s$ is a regularizing operator and $B_{s+1} = [P, A_s]$ is a pseudodifferential operator with respect to the variables x'' of order $s+1$ endowed with symbol $b(x, \xi'')$ with principal part equal to

$$c\left(x, \xi''\right) = -\frac{1}{i} \sum_{p=m+1}^{n} \left(x_0 + \lambda - \alpha\left(x'\right)\right)^2 \sum_{i,j=m+1}^{n} \partial_{x_p} b_j\left(x''\right)\xi_i''\xi_j''\partial_{\xi_p}\left(1 + \left|\xi''\right|^2\right)^{\frac{s}{2}}.$$

Hence,

$$b\left(x, \xi''\right) = c\left(x, \xi''\right) + d\left(x, \xi''\right),$$

where $c(x, \xi'')$ is the symbol of order s. Therefore, we have

$$B_{s+1} = C_{s+1} + D_s.$$

Then as a result

$$[P, A_s](u) = C_{s+1}u + D_s u. \tag{49}$$

Taking into account (41), (42), (43), and (49), we obtain

$$\begin{aligned}
\|Pv_s\| &\le \left\|\varphi\left(\frac{|x''|}{L}\right)A_s Pu\right\| + \left\|\varphi\left(\frac{|x''|}{L}\right)B_{s+1}u\right\| + \left\|\left[\varphi\left(\frac{|x''|}{L}\right), P\right](A_s u)\right\| \\
&\le c\|Pu\|_{H^{0,0,s}(\Omega)} + \left\|\varphi\left(\frac{|x''|}{L}\right)C_{s+1}u\right\| + \left\|\varphi\left(\frac{|x''|}{L}\right)D_s u\right\| + \varepsilon\|u\|_{H^{0,0,s}(\Omega)} \\
&\le c\left(\|Pu\|_{H^{0,0,s}(\Omega)} + \|u\|_{H^{0,0,s}(\Omega)}\right) + \varepsilon\|u\|_{H^{0,0,s}(\Omega)} + \left\|\varphi\left(\frac{|x''|}{L}\right)C_{s+1}u\right\|.
\end{aligned} \tag{50}$$

We remember that C_{s+1} is a pseudodifferential operator endowed with the symbol

$$c(x,\xi') = -\frac{1}{i}\sum_{p=m+1}^{n}\bigl(x_0+\lambda-\alpha(x')\bigr)^2\sum_{i,j=m+1}^{n}\partial_{x_p}b_j(x'')\xi_i''\xi_j''\partial_{\xi_p}\bigl(1+|\xi''|^2\bigr)^{\frac{1}{2}}.$$

Therefore, we have

$$\begin{aligned} C_{s+1}u &= \frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}c(x,\xi'')\widehat{u}(x_0,x',\xi'')\,d\xi'' \\ &= \frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}\chi\bigl(|\xi''|\bigr)c(x,\xi'')\widehat{u}(x_0,x',\xi'')\,d\xi'' \\ &\quad+\frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}\bigl(1-\chi\bigl(|\xi''|\bigr)\bigr)c(x,\xi'')\widehat{u}(x_0,x',\xi'')\,d\xi'', \end{aligned}$$

where $\chi\in C_0^\infty(\mathbb{R})$ such that $\chi(t)=1$ for $|t|<1$. Therefore, we have

$$\begin{aligned} C_{s+1}u &= Ru \\ &\quad+\frac{1}{(2\pi)^{n-m}}\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}\bigl(1-\chi\bigl(|\xi''|\bigr)\bigr)c(x,\xi'')\widehat{u}(x_0,x',\xi'')\,d\xi'', \end{aligned}$$

where R is a regularizing operator. On the other hand, we have

$$\begin{aligned} &\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}\bigl(1-\chi\bigl(|\xi''|\bigr)\bigr)c(x,\xi'')\widehat{u}(x_0,x',\xi'')\,d\xi'' \\ &\quad=\sum_{j=m+1}^{n}\int_{\mathbb{R}^{n-m}} e^{ix''\cdot\xi''}c'(x,\xi'')\widehat{\bigl(x_0+\lambda-\alpha(x')\bigr)^2\partial_{x_j}u}(x_0,x',\xi'')\,d\xi'', \end{aligned}$$

where $c'(x,\xi'')=\frac{(1-\chi(|\xi''|))c(x,\xi'')}{(x_0+\lambda-\alpha(x'))^2|\xi''|^2}(\xi_1+\cdots+\xi_n)$ is a symbol of order s. As a consequence, it follows that

$$\begin{aligned} C_{s+1}u &= Ru+\sum_{j=1}^{n}C_s'\bigl(x_0+\lambda-\alpha(x')\bigr)^2\partial_{x_j}u \\ &= Ru+\sum_{j=m+1}^{n}C_s'\bigl(x_0+\lambda-\alpha(x')\bigr)^2\partial_{x_j}\chi\biggl(\frac{|x_0+\lambda-\alpha(x')|}{\delta}\biggr)u \\ &\quad+\sum_{j=m+1}^{n}C_s'\bigl(x_0+\lambda-\alpha(x')\bigr)^2\partial_{x_j}\biggl[1-\chi\biggl(\frac{|x_0+\lambda-\alpha(x')|}{\delta}\biggr)\biggr]u. \end{aligned}$$

Then, taking into account Lemma 4.5,

$$\begin{aligned} \|C_{s+1}u\| &\le \|Ru\|+c\sum_{j=m+1}^{n}\biggl\|\frac{(x_0+\lambda-\alpha(x'))^2}{\delta}\delta\chi\biggl(\frac{|x_0+\lambda-\alpha(x')|}{\delta}\biggr)\partial_{x_j}u\biggr\|_{H^{0,0,s(\Omega)}} \\ &\quad+c\sum_{j=m+1}^{n}\biggl\|\bigl(x_0+\lambda-\alpha(x')\bigr)^2\partial_{x_j}\biggl[1-\chi\biggl(\frac{|x_0+\lambda-\alpha(x')|}{\delta}\biggr)\biggr]u\biggr\|_{H^{0,0,s}(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&\leq \|u\|_{H^{0,0,s}(\Omega)} + c\delta \sum_{j=m+1}^{n} \big\| \big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u \big\|_{H^{0,0,s}(\Omega)} \\
&\quad + c\varepsilon \left\| P\left(1 - \chi\left(\frac{|x_0 + \lambda - \delta(x')|}{\delta}\right)\right) u \right\| \\
&\leq \|u\|_{H^{0,0,s}(\Omega)} + c\delta \sum_{j=m+1}^{n} \|\big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u\|_{H^{0,0,s}(\Omega)} + c\varepsilon \|Pu\|_{H^{0,0,s}(\Omega)} \\
&\quad + c\varepsilon \left\| \left[P, \left(1 - \chi\left(\frac{|x_0 + \lambda - \alpha(x')|}{\delta}\right)\right)\right](u) \right\|_{H^{0,0,s}(\Omega)} \\
&\leq \|u\|_{H^{0,0,s}(\Omega)} + c\delta \sum_{j=m+1}^{n} \big\| \big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u \big\|_{H^{0,0,s}(\Omega)} \\
&\quad + c\varepsilon \left(\|Pu\|_{H^{0,0,s}(\Omega)} + \|\partial_{x_0} u\|_{H^{0,0,s}(\Omega)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega)} \right).
\end{aligned}
\tag{51}
$$

Making use of (50), (51), and Lemma 3.1, we have

$$
\begin{aligned}
\|Pv_s\| \leq{}& c\big(\|Pu\|_{H^{0,0,s}(\Omega_k)} + k\|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)}\big) \\
&+ c\varepsilon \left(\|Pu\|_{H^{0,0,s}(\Omega_k)} + \|\partial_{x_0} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega_k)} \right) \\
&+ c\delta \sum_{j=m+1}^{n} \big\| \big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u \big\|_{H^{0,0,s}(\Omega_k)}.
\end{aligned}
\tag{52}
$$

Finally, by (52), (38), (39), (40), and (41), for δ and ε small enough and, hence, k small enough, as a result

$$
\begin{aligned}
&\|\partial_{x_0} u\|_{H^{0,0,s(\Omega_k)}} + \sum_{j=1}^{m} \|\partial_{x_j} u\|_{H^{0,0,s}(\Omega_k)} + \sum_{j=m+1}^{n} \big\| \big(x_0 + \lambda - \alpha\big(x'\big)\big)\partial_{x_j} u \big\|_{H^{0,0,s}(\Omega_k)} \\
&\quad + \|u\|_{H^{0,0,s}(\Omega_k)} \leq c\|Pu\|_{H^{0,0,s}(\Omega_k)}, \quad \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq \Omega_k. \qquad \square
\end{aligned}
$$

5 A local existence theorem near the boundary and a regularity result

Let $\Omega_k = [0, k[\times \Omega_0$, with $k > 0$; the following local existence theorem near the boundary holds.

Theorem 5.1 *Let $f \in H^{0,0,s}(\Omega)$, with $s \geq 0$. Then there exists $w \in H^{0,0,s}(\Omega_k)$ such that*

$$
\big(w, {}^tPu\big) = (f, u), \quad \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq \Omega_k.
$$

Proof Let S be the space

$$
S = \big\{\psi \in C_0^\infty(\overline{\Omega}_k) : \psi = {}^tPu, \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq \Omega_k\big\}.
$$

Let T be the linear functional defined as

$$
T(\psi) = T\big({}^tPu\big) = (f, u), \quad \forall \psi \in S.
$$

Making use of Theorems 4.1 and 4.2, we have

$$\begin{aligned}
|T(\psi)| &= |(f,u)| \\
&\leq \|f\|_{H^{0,0,s}(\Omega)} \|u\|_{H^{0,0,-s}(\Omega_k)} \\
&\leq c\|f\|_{H^{0,0,s}(\Omega)} \|{}^tPu\|_{H^{0,0,-s}(\Omega_k)} \\
&= c'\|\psi\|_{H^{0,0,-s}(\Omega_k)}, \quad \forall \psi \in S,
\end{aligned}$$

where $c' = c\|f\|_{H^{0,0,s}(\Omega)}$. Hence T is continuous on S and can be extended to a linear continuous functional in $H^{0,0,-s}(\overline{\Omega}_k)$. Making use of the representation theorems, there exists $w \in H^{0,0,s}(\overline{\Omega}_k)$ such that

$$\begin{aligned}
T(\psi) &= (w,\phi) \\
&= (w, {}^tPu) \\
&= (f,u), \quad \forall u \in C_0^\infty(\Omega) : \operatorname{supp} u \subseteq \Omega_k. \qquad \square
\end{aligned}$$

Now, let us study the regularity of the solution w. To this aim, we set

$$L = D_{x_0}^2 - \operatorname{Div}_{x'}\bigl(A(x',x'')D_{x'}\bigr)$$

and, for every $x'' \in \Omega''$, we consider the Cauchy problem

$$\begin{cases} Lv = h, & \text{in }]0,k[\times\Omega', \\ v(0,x') = 0, & v_{x_0}(0,x') = 0. \end{cases}$$

Since L is a strictly hyperbolic operator, it is well known that if $h \in H^s$ then the solution $v \in H^{s+1}$. As a consequence, since $Pw = f$ in the sense of distributions, $Lw = h$, with $h = f + (x_0 + \lambda - \alpha(x'))^2 \operatorname{Div}_{x''}(B(x'')D_{x''})w - \gamma(x)w$. Moreover, having $f \in H^{s,2(r-s)}$, with $0 \leq s \leq r$ and $r \geq 2$, it follows that

$$w \in H^{1,2(r-1)}(\Omega_k). \tag{53}$$

Let us proceed by induction. We prove

$$w \in H^{s-1,2(r-1)}(\Omega_k) \quad \Longrightarrow \quad w \in H^{s,2(r-s)}(\Omega_k), \quad 2 \leq s \leq r.$$

Hence, we compute

$$\partial^{s-1,2(r-s+1)}Lw = \partial^{s-1,2(r-s+1)}h,$$

from which we have

$$\begin{aligned}
L\partial^{s-1,2(r-s+1)}w &= \partial^{s-1,2(r-s+1)}h - \bigl[\partial^{s-1,2(r-s+1)}, L\bigr]w \\
&= \partial^{s-1,2(r-s+1)}f + \bigl(x_0 + \lambda - \alpha(x')\bigr)^2 \operatorname{Div}_{x''}\bigl(B(x'')D_{x''}\bigr)\partial^{s-1,2(r-s+1)}w
\end{aligned}$$

$$+\left[\left(x_0+\lambda-\alpha\left(x'\right)\right)^2\operatorname{Div}_{x''}\left(B\left(x''\right)D_{x''}\right),\partial^{s-1,2(r-s+1)}\right]w$$
$$-\left[\partial^{s-1,2(r-s+1)},L\right]w.$$

This implies

$$w\in H^{s,2(r-s+1)}(\Omega_k)\subseteq H^{s,2(r-s)}(\Omega_k).$$

Since $s\leq r-1$, as a result

$$w\in H^r(\Omega_k).$$

Therefore, we proved that if w is solution to the equation:

$$\left(w,{}^tP\varphi\right)=(f,\varphi),\quad\forall\varphi\in C_0^\infty(\overline{\Omega}_k),\tag{54}$$

then the distribution $w\in H^{r+1}(\overline{\Omega}_k)$ ($r\geq 2$). Integrating by part the left-hand side of (54), as a result, for every $\varphi\in C_0^\infty(\overline{\Omega}_k)$ with $\operatorname{supp}\varphi\subseteq\Omega_k$,

$$(Pw,\varphi)=(f,\varphi),$$

and that implies

$$Pw=f,\quad\text{a.e. in }\Omega_k.\tag{55}$$

Moreover, integrating by parts the left-hand side of (54), for every $\varphi\in C_0^\infty(\overline{\Omega}_k)$ with $\varphi(0,x',x'')=0$, we have

$$(Pw,\varphi)-\int_{\Omega_0}w\left(0,x',x''\right)\varphi_{x_0}\left(0,x',x''\right)dx'\,dx''=(f,\varphi),$$

and combining with (55), it follows that

$$w\left(0,x',x''\right)=0.$$

Finally, integrating by part the left-hand side of (54), for every $\varphi\in C_0^\infty(\Omega_k)$ with $\varphi_{x_0}(0,x',x'')=0$, we obtain

$$(Pw,\varphi)-\int_{\Omega_0}w_{x_0}\left(0,x',x''\right)\varphi\left(0,x',x''\right)dx'\,dx''=(f,\varphi),$$

and making use of (55), as a result

$$w_{x_0}\left(0,x',x''\right)=0.$$

Hence, we proved that $w\in H^r(\Omega_k)$ ($r\geq 2$) is a solution to the problem

$$\begin{cases}Pw=f,\quad\text{in }\Omega_k,\\ w(0,x',x'')=0,\qquad w_{x_0}(0,x',x'')=0,\end{cases}$$

for k small enough.

6 A global existence result

Let $\overline{x}_0 > 0$ and let $\Omega_{\overline{x}_0} = [\overline{x}_0, +\infty[\times\Omega_0$, by means of the change of variables $x_0 = y_0 + \overline{x}_0$, the problem

$$\begin{cases} Pw = f, \quad \text{in } \Omega_{\overline{x}_0}, \\ w(\overline{x}_0, x', x'') = 0, \qquad w_{x_0}(\overline{x}_0, x', x'') = 0, \end{cases}$$

becomes

$$\begin{cases} Pv = f, \quad \text{in } \Omega, \\ v(0, x', x'') = 0, \qquad v_{y_0}(0, x', x'') = 0, \end{cases}$$

where $v(y_0, x', x'') = v(x_0 - \overline{x}_0, x', x'') = w(x_0, x', x'')$.

According to the results of Section 5, for k small enough, there exists a solution $v \in H^r(\Omega_k)$, $r \geq 2$, verifying the problem

$$\begin{cases} Pv = f, \quad \text{in } \Omega_k, \\ v(0, x', x'') = 0, \qquad v_{y_0}(0, x', x'') = 0. \end{cases}$$

Hence, there exists a solution $w \in H^r(\Omega_{\overline{x}_0,k})$, where $\Omega_{\overline{x}_0,k} = [\overline{x}_0, \overline{x}_0 + k[\times\Omega_0$ verifying the problem

$$\begin{cases} Pw = f, \quad \text{in } \Omega_{\overline{x}_0,k}, \\ w(\overline{x}_0, x', x'') = 0, \qquad w_{x_0}(\overline{x}_0, x', x'') = 0. \end{cases} \tag{56}$$

Now, if $B(x'')$ is constant and $\Omega_0 = \Omega' \times \mathbb{R}^{n-m}$, k does not depend on s. Then we can proceed in the following way. From the existence of a solution $w \in H^r(\Omega_{\overline{x}_0,k})$ to problem (56), it follows that also the problem

$$\begin{cases} Pw = f, \quad \text{in } \Omega_{\overline{x}_0,k}, \\ w(\overline{x}_0, x', x'') = g_1(x', x''), \qquad w_{x_0}(\overline{x}_0, x', x'') = g_2(x', x''), \end{cases} \tag{57}$$

where $f \in C^\infty(\Omega)$, $g_1 \in C^\infty(\Omega_0)$, $g_2 \in C^\infty(\Omega_0)$, admits a solution $w \in C^\infty(\overline{\Omega_{\overline{x}_0,k}})$. In fact, let $h(x_0, x', x'')$ be a function belonging to $C^\infty(\Omega_{\overline{x}_0,k})$ such that $h(\overline{x}_0, x', x'') = g_1(x', x'')$ and $h_{x_0}(\overline{x}_0, x', x'') = g_2(x', x'')$, the solution to (57) is $w = h + \overline{w}$, where $\overline{w}$ is solution to

$$\begin{cases} P\overline{w} = f + Ph, \quad \text{in } \Omega_{\overline{x}_0,k}, \\ \overline{w}(\overline{x}_0, x', x'') = 0, \qquad \overline{w}_{x_0}(\overline{x}_0, x', x'') = 0. \end{cases}$$

Set $\Omega_h = [0, h[\times\Omega_0$, with $h > 0$, by means of compactness theorems and the arbitrariness of $\overline{x}_0$, we can decompose $\overline{\Omega}_k$ in the union of a finite number of compacts $\overline{\Omega}_i = [k_{i-1}, k_i] \times \Omega_0$, for $i = 1, \dots, p$, where $k_0 = 0$, and such that there exists a solution $w_i \in C^\infty(\Omega_i)$ to the problem

$$\begin{cases} Pw_i = f, \quad \text{in } \Omega_i, \\ w_i(k_i, x', x'') = w_{i-1}(k_i, x', x''), \qquad \partial_{x_0} w_i(k_i, x', x'') = \partial_{x_0} w_{i-1}(k_i, x', x''), \end{cases}$$

where $i = 1,\dots,p-1$, $w_0(0,x',x'') = 0$ and $\partial_{x_0} w_0(0,x',x'') = 0$. By construction, it follows that the function

$$w(x_0,x',x'') = \sum_{i=0}^{p} w_i(x_0,x',x'')\chi_i(x_0,x',x''),$$

where

$$\chi_i(x_0,x',x'') = \begin{cases} 1 & \text{in } [k_i,k_{i+1}] \times \Omega_0, \\ 0 & \text{otherwise}, \end{cases}$$

is a solution to the problem

$$\begin{cases} Pw = f, & \text{in } \Omega_h, \\ w(0,x',x'') = 0, & w_{x_0}(0,x',x'') = 0, \end{cases}$$

with $f \in C^\infty(\overline{\Omega})$ and $w \in C^\infty(\Omega_h)$. For the arbitrariness of h, we have proved that under assumptions (i), (ii), (iii), and (iv), if $\Omega_0 = \Omega' \times \mathbb{R}^{n-m}$ and $B(x'')$ is a constant, then the problem

$$\begin{cases} Pu = f, & \text{in } \Omega, \\ u(0,x',x'') = 0, & u_{x_0}(0,x',x'') = 0, \end{cases}$$

with $f \in C^\infty(\overline{\Omega})$, admits a solution $u \in C^\infty(\overline{\Omega})$.

If $B(x'')$ is not constant, since c depends on s in (37), we proceed as follows. For every $h > 0$ and for every $\overline{x}_0 \in [0,h[$, we set $\Omega_{\overline{x}_0,k} = [\overline{x}_0,\overline{x}_0 + k[\times \Omega_0$. By means of a change of variables $x_0 = y_0 + \overline{x}_0$, we show, as done before, (37) for every $u \in C_0^\infty(\overline{\Omega}_{\overline{x}_0,k})$ and k small enough. Then it is possible to divide Ω_h in a finite number of subsets $\Omega_0 = [0,k_0[\times \Omega_0$, $\Omega_1 = [k_1,k_2[\times \Omega_0, \dots, \Omega_p = [k_p,h[\times \Omega_0$, with $k_{i+1} < k_i < k_j$, for every $i = 0,\dots,p$, $k_{p+1} = h$ and $j \geq i+2$, such that (48) holds in every Ω_i, namely (37) holds for every $u \in C_0^\infty(\overline{\Omega}_i)$, $i = 0,\dots,p$. Now, for every $u \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} u \subseteq \Omega_h$, as a result

$$\begin{aligned} \|u\|_{H^{0,0,s}([k_i,k_{i+2}[\times\Omega_0)} &\leq \|u\psi\|_{H^{0,0,s}([k_i,k_{i+1}[\times\Omega_0)} \\ &\leq c\|Pu\psi\|_{H^{0,0,s}([k_i,k_{i+1}[\times\Omega_0)} \\ &\leq c\|Pu\|_{H^{0,0,s}(\Omega_h)} \\ &\quad + c\big(\|\partial_{x_0} u\|_{H^{0,0,s}([k_{i+2},k_{i+4}[\times\Omega_0)} + \|u\|_{H^{0,0,s}([k_{i+2},k_{i+4}[\times\Omega_0)}\big), \end{aligned} \tag{58}$$

where $\psi \in C^\infty([0,h[)$, $\psi = 1$ on $[k_i,k_{i+2}]$ and $\operatorname{supp}\psi \subseteq [k_i,k_{i+1}]$, for every i odd with $i = -1,\dots,p-2$, $k_{-1} = 0$ and $k_{p+2} > k_{p+1}$. By (58) it follows that

$$\|u\|_{H^{0,0,s}([k_i,k_{i+2}[\times\Omega_0)} \leq c\|Pu\|_{H^{0,0,s}(\Omega_h)},$$

from which, adding with respect to i, with i odd and $i = -1,\dots,p-2$, as a result

$$\|u\|_{H^{0,0,s}(\Omega_h)} \leq c\|Pu\|_{H^{0,0,s}(\Omega_h)}.$$

Making using of the previous inequality and proceeding as in Section 5, we see that there exists $w \in H^{0,0,s}(\Omega_h)$ such that

$$\bigl(w, {}^tPu\bigr) = (f, u), \quad \forall u \in C_0^\infty(\overline{\Omega}_h), \tag{59}$$

and $w \in H^r(\Omega_h) \cap H^{r,2(r-s)}(\Omega_h)$, with $0 \le s \le r$, $r \ge 2$. Integrating by parts in (59) (see Section 5), for the arbitrariness of h, we can prove that for every $h > 0$ the problem

$$\begin{cases} Pw = f, & \text{in } \Omega_h, \\ w(0, x', x'') = 0, & w_{x_0}(0, x', x'') = 0, \end{cases}$$

with $f \in H^{s,2(r-s)}(\Omega)$, for every $0 \le s \le r$, $r \le 2$, admits a solution $w \in H^r(\Omega_h) \cap H^{r,2(r-s)}(\Omega_h)$, with $0 \le s \le r$, $r \ge 2$.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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