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# Convergence of splitting algorithms for the sum of two accretive operators with applications

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## Abstract

We study a splitting algorithm for problems involving the sum of two accretive operators. We prove the strong convergence of the algorithm. Applications to variational inequality, fixed point, equilibrium, and minimization problems are provided.

**Keywords:** accretive operator; fixed point; nonexpansive mapping; resolvent; zero point

## 1 Introduction

Many important real world problems have reformulations which require finding zero points of some nonlinear operators, for instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and optimization problems; see [1–11] and the references therein. It is well known that minimizing a convex function  $f$  can be reduced to finding zero points of the subdifferential mapping  $\partial f$ . Forward-backward splitting algorithms were proposed by Lions and Mercier [12], by Passty [13], and, in a dual form for convex programming, by Han and Lou [14]. The algorithms, which provide a range of approaches to solving large-scale optimization problems and variational inequalities, have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation, and this operator is decomposed as the sum of two nonlinear operators. This paper concerns a forward-backward splitting algorithm with computational errors designed to find zeros of the sum of two accretive operators  $A$  and  $B$ .

The paper is organized in the following way. In Section 2, we present the preliminaries that are needed in our work. In Section 3, we present a splitting algorithm for finding zeros of the sum of two accretive operators  $A$  and  $B$ . Convergence analysis of the algorithms is investigated. In Section 4, applications of our main results are provided.

## 2 Preliminaries

Let  $E$  be a real Banach space with the dual  $E^*$ . Given a continuous strictly increasing function  $\varphi : R^+ \rightarrow R^+$ , where  $R^+$  denotes the set of nonnegative real numbers, such that  $\varphi(0) = 0$  and  $\lim_{r \rightarrow \infty} \varphi(r) = \infty$ , we associate with it a (possibly multivalued) generalized

duality map  $\mathfrak{J}_\varphi : E \rightarrow 2^{E^*}$ , defined as  $\mathfrak{J}_\varphi(x) = \{x^* \in E^* : x^*(x) = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\}$ ,  $\forall x \in E$ . In this paper, we use the generalized duality map associated with the gauge function  $\varphi(t) = t^{q-1}$  for  $q > 1$ . Let  $\rho_E : [0, \infty) \rightarrow [0, \infty)$  be the modulus of smoothness of  $E$  by  $\rho_E(t) = \sup\{\frac{\|x+y\| - \|x-y\|}{2} - 1, x \in U_E, \|y\| \leq t\}$ . A Banach space  $E$  is said to be uniformly smooth if  $\frac{\rho_E(t)}{t} \rightarrow 0$  as  $t \rightarrow 0$ . Let  $q > 1$ .  $E$  is said to be  $q$ -uniformly smooth if there exists a fixed constant  $c > 0$  such that  $\rho_E(t) \leq ct^q$ . The modulus of convexity of  $E$  is the function  $\delta_E(\epsilon) : (0, 2] \rightarrow [0, 1]$  defined by  $\delta_E(\epsilon) = \inf\{1 - \frac{\|x+t\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}$ . Recall that  $E$  is said to be uniformly convex if  $\delta_E(\epsilon) > 0$  for any  $\epsilon \in (0, 2]$ . Let  $E$  be a smooth Banach space, and let  $C$  be a nonempty subset of  $E$ . Let  $\text{Proj}_C : E \rightarrow C$  be a retraction and  $\mathfrak{J}$  be the normalized duality mapping on  $E$ . Then the following are equivalent [15]: (1)  $\text{Proj}_C$  is sunny and nonexpansive; (2)  $\langle x - \text{Proj}_C x, \mathfrak{J}(y - \text{Proj}_C x) \rangle \leq 0, \forall x \in E, y \in C$ .

Let  $I$  denote the identity operator on  $E$ . An operator  $A \subset E \times E$  with domain  $D(A) = \{z \in E : Az \neq \emptyset\}$  and range  $R(A) = \bigcup\{Az : z \in D(A)\}$  is said to be accretive iff, for  $t > 0$  and  $x, y \in D(A)$ ,  $\|x - y\| \leq \|x - y + t(u - v)\|, \forall u \in Ax, v \in Ay$ . It follows from Kato [16] that  $A$  is accretive iff, for  $x, y \in D(A)$ , there exists  $j_q(x_1 - x_2)$  such that  $\langle u - v, j_q(x - y) \rangle \geq 0$ . An accretive operator  $A$  is said to be  $m$ -accretive iff  $R(I + rA) = E$  for all  $r > 0$ . In this paper, we use  $A^{-1}(0)$  to denote the set of zeros of  $A$ . For an accretive operator  $A$ , we can define a nonexpansive single-valued mapping  $J_r : R(I + rA) \rightarrow D(A)$  by  $J_r = (I + rA)^{-1}$  for each  $r > 0$ , which is called the resolvent of  $A$ . Recall that a single-valued operator  $A : C \rightarrow E$  is said to be  $\alpha$ -inverse strongly accretive if there exists a constant  $\alpha > 0$  and some  $j_q(x - y) \in \mathfrak{J}_q(x - y)$  such that  $\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|Ax - Ay\|^q, \forall x, y \in C$ .

Let  $T : C \rightarrow C$  be a mapping. Recall that  $T$  is said to be  $\kappa$ -contractive iff there exists a constant  $\kappa \in (0, 1)$  such that  $\|Tx - Ty\| \leq \kappa \|x - y\|, \forall x, y \in C$ .  $T$  is said to be nonexpansive iff  $\kappa = 1$ .  $T$  is said to be  $\kappa$ -strictly pseudocontractive iff there exists a constant  $\kappa \in (0, 1)$  such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q - \kappa \|(x - Tx) - (y - Ty)\|^q, \quad \forall x, y \in C$$

for some  $j_q(x - y) \in \mathfrak{J}_q(x - y)$ .  $T$  is said to be pseudocontractive iff  $\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q, \forall x, y \in C$  for some  $j_q(x - y) \in \mathfrak{J}_q(x - y)$ .

In order to obtain our main results, we also need the following lemmas.

**Lemma 2.1** [17] *Let  $E$  be a real  $q$ -uniformly smooth Banach space. Then the following inequality holds:  $\|x + y\|^q \leq \|x\|^q + q\langle y, \mathfrak{J}_q(x + y) \rangle$  and  $\|x + y\|^q \leq \|x\|^q + q\langle y, \mathfrak{J}_q(x) \rangle + K_q \|y\|^q, \forall x, y \in E$ , where  $K_q$  is some fixed positive constant.*

**Lemma 2.2** [18] *Let  $E$  be a real Banach space, and let  $C$  be a nonempty closed and convex subset of  $E$ . Let  $A : C \rightarrow E$  be a single-valued operator, and let  $B : E \rightarrow 2^E$  be an  $m$ -accretive operator. Then  $F(J_a(I - aA)) = (A + B)^{-1}(0)$ , where  $J_a(I - aA)$  is the resolvent of  $B$  for  $a > 0$ .*

**Lemma 2.3** [19] *Let  $E$  be a real Banach space, and  $A$  be an  $m$ -accretive operator. For  $\lambda > 0, \mu > 0$ , and  $x \in E$ , we have  $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)$ , where  $J_\lambda = (I + \lambda A)^{-1}$  and  $J_\mu = (I + \mu A)^{-1}$ .*

**Lemma 2.4** [20] *Let  $\{a_n\}$  be a sequence of nonnegative real numbers such that  $a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$ , where  $\{c_n\}$  is a sequence of nonnegative real numbers,  $\{t_n\} \subset (0, 1)$  and  $\{b_n\}$  is a number sequence. Assume that  $\sum_{n=0}^{\infty} t_n = \infty, \limsup_{n \rightarrow \infty} \frac{b_n}{t_n} \leq 0$ , and  $\sum_{n=0}^{\infty} c_n < \infty$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

**Lemma 2.5** [21] *Let  $q > 1$ . Then the following inequality holds:  $ab \leq \frac{a^q}{q} + \frac{q-1}{q}b^{\frac{q}{q-1}}$ , for arbitrary positive real numbers  $a$  and  $b$ .*

**Lemma 2.6** [22] *Let  $C$  be a nonempty closed convex subset of a real uniformly smooth Banach space  $E$ . Let  $f : C \rightarrow C$  be a contractive mapping, and let  $T : C \rightarrow C$  be a non-expansive mapping. For each  $t \in (0, 1)$ , let  $x_t$  be the unique solution of the equation  $x = tf(x) + (1-t)Tx$ . Then  $\{x_t\}$  converges strongly to a fixed point  $\bar{x} = Q_{F(T)}f(\bar{x})$ .*

### 3 Main results

**Theorem 3.1** *Let  $E$  be a real  $q$ -uniformly smooth Banach space with the constant  $K_q$ . Let  $B : E \rightarrow 2^E$  be an  $m$ -accretive operator such that  $\overline{D(B)}$  is convex. Let  $A : \overline{D(B)} \rightarrow E$  be an  $\alpha$ -inverse strongly accretive operator. Assume that  $(A + B)^{-1}(0) \neq \emptyset$ . Let  $f : \overline{D(B)} \rightarrow \overline{D(B)}$  be a fixed  $\kappa$ -contraction. Let  $\{r_n\}$ ,  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  be positive real number sequences, where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  are in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n J_{r_n}(x_n - r_n A x_n + e_n) + \gamma_n f_n, \quad \forall n \geq 0,$$

where  $J_{r_n} = (I + r_n B)^{-1}$ ,  $\{e_n\}$  is a sequence in  $E$ , and  $\{f_n\}$  is a bounded sequence in  $\overline{D(B)}$ . Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{e_n\}$ , and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (3)  $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ ;
- (4)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $r_n \leq (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ ;
- (5)  $\sum_{n=0}^{\infty} \|e_n\| < \infty$ ,  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x = \text{Proj}_{(A+B)^{-1}(0)} f(x)$ , where  $\text{Proj}_{(A+B)^{-1}(0)}$  is the unique sunny nonexpansive retraction of  $C$  onto  $(A + B)^{-1}(0)$ .

*Proof* First, we show that  $\{x_n\}$  is bounded. In view of Lemma 2.1, we find that

$$\begin{aligned} & \| (I - r_n A)x - (I - r_n A)y \|^q \\ & \leq \|x - y\|^q - q r_n \langle Ax - Ay, \mathfrak{J}_q(x - y) \rangle + K_q r_n^q \|Ax - Ay\|^q \\ & \leq \|x - y\|^q - q r_n \alpha \|Ax - Ay\|^q + K_q r_n^q \|Ax - Ay\|^q \\ & = \|x - y\|^q - (\alpha q - K_q r_n^{q-1}) r_n \|Ax - Ay\|^q. \end{aligned}$$

From restriction (4), we find that  $I - r_n A$  is nonexpansive. Fixing  $p \in (A + B)^{-1}(0)$ , we find from Lemma 2.2 that  $p = J_{r_n}(x_n - r_n A x_n)p$ . It follows from restriction (5) that

$$\begin{aligned} & \|x_{n+1} - p\| \\ & \leq \alpha_n \|f(x_n) - p\| + \beta_n \|J_{r_n}(x_n - r_n A x_n + e_n) - p\| + \gamma_n \|f_n - p\| \\ & \leq \alpha_n \kappa \|x_n - p\| + \alpha_n \|f(p) - p\| \\ & \quad + \beta_n \|(x_n - r_n A x_n + e_n) - (I - r_n A)p\| + \gamma_n \|f_n - p\| \\ & \leq (1 - \alpha_n(1 - \kappa)) \|x_n - p\| + \alpha_n \|f(p) - p\| + \|e_n\| + \gamma_n \|f_n - p\| \end{aligned}$$

$$\begin{aligned}
 &\leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - \kappa} \right\} + \|e_n\| + \gamma_n \|f_n - p\| \\
 &\leq \max \left\{ \|x_{n-1} - p\|, \frac{\|f(p) - p\|}{1 - \kappa} \right\} + \|e_{n-1}\| + \|e_n\| \\
 &\quad + \gamma_{n-1} \|f_{n-1} - p\| + \gamma_n \|f_n - p\| \\
 &\quad \vdots \\
 &\leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \kappa} \right\} + \sum_{i=0}^n \|e_i\| + \sum_{i=0}^n \gamma_i M \\
 &\leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \kappa} \right\} + \sum_{i=0}^{\infty} \|e_i\| + \sum_{i=0}^{\infty} \gamma_i M < \infty,
 \end{aligned}$$

where  $M = \sup_{n \geq 0} \{\|f_n - p\|\}$ . This shows that  $\{x_n\}$  is bounded. Set  $y_n = (I - r_n A)x_n + e_n$ . It follows that

$$\|y_{n-1} - y_n\| \leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \|Ax_{n-1}\| + \|e_n\| + \|e_{n-1}\|.$$

Using Lemma 2.3, we find that

$$\begin{aligned}
 &\|J_{r_{n-1}} y_{n-1} - J_{r_n} y_n\| \\
 &= \left\| J_{r_{n-1}} \left( \frac{r_{n-1}}{r_n} y_n + \left( 1 - \frac{r_{n-1}}{r_n} \right) J_{r_n} y_n \right) - J_{r_{n-1}} y_{n-1} \right\| \\
 &\leq \left\| \frac{r_{n-1}}{r_n} (y_n - y_{n-1}) + \left( 1 - \frac{r_{n-1}}{r_n} \right) (J_{r_n} y_n - y_{n-1}) \right\| \\
 &\leq \|y_n - y_{n-1}\| + \frac{|r_n - r_{n-1}|}{r_n} \|J_{r_n} y_n - y_n\| \\
 &\leq \|x_n - x_{n-1}\| + |r_n - r_{n-1}| \left( \|Ax_{n-1}\| + \frac{\|J_{r_n} y_n - y_n\|}{r_n} \right) \\
 &\quad + \|e_n\| + \|e_{n-1}\|. \tag{3.1}
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq \alpha_n \kappa \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
 &\quad + \gamma_n \|f_n - f_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|f_{n-1}\| \\
 &\quad + \beta_n \|J_{r_{n-1}} y_{n-1} - J_{r_n} y_n\| + |\beta_n - \beta_{n-1}| \|J_{r_{n-1}} y_{n-1}\|. \tag{3.2}
 \end{aligned}$$

Using (3.1) and (3.2), we find

$$\begin{aligned}
 \|x_{n+1} - x_n\| &\leq (1 - \alpha_n(1 - \kappa)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
 &\quad + \gamma_n \|f_n - f_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|f_{n-1}\| \\
 &\quad + |r_n - r_{n-1}| \left( \|Ax_{n-1}\| + \frac{\|J_{r_n} y_n - y_n\|}{r_n} \right) \\
 &\quad + \|e_n\| + \|e_{n-1}\| + |\beta_n - \beta_{n-1}| \|J_{r_{n-1}} y_{n-1}\|.
 \end{aligned}$$

Using restrictions (3), (4), and (5), we obtain from Lemma 2.4 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.3)$$

On the other hand, we have

$$\|J_{r_n} y_n - x_n\| \leq \frac{\|x_{n+1} - x_n\|}{\beta_n} + \frac{\alpha_n}{\beta_n} \|x_n - f(x_n)\| + \frac{\gamma_n}{\beta_n} \|x_n - f_n\|.$$

Using restrictions (2) and (5), we obtain from (3.3) that

$$\lim_{n \rightarrow \infty} \|J_{r_n} y_n - x_n\| = 0. \quad (3.4)$$

Since

$$\begin{aligned} \|J_{r_n}(x_n - r_n A x_n) - x_n\| &\leq \|J_{r_n}(x_n - r_n A x_n) - J_{r_n} y_n\| + \|J_{r_n} y_n - x_n\| \\ &\leq \|e_n\| + \|J_{r_n} y_n - x_n\|, \end{aligned}$$

we find from (3.4) and restriction (5) that

$$\lim_{n \rightarrow \infty} \|J_{r_n}(x_n - r_n A x_n) - x_n\| = 0. \quad (3.5)$$

Without loss of generality, let us assume that there exists a real number  $r$  such that  $r_n \geq r > 0$ . Since  $B$  is accretive, we have

$$\left\langle \frac{x_n - J_r(I - rA)x_n}{r} - \frac{x_n - J_{r_n}(I - r_n A)x_n}{r_n}, \mathfrak{J}_q(J_r(I - rA)x_n - J_{r_n}(I - r_n A)x_n) \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} &\|J_r(I - rA)x_n - J_{r_n}(I - r_n A)x_n\|^q \\ &\leq \frac{r_n - r}{r_n} \langle x_n - J_{r_n}(I - r_n A)x_n, \mathfrak{J}_q(J_r(I - rA)x_n - J_{r_n}(I - r_n A)x_n) \rangle \\ &\leq \|x_n - J_{r_n}(I - r_n A)x_n\| \|J_r(I - rA)x_n - J_{r_n}(I - r_n A)x_n\|^{q-1}. \end{aligned}$$

This implies from (3.5) that

$$\lim_{n \rightarrow \infty} \|J_r(x_n - rA x_n) - x_n\| = 0. \quad (3.6)$$

Since  $J_r(I - rA)$  is nonexpansive and  $f$  is contractive, we find that the mapping  $tf + (1 - t)J_r(I - rA)$  is contractive. Let  $x_t$  be the unique fixed point of the mapping  $tf + (1 - t)J_r(I - rA)$ , that is,  $x_t = tf(x_t) + (1 - t)J_r(I - rA)x_t$ ,  $\forall t \in (0, 1)$ . Setting  $x = \lim_{t \rightarrow 0} x_t$ , we have  $x = \text{Proj}_{(A+B)^{-1}(0)} f(x)$ , where  $\text{Proj}_{(A+B)^{-1}(0)}$  is the unique sunny nonexpansive retraction from  $C$  onto  $(A + B)^{-1}(0)$ .

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle f(x) - x, \mathfrak{J}_q(x_n - x) \rangle \leq 0.$$

Since

$$\begin{aligned} \|x_t - x_n\|^q &\leq t \langle f(x_t) - x_n, \mathfrak{J}_q(x_t - x_n) \rangle \\ &\quad + (1-t) \langle J_r(I-rA)x_t - x_n, \mathfrak{J}_q(x_t - x_n) \rangle \\ &\leq t \langle f(x_t) - x_t, \mathfrak{J}_q(x_t - x_n) \rangle + t \langle x_t - x_n, \mathfrak{J}_q(x_t - x_n) \rangle \\ &\quad + (1-t) \langle J_r(I-rA)x_t - J_r(I-rA)x_n, \mathfrak{J}_q(x_t - x_n) \rangle \\ &\quad + (1-t) \langle J_r(I-rA)x_n - x_n, \mathfrak{J}_q(x_t - x_n) \rangle \\ &\leq t \langle f(x_t) - x_t, \mathfrak{J}_q(x_t - x_n) \rangle + \|x_t - x_n\|^q \\ &\quad + \|J_r(I-rA)x_n - x_n\| \|x_t - x_n\|^{q-1}, \end{aligned}$$

we have

$$\langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t) \rangle \leq \frac{1}{t} \|J_r(I-rA)x_n - x_n\| \|x_t - x_n\|^{q-1}.$$

Fix  $t$  and let  $n \rightarrow \infty$ . It follows from (3.6) that

$$\limsup_{n \rightarrow \infty} \langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t) \rangle \leq 0. \quad (3.7)$$

Since the duality map  $\mathfrak{J}_q$  is single-valued and strong-weak\* uniformly continuous on bounded sets of a Banach space  $E$  with a uniformly Gâteaux differentiable norm, one has

$$\begin{aligned} &|\langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t) \rangle - \langle f(x) - x, \mathfrak{J}_q(x_n - x) \rangle| \\ &= |\langle f(x) - x, \mathfrak{J}_q(x_n - x) - \mathfrak{J}_q(x_n - x_t) \rangle \\ &\quad + \langle f(x) - x - (f(x_t) - x_t), \mathfrak{J}_q(x_n - x_t) \rangle| \\ &\leq |\langle f(x) - x, \mathfrak{J}_q(x_n - x) - \mathfrak{J}_q(x_n - x_t) \rangle| \\ &\quad + \|f(x) - x - (f(x_t) - x_t)\| \|x_n - x_t\|^{q-1}. \end{aligned}$$

Hence,  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  such that  $t \in (0, \delta)$ , one has

$$\langle f(x) - x, \mathfrak{J}_q(x_n - x) \rangle \leq \langle f(x_t) - x_t, \mathfrak{J}_q(x_n - x_t) \rangle + \epsilon.$$

Using (3.7), we see that

$$\limsup_{n \rightarrow \infty} \langle f(x) - x, \mathfrak{J}_q(x_n - \bar{x}) \rangle \leq 0. \quad (3.8)$$

Using Lemma 2.5, one has

$$\begin{aligned} \|x_{n+1} - x\|^q &\leq \alpha_n \langle f(x_n) - f(x), \mathfrak{J}_q(x_{n+1} - x) \rangle + \alpha_n \langle f(x) - x, \mathfrak{J}_q(x_{n+1} - \bar{x}) \rangle \\ &\quad + \beta_n \|J_n(x_n - r_n A x_n + e_n) - x\| \|x_{n+1} - x\|^{q-1} \\ &\quad + \gamma_n \|f_n - x\| \|x_{n+1} - x\|^{q-1} \\ &\leq (1 - \alpha_n(1 - \kappa)) \|x_n - x\| \|x_{n+1} - x\|^{q-1} \end{aligned}$$

$$\begin{aligned}
& + \alpha_n \langle f(x) - x, \mathfrak{J}_q(x_{n+1} - x) \rangle + \|e_n\| \|x_{n+1} - x\|^{q-1} \\
& + \gamma_n \|f_n - x\| \|x_{n+1} - x\|^{q-1} \\
\leq & (1 - \alpha_n(1 - \kappa)) \left( \frac{1}{q} \|x_n - x\|^q + \frac{q-1}{q} \|x_{n+1} - x\|^q \right) \\
& + \alpha_n \langle f(x) - x, \mathfrak{J}_q(x_{n+1} - x) \rangle + \|e_n\| \|x_{n+1} - x\|^{q-1} \\
& + \gamma_n \|f_n - x\| \|x_{n+1} - x\|^{q-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x\|^q \leq & (1 - \alpha_n(1 - \kappa)) \|x_n - x\|^q \\
& + q \alpha_n \langle f(x) - x, \mathfrak{J}_q(x_{n+1} - x) \rangle + q \|e_n\| \|x_{n+1} - x\|^{q-1} \\
& + q \gamma_n \|f_n - x\| \|x_{n+1} - x\|^{q-1}.
\end{aligned}$$

Using restrictions (2) and (5), we see from (3.8) that  $\{x_n\}$  converges strongly to  $x$ . This completes the proof.  $\square$

**Remark 3.2** The framework of the space in Theorem 3.1 can be applicable to  $L_p$ , where  $p > 1$ .

**Corollary 3.3** Let  $E$  be a real  $q$ -uniformly smooth Banach space with the constant  $K_q$ . Let  $B : E \rightarrow 2^E$  be an  $m$ -accretive operator such that  $\overline{D(B)}$  is convex. Assume that  $B^{-1}(0) \neq \emptyset$ . Let  $f : \overline{D(B)} \rightarrow \overline{D(B)}$  be a fixed  $\kappa$ -contraction. Let  $\{r_n\}$  and  $\{\alpha_n\}$  be positive real number sequences, where  $\{\alpha_n\}$  is in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad \forall n \geq 0,$$

where  $J_{r_n} = (I + r_n B)^{-1}$ . Assume that the sequences  $\{\alpha_n\}$  and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (2)  $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ ;
- (3)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x = \text{Proj}_{B^{-1}(0)} f(x)$ , where  $\text{Proj}_{B^{-1}(0)}$  is the unique sunny nonexpansive retraction of  $C$  onto  $B^{-1}(0)$ .

**Corollary 3.4** Let  $E$  be a real  $q$ -uniformly smooth Banach space with the constant  $K_q$ , and let  $C$  be a nonempty closed and convex subset of  $E$ . Let  $f : C \rightarrow C$  be a fixed  $\kappa$ -contraction. Let  $T : C \rightarrow C$  be an  $\alpha$ -strictly pseudocontractive mapping with a nonempty fixed point set. Let  $\{r_n\}$ ,  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  be positive number sequences, where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n (1 - r_n) x_n + r_n \beta_n T x_n + \gamma_n f_n, \quad \forall n \geq 0,$$

where  $\{f_n\}$  is a bounded sequence in  $C$ . Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ , and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (3)  $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ ;
- (4)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $r_n \leq (\frac{q\alpha}{Kq})^{\frac{1}{q-1}}$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ ;
- (5)  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x = \text{Proj}_{F(T)} f(x)$ , where  $\text{Proj}_{F(T)}$  is the unique sunny nonexpansive retraction of  $C$  onto  $F(T)$ .

*Proof* Putting  $A = I - T$ , we find that  $A$  is  $\alpha$ -inverse strongly accretive and  $F(T) = A^{-1}(0)$ . Notice that

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + \beta_n(1 - r_n)x_n + r_n \beta_n T x_n + \gamma_n f_n \\ &= \alpha_n f(x_n) + \beta_n((1 - r_n)x_n + r_n T x_n) + \gamma_n f_n \\ &= \alpha_n f(x_n) + \beta_n(x_n - r_n(I - T)x_n) + \gamma_n f_n \\ &= \alpha_n f(x_n) + \beta_n(x_n - r_n A x_n) + \gamma_n f_n. \end{aligned}$$

Using Theorem 3.1, we find the desired conclusion immediately.  $\square$

#### 4 Applications

In this section, we give some applications of our main results in the framework of Hilbert spaces.

From now on, we always assume that  $C$  is a nonempty closed and convex subset of a real Hilbert space  $H$  and  $P_C$  stands for the metric projection from  $H$  onto  $C$ . Let  $A : C \rightarrow H$  be a monotone operator. Recall that the classical variational inequality is to find  $x \in C$  such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (4.1)$$

The solution set of the variational inequality is denoted by  $\text{VI}(C, A)$ .

Let  $i_C$  be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

It is easy to see that  $i_C$  is a proper lower and semicontinuous convex function on  $H$ , and the subdifferential  $\partial i_C$  of  $i_C$  is maximal monotone. Define the resolvent  $J_r := (I + r\partial i_C)^{-1}$  of the subdifferential operator  $\partial i_C$ . Letting  $x = J_r y$ , we find that

$$\begin{aligned} y \in x + r\partial i_C x &\iff y \in x + rN_C x \\ &\iff \langle y - x, v - x \rangle \leq 0, \quad \forall v \in C \\ &\iff x = P_C y, \end{aligned}$$

where  $N_C x := \{e \in H : \langle e, v - x \rangle \leq 0, \forall v \in C\}$ .

Putting  $B = \partial i_C$  in Theorem 3.1, we find that  $J_{r_n} = P_C$ . Hence, the following result can be obtained immediately.

**Theorem 4.1** Let  $f : C \rightarrow C$  be a fixed  $\kappa$ -contraction. Let  $A : C \rightarrow E$  be an  $\alpha$ -inverse strongly monotone operator with  $\text{VI}(C, A) \neq \emptyset$ . Let  $\{r_n\}$  be a positive number sequence. Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(x_n - r_n A x_n + e_n) + \gamma_n f_n, \quad \forall n \geq 0,$$

where  $\{e_n\}$  is a sequence in  $H$  and  $\{f_n\}$  is a bounded sequence in  $C$ . Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{e_n\}$ , and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (3)  $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ ;
- (4)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $r_n \leq 2\alpha$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ ;
- (5)  $\sum_{n=0}^{\infty} \|e_n\| < \infty$ ,  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to  $x = P_{\text{VI}(C, A)} f(x)$ , where  $P_{\text{VI}(C, A)}$  is the unique metric projection from  $C$  onto  $\text{VI}(C, A)$ .

Next, we consider the problem of finding a solution of a Ky Fan inequality [23], which is known as an equilibrium problem in the terminology of Blum and Oettli; see [24] and the references therein.

Let  $F$  be a bifunction of  $C \times C$  into  $\mathbb{R}$ , where  $\mathbb{R}$  denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (4.2)$$

The solution set of the problem is denoted by  $\text{EP}(F)$  in this section.

To study the equilibrium problem (4.2), we may assume that  $F$  satisfies the following restrictions:

- (A1)  $F(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $F$  is monotone, i.e.,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A3) for each  $x, y, z \in C$ ,  $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ ;
- (A4) for each  $x \in C$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous.

The following lemma can be found in [24].

**Lemma 4.2** Let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4). Then, for any  $r > 0$  and  $x \in H$ , there exists  $z \in C$  such that  $F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0$ ,  $\forall y \in C$ . Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (4.3)$$

for all  $r > 0$  and  $x \in H$ . Then (1)  $T_r$  is single-valued and firmly nonexpansive; (2)  $F(T_r) = \text{EP}(F)$  is closed and convex.

**Lemma 4.3** Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  which satisfies (A1)-(A4), and let  $A_F$  be a multivalued mapping of  $H$  into itself defined by

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \quad (4.4)$$

Then  $A_F$  is a maximal monotone operator with the domain  $D(A_F) \subset C$ ,  $EP(F) = A_F^{-1}(0)$ , and  $T_r x = (I + rA_F)^{-1}x$ ,  $\forall x \in H$ ,  $r > 0$ , where  $T_r$  is defined as in (4.3).

Based on Lemmas 4.2 and 4.3, we find from Theorem 3.1 the following immediately.

**Theorem 4.4** Let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4) such that  $EP(F)$  is not empty. Let  $f : C \rightarrow C$  be a fixed  $\kappa$ -contraction. Let  $\{r_n\}$  be a positive number sequence. Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n T_{r_n}(x_n - r_n A x_n + e_n) + \gamma_n f_n, \quad \forall n \geq 0,$$

where  $J_{r_n} = (I + r_n A_F)^{-1}$ ,  $\{e_n\}$  is a sequence in  $H$  and  $\{f_n\}$  is a bounded sequence in  $C$ . Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{e_n\}$ , and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (3)  $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ ;
- (4)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $r_n \leq 2\alpha$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ ;
- (5)  $\sum_{n=0}^{\infty} \|e_n\| < \infty$ ,  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to some point in  $EP(F)$ .

For any matrix  $D \in \mathfrak{R}^{m \times n}$ , we denote its transpose by  $D^T$  and its operator norm by  $\|D\| = \max_{x \in \mathfrak{R}^n: \|x\|=1} \|Dx\|$ .

Consider the inclusion problem [10]  $0 \in A(x_1, x_2, y) + B(x_1, x_2, y)$ , where  $A(x_1, x_2, y) = (D^T y, E^T y, -Dx_1 - Ex_2)$ ,  $B(x_1, x_2, y) = T_1 x_1 \times T_2 x_2 \times \{b\}$  and  $T_1$  and  $T_2$  are maximal monotone mappings on  $\mathfrak{R}^{n_1}$  and  $\mathfrak{R}^{n_2}$ , respectively, and  $D \in \mathfrak{R}^{m \times n_1}$ ,  $E \in \mathfrak{R}^{m \times n_2}$ ,  $b \in \mathfrak{R}^m$ . Then,  $A$  and  $B$  are maximal monotone and  $A$  is Lipschitz continuous on  $\mathfrak{R}^{m+n_1+n_2}$  with the constant

$$\eta = \sqrt{\|D^T\|^2 + \|E^T\|^2 + \|D\|^2 + \|E\|^2}.$$

The special case where  $T_1 = \partial f_1$ ,  $T_2 = \partial f_2$  yields the following convex program:

$$\begin{cases} \text{minimize } f_1(x_1) + f_2(x_2) \\ \text{subject to } Dx_1 + Ex_2 = b, \end{cases}$$

where  $f_1$  and  $f_2$  are closed proper convex functions on, respectively,  $\mathfrak{R}^{n_1}$  and  $\mathfrak{R}^{n_2}$ . The special case where  $n_1 = n_2$ ,  $D = -E = I$  and  $b = 0$  yields the inclusion  $0 \in T_1 x + T_2 x$ .

Finally, we consider finding minimizers of proper lower semicontinuous convex functions.

For a proper lower semicontinuous convex function  $h : H \rightarrow (-\infty, \infty]$ , the subdifferential mapping  $\partial h$  of  $h$  is defined by

$$\partial h(x) = \{x^* \in H : h(x) + \langle y - x, x^* \rangle \leq h(y), \forall y \in H\}, \quad \forall x \in H.$$

Rockafellar [25] proved that  $\partial h$  is a maximal monotone operator. It is easy to verify that  $0 \in \partial h(v)$  if and only if  $h(v) = \min_{x \in H} h(x)$ .

**Theorem 4.5** Let  $f : C \rightarrow C$  be a fixed  $\kappa$ -contraction. Let  $h : H \rightarrow (-\infty, +\infty]$  be a proper convex lower semicontinuous function such that  $(\partial h)^{-1}(0)$  is not empty. Let  $\{r_n\}$  be a positive number sequence. Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$ . Let  $\{x_n\}$  be a sequence generated in the following iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n \arg \min_{z \in H} \left\{ h(z) + \frac{\|z - x_n - e_n\|^2}{2r_n} \right\} + \gamma_n f_n, \quad \forall n \geq 0,$$

where  $\{e_n\}$  is a sequence in  $H$  and  $\{f_n\}$  is a bounded sequence in  $C$ . Assume that the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{e_n\}$ , and  $\{r_n\}$  satisfy the following restrictions:

- (1)  $\alpha_n + \beta_n + \gamma_n = 1$ ;
- (2)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (3)  $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ ;
- (4)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $r_n \leq 2\alpha$ ,  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ ;
- (5)  $\sum_{n=0}^{\infty} \|e_n\| < \infty$ ,  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

Then the sequence  $\{x_n\}$  converges strongly to some minimizer of  $h$ .

*Proof* Since  $h : H \rightarrow (-\infty, \infty]$  is a proper convex and lower semicontinuous function, we see that the subdifferential  $\partial h$  of  $h$  is maximal monotone. Putting  $A = 0$  and  $y_n = J_{r_n}(x_n + e_n)$ , we see that

$$y_n = \arg \min_{z \in H} \left\{ h(z) + \frac{\|z - x_n - e_n\|^2}{2r_n} \right\}$$

is equivalent to

$$0 \in \partial h(y_n) + \frac{1}{r_n}(y_n - x_n - e_n).$$

It follows that

$$x_n + e_n \in y_n + r_n \partial h(y_n).$$

By using Theorem 3.1, we draw the desired conclusion immediately.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

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#### Acknowledgements

The authors are grateful to the anonymous referees for useful suggestions, which improved the contents of the article.

Received: 15 March 2014 Accepted: 25 June 2014 Published: 15 August 2014

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doi:10.1186/1687-1812-2014-166

**Cite this article as:** Qin et al.: Convergence of splitting algorithms for the sum of two accretive operators with applications. *Fixed Point Theory and Applications* 2014 **2014**:166.

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