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Nonoscillation for higher-order nonlinear delay dynamic equations on time scales

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Abstract

In this paper, we investigate the nonoscillation of the higher-order nonlinear delay dynamic equation

$$(a_{n-1}(t)(a_{n-2}(t)(\cdots(a_1(t)x^\Delta(t))^\Delta\cdots)^\Delta)^\Delta)^\Delta+u(t)g(x(\delta(t)))=R(t)$$
$$\text{for } t\in[t_0,\infty)_\mathbb{T},$$

where $\mathbb{T}$ is a scale with $\sup\mathbb{T}=\infty$, $t_0\in\mathbb{T}$, and $[t_0,\infty)_\mathbb{T}=\{t\in\mathbb{T}:t\geq t_0\}$. We obtain some sufficient conditions for all solutions of this equation to be nonoscillatory.

MSC: 34K11; 39A10; 39A99

Keywords: nonoscillation; dynamic equation; time scale

1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus, the set $\mathbb{R}$ of all real numbers, the set $\mathbb{N}$ of all natural numbers, and the set $\mathbb{Z}$ of all integers are examples of time scales. On a time scale $\mathbb{T}$, the forward jump operator, the backward jump operator, and the graininess function are defined as

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},\qquad\rho(t)=\sup\{s\in\mathbb{T}:s<t\},\quad\text{and}\quad\mu(t)=\sigma(t)-t,$$

respectively.

In this paper, we investigate the nonoscillation of the higher-order nonlinear delay dynamic equation

$$\bigl(a_{n-1}(t)\bigl(a_{n-2}(t)\bigl(\cdots\bigl(a_1(t)x^\Delta(t)\bigr)^\Delta\cdots\bigr)^\Delta\bigr)^\Delta\bigr)^\Delta+u(t)g\bigl(x\bigl(\delta(t)\bigr)\bigr)=R(t)$$
$$\text{for } t\in[t_0,\infty)_\mathbb{T},\tag{1.1}$$

where $t_0\in\mathbb{T}$, the time scale interval $[t_0,\infty)_\mathbb{T}\equiv\{t\in\mathbb{T}:t\geq t_0\}$, $a_i\in C_{rd}([t_0,\infty)_\mathbb{T},(0,\infty))$ $(1\leq i\leq n-1)$, $u,R\in C_{rd}([t_0,\infty)_\mathbb{T},\mathbb{R})$, $\delta\in C_{rd}([t_0,\infty)_\mathbb{T},\mathbb{T})$ is surjective with $\delta(t)\leq t$ and $\delta(t)\to\infty$ as $t\to\infty$, and $g\in C([t_0,\infty)_\mathbb{T}\times\mathbb{R},\mathbb{R})$. Our goal is to obtain sufficient conditions for all solutions of (1.1) to be nonoscillatory.

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We define

$$R_i\bigl(t,x(t)\bigr) = \begin{cases} x(t) & \text{if } i = 0, \\ a_i(t)R_{i-1}^{\triangle}(t,x(t)) & \text{if } 1 \le i \le n-1. \end{cases} \tag{1.2}$$

Then (1.1) reduces to the equation

$$R_{n-1}^{\triangle}\bigl(t,x(t)\bigr) + u(t)g\bigl(x\bigl(\delta(t)\bigr)\bigr) = R(t). \tag{1.3}$$

We can suppose the $\sup\mathbb{T} = \infty$ since we are interested in the oscillatory behavior of solutions near infinity. By a solution of (1.1) we mean a nontrivial real-valued function $x \in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $T_x \ge t_0$, such that $R_{n-1}(t,x(t)) \in C_{rd}^1([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. Since we are working on a time scale, the notion of oscillation takes the form of what is known as a *generalized zero* of a function. We say that $x(t)$ has a generalized zero at a point T if $x(T)x(\sigma(T)) \le 0$. A function is said to be *oscillatory* if it has arbitrarily large generalized zeros and *nonoscillatory* otherwise.

In order to create a theory that can unify discrete and continuous analysis, the theory of time scale was initiated by Hilger's landmark paper [1], which has received a lot of attention. There exist a variety of interesting time scales, and they give rise to many applications (see [2]). We refer the reader to [3, 4] for further results on time-scale calculus. In the thousands of papers in the literature, finding sufficient conditions for all solutions of an equation to be oscillatory have been a major focus of study (see [5–28]), but finding necessary and sufficient conditions for the existence of a nonoscillatory bounded solution of an equation are more rare (see [29]).

Zhu and Wang [21] studied the existence of nonoscillatory solutions to neutral dynamic equation

$$\bigl[x(t) + p(t)x\bigl(g(t)\bigr)\bigr]^{\triangle} + f\bigl(t,x\bigl(h(t)\bigr)\bigr) = 0.$$

Karpuz and Öcalan [22] studied the asymptotic behavior of delay dynamic equations of the form

$$\bigl[x(t) + A(t)x\bigl(\alpha(t)\bigr)\bigr]^{\triangle} + B(t)F\bigl(x\bigl(\beta(t)\bigr)\bigr) - C(t)G\bigl(x\bigl(\gamma(t)\bigr)\bigr) = \varphi(t).$$

Wu *et al.* [25] investigated the oscillation of the higher-order dynamic equation

$$\bigl\{r_n(t)\bigl[\bigl(r_{n-1}(t)\bigl(\cdots\bigl(r_1(t)x(t)^{\triangle}\bigr)^{\triangle}\cdots\bigr)^{\triangle}\bigr)^{\triangle}\bigr]^{\gamma}\bigr\}^{\triangle} + F\bigl(t,x\bigl(\tau(t)\bigr)\bigr) = 0.$$

Sun *et al.* [26] obtained some necessary and sufficient conditions for the existence of nonoscillatory solution for the higher-order equation

$$\bigl\{a(t)\bigl[\bigl(x(t) - p(t)x\bigl(\tau(t)\bigr)\bigr)^{\triangle^m}\bigr]^{\alpha}\bigr\}^{\triangle} + f\bigl(t,x\bigl(\delta(t)\bigr)\bigr) = 0.$$

2 Auxiliary results

We state the following conditions, which are needed in the sequel.

(H_1) There exist constants $\alpha, \beta \ge 0$ and $\gamma \ge 0$ such that $|g(u)| \le \alpha|u|^{\gamma} + \beta$.

(H_2) $\int_{t_0}^{\infty} \frac{\Delta s_1}{a_1(s_1)} \int_{t_0}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |R(s_n)|\Delta s_n < \infty$.

(H_3) $\int_{t_0}^{\infty} \frac{\Delta s_1}{a_1(s_1)} \int_{t_0}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} |u(s_n)| \Delta s_n < \infty$.

We shall employ the following lemma.

Lemma 2.1 *Let* $\mathbb{R}_+ \equiv [0,\infty)$ *and* $H = \{(t, s_1, s_2, \ldots, s_{n-1}) : 0 \le s_{n-1} \le s_{n-2} \le \cdots \le s_1 \le t < \infty\}$. *Suppose that* $r \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_+)$, $h \in C_{rd}(H, \mathbb{R}_+)$, *and that* $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is nondecreasing with* $p(r) > 0$ *for* $r > 0$. *If there exists a constant* $c > 0$ *such that*

$$r(t) \le c + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n) p\bigl(r(s_n)\bigr) \Delta s_n, \tag{2.1}$$

then

$$r(t) \le P^{-1}\biggl(P(c) + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n) \Delta s_n\biggr),$$

where

$$P(w) = \int_{w_0}^{w} \frac{ds}{p(s)}, \quad w_0, w > 0,$$

P^{-1} *is the inverse function of* P, *and*

$$P(c) + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n) \Delta s_n \in \operatorname{Dom}\bigl(P^{-1}\bigr). \tag{2.2}$$

Proof Let $z(t)$ denote the right side of inequality (2.1). Then $z(t_0) = c$, $r(t) \le z(t)$, and

$$\begin{aligned} z^{\Delta}(t) &= \int_{t_0}^{t} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(t, s_2, \ldots, s_n) p\bigl(r(s_n)\bigr) \Delta s_n \\ &\le p\bigl(z(t)\bigr) \int_{t_0}^{t} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(t, s_2, \ldots, s_n) \Delta s_n. \end{aligned}$$

Since $z^{\Delta}(t) \ge 0$ and p is nondecreasing, we obtain

$$\frac{z^{\Delta}(t)}{p(z(t))} \le \int_{t_0}^{t} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(t, s_2, \ldots, s_n) \Delta s_n. \tag{2.3}$$

Noting that

$$P^{\Delta}\bigl(z(t)\bigr) = z^{\Delta}(t) \int_0^1 \frac{dh}{p[hz(\sigma(t)) + (1-h)z(t)]} \le \frac{z^{\Delta}(t)}{p(z(t))},$$

we have

$$P\bigl(z(t)\bigr) \le P\bigl(z(t_0)\bigr) + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n) \Delta s_n.$$

Since $P(w)$ is increasing, we have

$$z(t) \le P^{-1}\biggl(P(c) + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n) \Delta s_n\biggr).$$

The proof is complete. □

Notice that taking $p(v) = v^{\xi}$ and $\xi > 1$ in Lemma 2.1, we have

$$P\big(z(t)\big) - P\big(z(t_0)\big) = \frac{1}{1-\xi}\big[z^{1-\xi}(t) - z^{1-\xi}(t_0)\big].$$

So

$$\frac{1}{1-\xi}z^{1-\xi}(t) \le \frac{1}{1-\xi}z^{1-\xi}(t_0) + \int_{t_0}^{t}\Delta s_1\int_{t_0}^{s_1}\Delta s_2\int_{t_0}^{s_2}\cdots\int_{t_0}^{s_{n-1}} h(s_1,s_2,\dots,s_n)\Delta s_n,$$

that is,

$$z^{1-\xi}(t) \ge z^{1-\xi}(t_0) + (1-\xi)\int_{t_0}^{t}\Delta s_1\int_{t_0}^{s_1}\Delta s_2\int_{t_0}^{s_2}\cdots\int_{t_0}^{s_{n-1}} h(s_1,s_2,\dots,s_n)\Delta s_n.$$

We have

$$r(t) \le \left[c^{1-\xi} - (\xi-1)\int_{t_0}^{t}\Delta s_1\int_{t_0}^{s_1}\Delta s_2\int_{t_0}^{s_2}\cdots\int_{t_0}^{s_{n-1}} h(s_1,s_2,\dots,s_n)\Delta s_n\right]^{\frac{-1}{\xi-1}},$$

provided that

$$\int_{t_0}^{t}\Delta s_1\int_{t_0}^{s_1}\Delta s_2\int_{t_0}^{s_2}\cdots\int_{t_0}^{s_{n-1}} h(s_1,s_2,\dots,s_n)\Delta s_n < \frac{c^{1-\xi}}{\xi-1}. \tag{2.4}$$

3 Main results

Now, we state and prove our main results.

Theorem 3.1 *Assume that conditions* (H_1)-(H_3) *hold and for some* $k \ge 0$,

$$\int_{t_0}^{\infty}\frac{\Delta s_1}{a_1(s_1)}\int_{t_0}^{s_1}\frac{\Delta s_2}{a_2(s_2)}\cdots\int_{t_0}^{s_{n-2}}\frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})}\int_{t_0}^{s_{n-1}}\big|u(s_n)\big|\delta^{k\gamma}(s_n)\Delta s_n < \infty. \tag{3.1}$$

If $x(t)$ *is an oscillatory solution of* (1.1) *such that*

$$\big|x(t)\big| = O\big(t^k\big), \quad t \to \infty, \tag{3.2}$$

then $x(t) \to 0$ *as* $t \to \infty$.

Proof We will show $\limsup_{t\longrightarrow\infty} x(t) = 0$ and $\liminf_{t\longrightarrow\infty} x(t) = 0$. Suppose that $\limsup_{t\longrightarrow\infty} x(t) = L > 0$. Then for any $t_1 \ge t_0$, there exists $t_2 \ge t_1$ such that $x(t_2) > \frac{L}{2}$. In view of conditions (H_2), (H_3), (3.1), and (3.2), there exist $T_0 \ge t_0$ and $K > 0$ such that $|x(t)| \le Kt^k$ $(t \ge T_0)$ and

$$\begin{aligned}&\int_{T_0}^{\infty}\frac{\Delta s_1}{a_1(s_1)}\int_{T_0}^{s_1}\frac{\Delta s_2}{a_2(s_2)}\cdots\int_{T_0}^{s_{n-2}}\frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})}\\ &\quad\times\int_{T_0}^{s_{n-1}}\big\{\big|R(s_n)\big| + \big|u(s_n)\big|\big[\alpha K^{\gamma}\delta^{k\gamma}(s_n) + \beta\big]\big\}\Delta s_n < \frac{L}{4}.\end{aligned} \tag{3.3}$$

Since $x(t)$ is an oscillatory solution of (1.1), every $R_i(t,x(t))$ is oscillatory for $i = 1,2,\dots,n-1$. Choose $T_0 < T_1 \le T_2 \le \cdots \le T_{n-1}$ such that

$$R_{n-i}\bigl(T_i, x(T_i)\bigr)R_{n-i}\bigl(\sigma(T_i), x\bigl(\sigma(T_i)\bigr)\bigr) \le 0, \quad i = 1,2,\dots,n-1, \tag{3.4}$$

and

$$R_{n-i}\bigl(T_i, x(T_i)\bigr) \le 0, \quad i = 1,2,\dots,n-1. \tag{3.5}$$

Integrating (1.1) from T_i to t, $i = 1,2,\dots,n-1$, successively $n-1$ times with $t > T_{n-1}$, we obtain

$$\begin{aligned} a_1x^{\Delta}(t) &= a_1(T_{n-1})x^{\Delta}(T_{n-1}) + \int_{T_{n-1}}^{t} \frac{R_2(T_{n-2},x(T_{n-2}))}{a_2(s_{n-2})}\Delta s_{n-2} \\ &\quad + \int_{T_{n-1}}^{t} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\int_{T_{n-2}}^{s_{n-2}} \frac{R_3(T_{n-3},x(T_{n-3}))}{a_3(s_{n-3})}\Delta s_{n-3} \\ &\quad + \cdots \\ &\quad + \int_{T_{n-1}}^{t} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\int_{T_{n-2}}^{s_{n-2}} \frac{\Delta s_{n-3}}{a_3(s_{n-3})}\cdots\int_{T_2}^{s_2} \frac{R_{n-1}(T_1,x(T_1))}{a_{n-1}(s_1)}\Delta s_1 \\ &\quad + \int_{T_{n-1}}^{t} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\int_{T_{n-2}}^{s_{n-2}} \frac{\Delta s_{n-3}}{a_3(s_{n-3})}\cdots\int_{T_1}^{s_1} \bigl[R(s) - u(s)g\bigl(x\bigl(\delta(s)\bigr)\bigr)\bigr]\Delta s \\ &\le \int_{T_{n-1}}^{t} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\int_{T_{n-2}}^{s_{n-2}} \frac{\Delta s_{n-3}}{a_3(s_{n-3})}\cdots\int_{T_1}^{s_1} \bigl[R(s) - u(s)g\bigl(x\bigl(\delta(s)\bigr)\bigr)\bigr]\Delta s. \end{aligned} \tag{3.6}$$

Choose $T_n > T_{n-1}$ so that

$$x(T_n)x\bigl(\sigma(T_n)\bigr) \le 0 \quad \text{and} \quad x(T_n) \le 0.$$

Take $T_{n+1} \ge T_n$ such that

$$x(T_{n+1}) \ge \frac{L}{2} \quad \text{and} \quad x(t) > 0, \quad t \in (T_n, T_{n+1}).$$

Note that such T_{n+1} exists since $\limsup_{t\longrightarrow\infty} x(t) > \frac{L}{2}$. Dividing (3.6) by $a_1(t)$ and integrating once more from T_n to T_{n+1}, we have

$$\begin{aligned} \frac{L}{2} \le x(T_{n+1}) &\le \int_{T_n}^{T_{n+1}} \frac{\Delta s_{n-1}}{a_1(s_{n-1})}\int_{T_{n-1}}^{s_{n-1}} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\int_{T_{n-2}}^{s_{n-2}} \frac{\Delta s_{n-3}}{a_3(s_{n-3})} \\ &\quad \cdots\int_{T_1}^{s_1} \bigl[R(s) - u(s)g\bigl(x\bigl(\delta(s)\bigr)\bigr)\bigr]\Delta s. \end{aligned} \tag{3.7}$$

It follows from (H_1) that

$$\begin{aligned} \frac{L}{2} &\le \int_{T_n}^{T_{n+1}} \frac{\Delta s_{n-1}}{a_1(s_{n-1})}\int_{T_{n-1}}^{s_{n-1}} \frac{\Delta s_{n-2}}{a_2(s_{n-2})}\cdots\int_{T_1}^{s_1} \bigl[\bigl|R(s)\bigr| + \bigl|u(s)\bigr|\bigl|g\bigl(x\bigl(\delta(s)\bigr)\bigr)\bigr|\bigr]\Delta s \\ &\le \int_{T_n}^{T_{n+1}} \frac{\Delta s_{n-1}}{a_1(s_{n-1})}\int_{T_{n-1}}^{s_{n-1}} \frac{\Delta s_{n-2}}{a_2(s_{n-2})} \end{aligned}$$

$$
\begin{aligned}
&\cdots \int_{T_1}^{s_1} \bigl\{\bigl|R(s)\bigr| + \bigl|u(s)\bigr|\bigl[\alpha\bigl|x\bigl(\delta(s)\bigr)\bigr|^{\gamma} + \beta\bigr]\bigr\}\Delta s \\
&\leq \int_{T_n}^{T_{n+1}} \frac{\Delta s_{n-1}}{a_1(s_{n-1})} \int_{T_{n-1}}^{s_{n-1}} \frac{\Delta s_{n-2}}{a_2(s_{n-2})} \cdots \int_{T_1}^{s_1} \bigl\{\bigl|R(s)\bigr| + \bigl|u(s)\bigr|\bigl[\alpha K^{\gamma}\delta^{k\gamma}(s) + \beta\bigr]\bigr\}\Delta s.
\end{aligned}
$$

In view of (3.3), we have a contradiction.

In a similar fashion, we can show that $\liminf_{t\longrightarrow\infty} x(t) = 0$. The proof is complete. □

Theorem 3.2 *Assume that conditions* (H_1)-(H_3) *hold with* $\gamma \geq 1$. *Then every oscillatory solution of* (1.1) *is bounded.*

Proof Let $x(t)$ be an oscillatory solution of (1.1), and $d > 0$ be a constant.

If $\gamma > 1$, then it follows from conditions (H_2) and (H_3) that there exists $T^* \geq t_0$ such that

$$
\int_{T^*}^{\infty} \frac{\Delta s_1}{a_1(s_1)} \int_{T^*}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T^*}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T^*}^{s_{n-1}} \bigl[\bigl|R(s_n)\bigr| + \beta\bigl|u(s_n)\bigr|\bigr]\Delta s_n < d \tag{3.8}
$$

and

$$
\int_{T^*}^{\infty} \frac{\Delta s_1}{a_1(s_1)} \int_{T^*}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T^*}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T^*}^{s_{n-1}} \alpha\bigl|u(s_n)\bigr|\Delta s_n < \frac{d^{1-\gamma}}{2(\gamma-1)}. \tag{3.9}
$$

We will show that eventually for any interval on which $x(t)$ is positive, we have that $x(t)$ is bounded by a constant independent of $x(t)$. Choose $T^* < T_1 \leq T_2 \leq \cdots \leq T_{n-1} \leq T_n$ so that (3.4)-(3.5) are satisfied, $\delta(t) > T_{n-1}$ for $t \geq T_n$, and $x(\delta(T_n))x(\delta(\sigma(T_n))) \leq 0$ with $x(\delta(T_n)) \leq 0$. As in the proof of Theorem 3.1, using (3.8), we have

$$
\begin{aligned}
x\bigl(\delta(t)\bigr) &\leq \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \\
&\quad \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \bigl\{\bigl|R(s_n)\bigr| + \bigl|u(s_n)\bigr|\bigl[\alpha\bigl|x\bigl(\delta(s_n)\bigr)\bigr|^{\gamma} + \beta\bigr]\bigr\}\Delta s_n \\
&= \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \bigl[\bigl|R(s_n)\bigr| + \beta\bigl|u(s_n)\bigr|\bigr]\Delta s_n \\
&\quad + \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \bigl|u(s_n)\bigr|\alpha\bigl|x\bigl(\delta(s_n)\bigr)\bigr|^{\gamma}\Delta s_n \\
&\leq d + \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \\
&\quad \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \bigl|u(s_n)\bigr|\alpha\bigl|x\bigl(\delta(s_n)\bigr)\bigr|^{\gamma}\Delta s_n.
\end{aligned} \tag{3.10}
$$

We can apply Lemma 2.1 with $c = d$, $h(s_1, s_2, \ldots, s_n) = \frac{\alpha|u(s_n)|}{a_1(s_1)a_2(s_2)\cdots a_{n-1}(s_{n-1})}$, $\xi = \gamma$, and $p(s) = s^{\gamma}$. From condition (3.9) we have

$$
\begin{aligned}
&d^{1-\gamma} - (\gamma-1)\alpha \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \bigl|u(s_n)\bigr|\Delta s_n \\
&\quad > d^{1-\gamma} - (\gamma-1)\frac{d^{1-\gamma}}{2(\gamma-1)} = \frac{d^{1-\gamma}}{2} > 0.
\end{aligned}
$$

Thus, (2.4) holds. It follows from Lemma 2.1 that

$$\begin{aligned} &x(\delta(t)) \\ &\quad \le \left[d^{1-\gamma} - (\gamma-1)\alpha \int_{T_1}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1}^{s_{n-1}} \left|u(s_n)\right| \Delta s_n\right]^{\frac{-1}{\gamma-1}} \\ &\quad \le \frac{2^{\frac{1}{\gamma-1}}}{d}. \end{aligned}$$

So $x(\delta(t))$ is bounded. A similar argument holds for intervals where $x(t)$ is negative.

If $\gamma = 1$, then choose $\hat{T} \ge t_0$ so that (3.8) holds with T^* replaced by $\hat{T}$ and

$$\int_{\hat{T}}^{\infty} \frac{\Delta s_1}{a_1(s_1)} \int_{\hat{T}}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{\hat{T}}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{\hat{T}}^{s_{n-1}} \left|u(s_n)\right| \Delta s_n < \frac{1}{\alpha+1}.$$

Choose $T^* < T_1' \le T_2' \le \cdots \le T_{n-1}' \le T_n'$ so that $R_{n-i}(T_i', x(T_i'))R_{n-i}(\sigma(T_i'), x(\sigma(T_i'))) \le 0$ with $R_{n-i}(T_i', x(T_i')) \ge 0$ for $1 \le i \le n$ and $\delta(t) > T_{n-1}'$ for $t \ge T_n'$. As in the proof of Theorem 3.1, using (3.8), we have

$$x(\delta(t)) \ge -d - \int_{T_1'}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{T_1'}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{T_1'}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{T_1'}^{s_{n-1}} \left|u(s_n)\right| \alpha \left|x(\delta(s_n))\right| \Delta s_n.$$

Combining (3.10) with this inequality, we obtain

$$\begin{aligned} \left|x(\delta(t))\right| \le d &+ \int_{L}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{L}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \\ &\cdots \int_{L}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{L}^{s_{n-1}} \left|u(s_n)\right| \alpha \left|x(\delta(s_n))\right| \Delta s_n, \end{aligned} \tag{3.11}$$

where $L = \min\{T_1, T_1'\}$. Denoting by $z(t)$ the right side of inequality (3.11), we see that $|x(\delta(t))| \le z(t)$, $z(\delta(t)) \le z(t)$, and

$$\begin{aligned} z(t) &= d + \int_{L}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{L}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{L}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{L}^{s_{n-1}} \left|u(s_n)\right| \alpha \left|x(\delta(s_n))\right| \Delta s_n \\ &\le d + \int_{L}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{L}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{L}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{L}^{s_{n-1}} \left|u(s_n)\right| \alpha z(s_n) \Delta s_n \\ &\le d + z(t) \int_{L}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{L}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{L}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{L}^{s_{n-1}} \left|u(s_n)\right| \alpha \Delta s_n \\ &\le d + \frac{\alpha}{\alpha+1} z(t), \end{aligned}$$

which implies $x(\delta(t)) \le d(\alpha+1)$. The proof is complete. □

After seeing the proof of Theorem 3.2, the proof of the following Theorem 3.3 becomes obvious.

Theorem 3.3 *Assume that conditions* (H_1)-(H_3) *hold with* $\gamma \ge 1$. *If* (3.1) *holds, then every oscillatory solution of* (1.1) *converges to zero as* $t \to \infty$.

In a similar fashion as before, we can show the following theorem.

Theorem 3.4 *Assume that conditions* (H_1)-(H_3) *hold with* $0 < \gamma < 1$. *If* (3.1) *holds, then every oscillatory solution of* (1.1) *is bounded and converges to zero as* $t \to \infty$.

Proof Notice that taking $p(v) = v^{\xi}$ and $0 < \xi < 1$ in Lemma 2.1, we have

$$P(z(t)) - P(z(t_0)) = \frac{1}{1-\xi}\left[z^{1-\xi}(t) - z^{1-\xi}(t_0)\right].$$

So

$$\frac{1}{1-\xi}z^{1-\xi}(t) \leq \frac{1}{1-\xi}z^{1-\xi}(t_0) + \int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n)\Delta s_n,$$

that is,

$$z(t) \leq \left[z^{1-\xi}(t_0) + (1-\xi)\int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n)\Delta s_n\right]^{\frac{1}{1-\xi}}.$$

We have

$$r(t) \leq \left[z^{1-\xi}(t_0) + (1-\xi)\int_{t_0}^{t} \Delta s_1 \int_{t_0}^{s_1} \Delta s_2 \int_{t_0}^{s_2} \cdots \int_{t_0}^{s_{n-1}} h(s_1, s_2, \ldots, s_n)\Delta s_n\right]^{\frac{1}{1-\xi}}.$$

Further, the proof is similar to that of Theorem 3.2, so we have

$$x(\delta(t)) \leq \left[d^{1-\gamma} + (1-\gamma)\alpha \int_{t_0}^{\delta(t)} \frac{\Delta s_1}{a_1(s_1)} \int_{t_0}^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_{t_0}^{s_{n-1}} \left|u(s_n)\right|\Delta s_n\right]^{\frac{1}{1-\gamma}}.$$

So we can conclude that every oscillatory solution of (1.1) is bounded, and by Theorem 3.1 $x(t)$ converges to zero as $t \to \infty$. The proof is complete. □

Theorem 3.5 *Assume that conditions* (H_1)-(H_3) *hold with* $g(0) = 0$. *If there exists* $N > 0$ *such that for all large* T, *either*

$$\liminf_{t \longrightarrow \infty} \int_{T}^{t} \left[R(s) - N\left|u(s)\right|\right]\Delta s > 0 \tag{3.12}$$

or

$$\limsup_{t \longrightarrow \infty} \int_{T}^{t} \left[R(s) + N\left|u(s)\right|\right]\Delta s < 0, \tag{3.13}$$

then all solutions of (1.1) *are nonoscillatory.*

Proof For contradiction, let $x(t)$ be an oscillatory solution of (1.1). By Theorem 3.3 and Theorem 3.4, $x(t)$ converges to 0 as $t \to \infty$. Hence, there exists $T_0 \geq t_0$ such that

$|g(x(\delta(t)))| \leq N$ for $t \geq T_0$. From (1.3) we have

$$R(t) - N\big|u(t)\big| \leq R_{n-1}^{\Delta}\big(t, x(t)\big) \leq R(t) + N\big|u(t)\big|. \tag{3.14}$$

If (3.12) holds, then we choose $T \geq T_0$ such that $\delta(t) \geq T_0$ for $t \geq T$,

$$R_{n-2}\big(T, x(T)\big)R_{n-2}\big(\sigma(T), x\big(\sigma(T)\big)\big) \leq 0, \qquad R_{n-2}\big(T, x(T)\big) \geq 0, \tag{3.15}$$

and integrating the left inequality in (3.14) from T to t, we obtain

$$R_{n-2}\big(T, x(T)\big) + \int_T^t \big[R(s) - N\big|u(s)\big|\big]\Delta s \leq R_{n-2}\big(t, x(t)\big).$$

This is a contradiction since if $x(t)$ is oscillatory, then $R_{n-2}(t, x(t))$ is also oscillatory.

If (3.13) holds, then we choose T so that the second inequality in (3.15) is reversed. This completes the proof of the theorem. □

4 Example

In this section, we give an example to illustrate our main results.

Lemma 4.1 [23, 24] *Assume that $s, t \in \mathbb{T}$ and $g \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$. Then*

$$\int_s^t \left[\int_\eta^t g(\eta, \zeta)\Delta\zeta\right]\Delta\eta = \int_s^t \left[\int_s^{\sigma(\zeta)} g(\eta, \zeta)\Delta\eta\right]\Delta\zeta.$$

Example 4.1 Let $\mathbb{T} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ with $q > 1$. Consider the higher-order dynamic equation

$$\big(t\big(t\big(\cdots\big(t^{2+\frac{1}{\gamma}}x^{\Delta}\big)^{\Delta}\cdots\big)^{\Delta}\big)^{\Delta}\big)^{\Delta} + \frac{1}{t^{1+k\gamma+\frac{1}{\gamma}}}\left|x\left(\frac{t}{q}\right)\right|^{\gamma} \operatorname{sgn} x\left(\frac{t}{q}\right) = \frac{1}{t^{1+\frac{1}{\gamma}}}, \tag{4.1}$$

where $t \in [q, \infty)_{\mathbb{T}}$, $\gamma > 0$, $k \geq 0$, $a_1(t) = t^{2+\frac{1}{\gamma}}$, $a_i(t) = t$ $(2 \leq i \leq n-1)$, $u(t) = \frac{1}{t^{1+k\gamma+\frac{1}{\gamma}}}$, $\delta(t) = \frac{t}{q}$, $R(t) = \frac{1}{t^{1+\frac{1}{\gamma}}}$, and $g(u) = |u|^{\gamma}\operatorname{sgn}(u)$.

It is easy to verify that $R(t)$ and $u(t)$ satisfy the condition (3.12). We will use the following inequality: if $s > t \geq q$, then

$$\begin{aligned}
\int_t^s \frac{1}{\tau}\Delta\tau &= \int_t^{qt} \frac{1}{\tau}\Delta\tau + \int_{qt}^{q^2t} \frac{1}{\tau}\Delta\tau + \cdots + \int_{q^{n-1}t}^{q^nt=s} \frac{1}{\tau}\Delta\tau \\
&= \frac{qt - t}{t} + \frac{q^2t - qt}{qt} + \cdots + \frac{q^nt - q^{n-1}t}{q^{n-1}t} \\
&= n(q-1) \leq q^{n+1} \leq q^nt = s.
\end{aligned}$$

Applying Lemma 4.1 and the last inequality, we have

$$\begin{aligned}
&\int_q^\infty \frac{\Delta s_1}{a_1(s_1)} \int_q^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_q^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_q^{s_{n-1}} \big|u(s_n)\big|\Delta s_n \\
&\quad \leq \int_q^\infty \frac{\Delta s_1}{a_1(s_1)} \int_q^{s_1} \frac{\Delta s_2}{a_2(s_2)} \cdots \int_q^{s_{n-2}} \frac{\Delta s_{n-1}}{a_{n-1}(s_{n-1})} \int_q^{s_{n-1}} \big|R(s_n)\big|\Delta s_n
\end{aligned}$$

$$
\begin{aligned}
&= \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-2}} \frac{\Delta s_{n-1}}{s_{n-1}} \int_q^{s_{n-1}} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&= \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-3}} \frac{\Delta s_{n-2}}{s_{n-2}} \int_q^{s_{n-2}} \Delta s_{n-1} \int_q^{s_{n-1}} \frac{1}{s_{n-1}} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&\le \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-3}} \frac{\Delta s_{n-2}}{s_{n-2}} \int_q^{s_{n-2}} \Delta s_{n-1} \int_q^{\sigma(s_{n-1})} \frac{1}{s_{n-1}} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&= \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-3}} \frac{\Delta s_{n-2}}{s_{n-2}} \int_q^{s_{n-2}} \frac{\Delta s_n}{s_n^{1+\frac{1}{\gamma}}} \int_{s_n}^{s_{n-2}} \frac{1}{s_{n-1}} \Delta s_{n-1} \\
&\le \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-3}} \frac{\Delta s_{n-2}}{s_{n-2}} \int_q^{s_{n-2}} \frac{s_{n-2}}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&= \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-4}} \frac{\Delta s_{n-3}}{s_{n-3}} \int_q^{s_{n-3}} \Delta s_{n-2} \int_q^{s_{n-2}} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&\le \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-4}} \frac{\Delta s_{n-3}}{s_{n-3}} \int_q^{s_{n-3}} \Delta s_{n-2} \int_q^{\sigma(s_{n-2})} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&= \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-4}} \frac{\Delta s_{n-3}}{s_{n-3}} \int_q^{s_{n-3}} \Delta s_n \int_{s_n}^{s_{n-3}} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_{n-2} \\
&\le \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{\Delta s_2}{s_2} \cdots \int_q^{s_{n-3}} \frac{s_{n-3}}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&\cdots \\
&\le \int_q^\infty \frac{\Delta s_1}{s_1^{2+\frac{1}{\gamma}}} \int_q^{s_1} \frac{s_1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \le \int_q^\infty \frac{\Delta s_1}{s_1^{1+\frac{1}{\gamma}}} \int_q^{\sigma(s_1)} \frac{1}{s_n^{1+\frac{1}{\gamma}}} \Delta s_n \\
&= \int_q^\infty \frac{\Delta s_n}{s_n^{1+\frac{1}{\gamma}}} \int_{s_n}^\infty \frac{1}{s_1^{1+\frac{1}{\gamma}}} \Delta s_1 \\
&\le \int_q^\infty \frac{\Delta s_n}{s_n^{1+\frac{1}{\gamma}}} \int_q^\infty \frac{1}{s_1^{1+\frac{1}{\gamma}}} \Delta s_1 = \left(\frac{q-1}{q^{\frac{1}{\gamma}}-1}\right)^2 < \infty.
\end{aligned}
$$

Thus, conditions (H_1)-(H_3) and (3.1) hold. Then it follows from Theorem 3.5 that every solution $x(t)$ of (4.1) is nonoscillatory.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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