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Research Article

Some Variational Results Using Generalizations of Sequential Lower Semicontinuity

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Kirk and Saliga and then Chen et al. introduced lower semicontinuity from above, a generalization of sequential lower semicontinuity, and they showed that well-known results, such as Ekeland's variational principle and Caristi's fixed point theorem, remain still true under lower semicontinuity from above. In a previous paper we introduced a new concept that generalizes lower semicontinuity from above. In the present one we continue such study, also introducing other two new generalizations of lower semicontinuity from above; we study such extensions, compare each other five concepts (sequential lower semicontinuity, lower semicontinuity from above, the one by us previously introduced, and the two here defined) and, in particular, we show that the above quoted well-known results remain still true under one of our such generalizations.

1. Introduction

In [1] Chen et al. proposed the following generalization ([1, Definitions 1.2 and 1.5]).

Definition 1.1. Let (X,τ) be a topological space. Let $x \in X$. A function $f : X \to [-\infty,+\infty]$ is said to be *sequentially lower semicontinuous from above at* x ("d-$slsc$ at x") if $(x_n)_{n\in\mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $(f(x_n))_{n\in\mathbb{N}}$ weakly decreasing sequence, implies $f(x) \leq \lim_{n\to+\infty} f(x_n)$. Moreover f is said to be *sequentially lower semicontinuous from above* ("d-$slsc$") if it is sequentially lower semicontinuous from above at x for every $x \in X$.

Actually the same definition was previously considered by Kirk and Saliga in [2, Section 2, definition above Theorem 2.1]. Both in [1, 2] this concept is called *lower semicontinuity from above*; furthermore also Borwein and Zhu in [3, Exercise 2.1.4] used the same concept, naming it *partial lower semicontinuity*; here we are calling it *sequential lower semicontinuity from above*, as it is a generalization of *sequential lower semicontinuity*.

Moreover the authors of [1] conjectured that, for convex functions on normed spaces, sequential lower semicontinuity from above is equivalent to weak sequential lower semicontinuity from above (see [1, some rows below Definition 1.5]). We exhibited some examples showing that such conjecture is false (see [4, Example 3.1 and Examples sketched in Remark 3.1]).

In [4] we defined the following new concept, that generalizes sequential lower semicontinuity from above.

Definition 1.2 (see [4, Definition 4.1]). Let (X, τ) be a topological space. Let f be a function, $f : X \to [-\infty, +\infty]$. Then f is said to be

(i) *inf-sequentially lower semicontinuous at* $x \in X$ ("*i-slsc* at x") if $(x_n)_{n\in\mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $\lim_{n\to+\infty} f(x_n) = \inf f$, implies $f(x) = \inf f$ (equivalently, in the above condition the part $\lim_{n\to+\infty} f(x_n) = \inf f$ can be replaced by $f(x_n) \searrow \inf f$);

(ii) *inf-sequentially lower semicontinuous* ("*i-slsc*") if it is *i-slsc* at x for every $x \in X$.

In particular we showed that for convex functions on normed spaces, such concept is equivalent to its weak counterpart ([4, Theorem 4.1]).

Here, with the purpose to continue the study already begun in [4], we define other two new concepts (Definitions 3.1), called by us *below sequential lower semicontinuity from above* (*bd-slsc*) and *uniform below sequential lower semicontinuity from above* (*ubd-slsc*), that generalize sequential lower semicontinuity from above and we show the following.

(a) As it already happened for *i-slsc*, for convex functions on normed spaces, one of such new concepts (*bd-slsc*) is equivalent to its weak counterpart (Theorem 4.1 and part (e) of Remarks 3.2); also by means of such result, it can be seen that, for convex functions on normed spaces and indifferently with respect to the topology induced by norm or to the weak topology, *i-slsc* and *bd-slsc* are each other equivalent (part (e) of Remarks 3.2, part (b) of Theorem 3.4 and Corollary 4.2).

(b) Some results listed in [1, 2], such as Ekeland's variational principle and Caristi's fixed point theorem, remain still true under an hypothesis of *ubd-slsc* (Section 5).

Moreover we study the five concepts of sequential lower semicontinuity, lower semicontinuity from above, inf-sequential lower semicontinuity, below sequential lower semicontinuity from above and uniform below sequential lower semicontinuity from above, supplying further results and examples, with the purpose of getting a comparison between such five concepts, both in the general case (Section 3) and in the case of convex functions (Section 4). In particular, in Theorem 4.10 we prove that every convex *ubd-slsc* function on a Banach space is continuous in the points of the interior of its effective domain.

Finally, in Section 5, we also give some examples to show that, in the generalization of Ekeland's variational principle by us proved, some hypotheses cannot be weakened.

2. Notations and Preliminaries

Notations 1. In the sequel, unless otherwise specified, all linear spaces will be considered on the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. As convention, in $[-\infty, +\infty]$, $\inf\emptyset = +\infty$ and the product $0 \cdot (+\infty)$ is considered equal to 0. By $\mathbb{N}$ we denote the set of natural numbers (0 included), while $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n > 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$; $\delta_{n,m}$ is the Kronecker symbol. If Z is a linear space on $\mathbb{R}$ or on $\mathbb{C}$, let $\dim Z$ denote the algebraic dimension of Z, and, if $A \subseteq Z$, let $\operatorname{sp} A$ and $\operatorname{co} A$ denote, respectively, the linear subspace of Z that is generated by A and the convex hull of A; if $x,\ y \in Z$ let $[x,y] := \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ and, if $x \neq y$, let $]x,y] := \{\lambda x+(1-\lambda)y : \lambda \in [0,1[\}$; if $0 \in A$, then the *Minkowski functional (or gauge) of* A is the function $g_A : Z \to [0,+\infty]$ defined by $g_A(x) := \inf\{\alpha \in \mathbb{R}_+ : x \in \alpha A\}$ for every $x \in Z$. If Z is a topological linear space, let Z' denote the continuous dual of Z; if $A \subseteq Z$, let $\overline{\operatorname{co}}\, A$ be the closure of $\operatorname{co} A$. If Z is a normed space, then $|z|_Z$ indicates the norm in Z of an element $z \in Z$ and $S_Z(a,r) := \{z \in Z : |z-a|_Z < r\}(a \in Z,\ r \in \mathbb{R}_+)$. Let ℓ^2 and c_0, respectively, denote the real, or complex, Banach spaces of the sequences whose squares of moduli of coordinates are summable, and of the infinitesimal sequences. If A and B are sets, if $C \subseteq A$ and $f\colon A \to B$ is a function, then $\#A$ denotes the cardinality of A, $f_{|C}$ means the restriction of f to C; if $g : A \to [-\infty,+\infty]$ is a function, then $\operatorname{dom} g := \{x \in A : g(x) < +\infty\}$ denotes the effective domain of g. If Z is a topological space and if $A \subseteq Z$, let ∂A be the boundary of A. Let E denote the integer part function. If $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of elements of $[-\infty,+\infty]$ and if $\ell \in [-\infty,+\infty]$, then $\tau_n \searrow \ell$ means that $(\tau_n)_{n\in\mathbb{N}}$ is a weakly decreasing sequence with $\lim_{n\to+\infty}\tau_n = \ell$. Henceforth we will shorten both lower semicontinuous and lower semicontinuity in "*lsc*", both sequentially lower semicontinuous and sequential lower semicontinuity in "*slsc*".

Definitions 2.1. Let X be a linear space on $\mathbb{F}$, $A \subseteq X, y \in A$. Then (in accordance with [5])

(a) the point y is said to be an *internal point* of A if for every $x \in X$ there exists an $\alpha_x \in \mathbb{R}_+$ such that $y + \lambda x \in A$ for all $\lambda \in [0,\alpha_x]$;

(b) (see [5, page 8, above Exercise 1.1.20, and page 9, between the two Examples]) A is said to be *absorbing* if *0* is an internal point of A, that is, if for every $x \in X$ there exists an $\alpha_x \in \mathbb{R}_+$ such that $\lambda x \in A$ for all $\lambda \in [0,\alpha_x]$;

(c) the set A is said to be *balanced* if $\lambda x \in A$ for every $x \in A$ and $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$;

(d) the point y is said to be an *extreme point* of A if $x,\ z \in A, \lambda \in\ ]0,1[$ for which $y = \lambda x + (1-\lambda)z$ implies $x = z = y$.

Example 2.2. For every infinite dimensional X normed space on $\mathbb{F}$ there exists C an absorbing balanced convex subset of $\overline{S_X(0,1)}, C$ without interior points.

Let $T\colon X \to X$ be a linear bijective not continuous operator (e.g., let $e_n \in X$ be such that $|e_n|_X = 1(n \in \mathbb{N}), e_n \neq e_m$ if $n,m \in \mathbb{N},\ n \neq m, \{e_n : n \in \mathbb{N}\}$ linearly independent set of vectors, B a Hamel basis of X such that $\{e_n : n \in \mathbb{N}\} \subseteq B, T(e_n) := ne_n$ for every $n \in \mathbb{N}, T(b) := b$ for every $b \in B \setminus \{e_n : n \in \mathbb{N}\}, T$ extended for linearity to all X). Let $C := T^{-1}(\overline{S_X(0,1)}) \cap \overline{S_X(0,1)}$. Then C is a balanced convex as intersection of two balanced convex sets, is bounded because contained in $\overline{S_X(0,1)}$, is absorbing as intersection of two absorbing sets ($T^{-1}(\overline{S_X(0,1)})$ is absorbing because if $x \in X \setminus \{0\}$ then $T(x) \in X \setminus \{0\}$ and so $\lambda x \in T^{-1}(\overline{S_X(0,1)})$ for every $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1/|T(x)|_X$). Moreover, if by absurd there existed an interior point of C, then, being C balanced and convex, 0 should be an interior point of C and therefore T should be a continuous operator, that is not possible.

Here we are providing also another example, which will be useful in the construction of the subsequent Example 4.8; to this purpose we describe expressly a set C, that can be obtained using Theorem 1 of [6], provided in the proof of such theorem a bounded Hamel basis (in case constituted by elements having norm less or equal to 1) is considered, and with a little change in the complex case for showing that "0 is not an interior point of C" (and so there are no interior points of C, as above noted).

Let $e_n \in X$ be such that $|e_n|_X \leq 1 (n \in \mathbb{N}), e_n \neq e_m$ if $n, m \in \mathbb{N}, n \neq m, \{e_n : n \in \mathbb{N}\}$ linearly independent set of vectors, B a Hamel basis of X such that $\{e_n : n \in \mathbb{N}\} \subseteq B, |b|_X \leq 1$ for every $b \in B, D := \{(1/(n+1))e_n : n \in \mathbb{N}\} \cup (B \setminus \{e_n : n \in \mathbb{N}\})$. Then it is enough to define

$$C := \operatorname{co}\{\alpha d : \alpha \in \mathbb{F}, |\alpha| = 1, \ d \in D\}. \tag{2.1}$$

Such a C is obviously a balanced set; for obtaining the remaining properties the same proof of Theorem 1 of [6] still works, with the unique following little change if $\mathbb{F} = \mathbb{C}$ for showing that $cd \notin C$ when $c > 1$ and $d \in D$ (i.e., one of the points of the demonstration of [6]): if by absurd such a $cd \in C$ then, using that D is a Hamel basis, there should exist $m \in \mathbb{Z}_+, \lambda_1, \ldots, \lambda_m \in [0,1]$, with $\sum_{j=1}^m \lambda_j = 1, \alpha_1, \ldots, \alpha_m \in \mathbb{C}$, with $|\alpha_j| = 1$ for every $j \in \{1, \ldots, m\}$ such that $cd = \sum_{j=1}^m \lambda_j \alpha_j d$, therefore $c = \sum_{j=1}^m \lambda_j \alpha_j$, that is impossible because $|\sum_{j=1}^m \lambda_j \alpha_j| \leq \sum_{j=1}^m \lambda_j |\alpha_j| = 1$ while $c > 1$.

Theorem 2.3. *Let X be a real normed space with algebraic dimension greater or equal to 2; then $\partial S_X(0,1)$ is an arcwise connected set.*

Proof. Let $x, y \in \partial S_X(0,1), x \neq y$. We will distinguish two cases:

(a) $x \neq -y$,

(b) $x = -y$.

In the case (a) the arc $\gamma : t \in [0,1] \mapsto ((1-t)x + ty)/|(1-t)x + ty|_X$ connects x to y: in fact $\gamma(0) = x, \gamma(1) = y$; moreover γ is defined and continuous, as, if by absurd there exists $t \in [0,1]$ such that $(1-t)x = -ty$, then $1 - t = (1-t)|x|_X = t|y|_X = t$, consequently $t = 1/2$ and so $(1/2)x = -(1/2)y$, that is in contradiction with the assumption of (a).

In the case (b), being $\dim X \geq 2$, there exists $z \in \partial S_X(0,1)$ linearly independent from x and y, hence $z \neq -x$ and $z \neq -y$; then an arc connecting x to y can be found joining together two arcs, one connecting x to z and another connecting z to y, both of them existing in consequence of (a). □

Lemma 2.4. *Let $I \subseteq \mathbb{R}$ be an interval, $x \in [-\infty, +\infty]$ an extreme of I, $\ell \in [-\infty, +\infty[, f : I \to [-\infty, +\infty]$ a convex function, $x_n \in I$, $x_n \neq x (n \in \mathbb{N})$ such that $x_n \to x$ and $f(x_n) \searrow \ell$. Then $\ell = \inf\{f(y) : y \in I \setminus \{x\}\}$.*

Proof. If $\ell = -\infty$, then the conclusion is obvious; therefore we can suppose $\ell \in \mathbb{R}$. Let $z \in I \setminus \{x\}$. Then there exist $n_z \in \mathbb{N}$ such that $f(x_{n_z}) \in \mathbb{R}, x_{n_z} \in]x, z[$ and $m_z \in \mathbb{N}, m_z > n_z$ such that $x_{m_z} \in]x, x_{n_z}[$; therefore, being $\ell \leq f(x_{m_z}) \leq f(x_{n_z})$, we deduce that $f(x_{m_z}) \in \mathbb{R}$ and, for the convexity of f, we get $f(x_{n_z}) \leq ((x_{m_z} - x_{n_z})/(x_{m_z} - z)) f(z) + ((x_{n_z} - z)/(x_{m_z} - z)) f(x_{m_z})$, whence, using in the second inequality that $f(x_{n_z}) \geq f(x_{m_z})$, we obtain $f(z) \geq ((x_{m_z} - z)/(x_{m_z} - x_{n_z})) f(x_{n_z}) - ((x_{n_z} - z)/(x_{m_z} - x_{n_z})) f(x_{m_z}) \geq f(x_{m_z}) \geq \ell$. □

Theorem 2.5. *Let X be an infinite dimensional normed space. Then there exists $A \subseteq X$, A infinite, countable and linearly independent set such that, for every $M, N \subseteq A, M \cap N = \emptyset$, it is $\overline{\operatorname{sp} M} \cap \overline{\operatorname{sp} N} = \{0\}$.*

Proof. It is enough to use, with respect to an infinite dimensional closed separable subspace Y of X, metrizability and compactness of $\overline{S_{Y'}(0,1)}$ with respect to the weak $*$ topology (see, e.g., [7, proof of Theorem V.5.1], [8, Theorem III.10.2]), and [9, proof of Proposition 1.f.3]. □

3. New Weaker Concepts of Sequential Lower Semicontinuity

Definitions 3.1. Let (X,τ) be a topological space. Let f be a function, $f : X \to [-\infty,+\infty]$. Then, if f is not $+\infty$ identically, f is said to be

(i) *below sequentially lower semicontinuous from above at* $x \in X$ ("*bd-slsc* at x") if there exists $a_x \in]\inf_X f,+\infty]$ such that: $(x_n)_{n\in\mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $f(x_n) \searrow \lim_{n\to+\infty} f(x_n) \le a_x$, implies $f(x) \le \lim_{n\to+\infty} f(x_n)$;

(ii) *below sequentially lower semicontinuous from above* ("*bd-slsc*") if it is *bd-slsc* at x for every $x \in X$;

(iii) *uniformly below sequentially lower semicontinuous from above* ("*ubd-slsc*") if there exists $a \in]\inf_X f,+\infty]$ such that: $x \in X$, $(x_n)_{n\in\mathbb{N}}$ sequence of elements of X for which $x_n \to x$ and $f(x_n) \searrow \lim_{n\to+\infty} f(x_n) \le a$, imply $f(x) \le \lim_{n\to+\infty} f(x_n)$.

When f has value $+\infty$ constantly, all these properties are assumed to hold in a vacuous way.

In the above definitions, equivalently, we can replace the part "$f(x_n) \searrow \lim_{n\to+\infty} f(x_n) \le a_x$ (resp., $\le a$)" with the following: "$(f(x_n))_{n\in\mathbb{N}}$ weakly decreasing sequence and $f(x_n) \le a_x$ (resp., $\le a$) for every $n \in \mathbb{N}$." Indeed one of the two implications is obvious (in both cases) and the other can be proved, for example in the case of (i) (being the other case similar), in this way: if the above variant of definition (i) is verified relatively to a certain value of a_x and if b_x is such that $\inf f < b_x < a_x$, then, for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X for which $x_n \to x$ and $f(x_n) \searrow \lim_{n\to+\infty} f(x_n) \le b_x$, there exists $k \in \mathbb{N}$ such that $f(x_n) \le a_x$ for every $n > k$ and so we conclude, applying such variant to the sequence $(x_n)_{n>k}$.

Remarks 3.2. Here we will note some easy comparison between Definitions 3.1 and other previously considered generalizations of sequential lower semicontinuity (Definitions 1.1 and 1.2).

Let (X,τ) be a topological space. Let $x \in X$. Let f be a function, $f : X \to [-\infty,+\infty]$. We note that

(a) if f is *d-slsc* at x, then f is *bd-slsc* at x;

(b) if f is *bd-slsc* at x, then f is *i-slsc* at x, because if f is not constantly $+\infty$, if $x_n \to x$ and $f(x_n) \searrow \inf f$, then $\lim_{n\to+\infty} f(x_n) \le a_x$ and $f(x) \le \inf f$;

(c) if f is *d-slsc*, then f is *ubd-slsc*, because if f is not constantly $+\infty$ it follows that (iii) of Definitions 3.1 is true with respect to an arbitrary value of $a > \inf f$;

(d) if f is *ubd-slsc*, then f is *bd-slsc*;

(e) if f is *bd-slsc*, then f is *i-slsc*, for (b);

(f) "f is *ubd-slsc* and (iii) of Definitions 3.1 is verified with respect to $a = +\infty$" if and only if f is *d-slsc*.

Remarks 3.3. Let(X,τ) be a topological space. Let $x \in X$. Let f be a function, $f : X \to [-\infty,+\infty]$. We observe that:

(a) if $\liminf_{y\to x} f(y) > \inf_X f$, then f is *bd-slsc* at x, because it suffices to consider $a_x \in]\inf_X f, \liminf_{y\to x} f(y)[$;

(b) for verifying that f satisfies *ubd-slsc* with respect to a certain $a > \inf_X f$ (a as in (iii) of Definitions 3.1), it is enough to prove that $f_{|\{z\in X : \liminf_{y\to z} f(y)\le a\}}$ is *ubd-slsc*.

Theorem 3.4. *Let (X,τ) be a topological space satisfying the first axiom of countability. Let $x \in X$. Let f be a function, $f : X \to [-\infty,+\infty]$. The following implications are true:*

(a) *if f is i-slsc at x, then f is bd-slsc at x;*

(b) *if f is i-slsc, then f is bd-slsc (so, under hypothesis of fulfillment of the first axiom of countability, a vice-versa of part (e) of Remarks 3.2 is true).*

Proof. It is sufficient to show the part (a) of the theorem.

By absurd, we suppose that f is not *bd-slsc* at x; then f is not $+\infty$ constantly and, choosing a sequence $(a_k)_{k\in\mathbb{N}}$, with $a_k > \inf f$ for every $k \in \mathbb{N}$, such that $a_k \to \inf f$, for every $k \in \mathbb{N}$ there exists a sequence $(x_{n,k})_{n\in\mathbb{N}}$ of elements of X for which $x_{n,k} \to x$, $(f(x_{n,k}))_{n\in\mathbb{N}}$ is a weakly decreasing sequence with $\lim_{n\to+\infty} f(x_{n,k}) \le a_k$ and $f(x) > \lim_{n\to+\infty} f(x_{n,k})$; moreover, from the hypothesis, we deduce that $\inf f < \lim_{n\to+\infty} f(x_{n,k})$ for every $k \in \mathbb{N}$. On the other hand, since τ verifies the first axiom of countability, there exists $\{U_h : h \in \mathbb{N}\}$ base for the neighbourhood system of x, with $U_{h+1} \subseteq U_h$ for every $h \in \mathbb{N}$.

Now we will define a sequence $(x_n)_{n\in\mathbb{N}}$ by means of which we will produce a contradiction. Let $x_0 = x_{n_0,0}$, where $n_0 = \min\{n \in \mathbb{N} : x_{n,0} \in U_0,\ \inf f < f(x_{n,0}) < a_0 + 1\}$ and, for $m \in \mathbb{N}$, chosen $x_0,\dots,x_m$ with $\inf f < f(x_h)$ $(h \in \{0,\dots,m\})$, let $x_{m+1} = x_{n_{m+1},k(m+1)}$, where $k(m+1) = \min\{k \in \mathbb{N} : k \ge m+1,\ a_k < f(x_m)\}$ and $n_{m+1} = \min\{n \in \mathbb{N} : x_{n,k(m+1)} \in U_{m+1}, \inf f < f(x_{n,k(\mathrm{m}+1)}) < a_{k(m+1)} + 1/(m+2),\ f(x_{n,k(m+1)}) \le f(x_m)\}$ (these choices are possible, because $f(x_{n,k}) \ge \lim_{p\to+\infty} f(x_{p,k}) > \inf f$ for every $n,\ k \in \mathbb{N}$). For construction $x_n \to x$, $f(x_n) \searrow \inf f$; furthermore $f(x) > \inf f$, because $f(x) > \lim_{n\to+\infty} f(x_{n,k}) > \inf f$ $(k \in \mathbb{N})$; but these facts are in contradiction with the *i-slsc* of f at x. □

Theorem 3.5. *Let X be a topological linear space and let $f : X \to [0,+\infty]$ be a function such that*

$$f(\alpha x) = \alpha f(x) \quad \text{for every } \alpha \in [0,+\infty[\text{ and for every } \ x \in X. \tag{3.1}$$

Suppose that f is ubd-slsc. Then f is slsc too. Therefore, for such a function f, ubd-slsc, d-slsc, and slsc are each other equivalent conditions (see part (c) of Remarks 3.2).

Proof. Let a be relative to the *ubd-slsc* of f as in (iii) of Definitions 3.1. Then, since $\inf_X f = f(0) = 0$, we get that $a > 0$.

Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X such that $x_n \to x$; let $\alpha := \liminf_{n\to+\infty} f(x_n)$. We will conclude if we will prove that $f(x) \le \alpha$.

Now, since the desired conclusion is obvious if $\alpha = +\infty$, it is enough that we distinguish two cases:

(i) $\alpha = 0$,

(ii) $\alpha \in \mathbb{R}_+$.

In the case (i), being $\alpha = 0 = \inf f$, then there exists a subsequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ of $(f(x_n))_{n\in\mathbb{N}}$ constantly 0 or strictly decreasing to 0: anyhow such subsequence is weakly decreasing and converging to $0 < a$; so, applying *ubd-slsc* of f, the desired result follows.

In the second case, (ii), let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{k\to+\infty} f(x_{n_k}) = \alpha$; let $k_0 \in \mathbb{N}$ be such that $f(x_{n_k}) \in \mathbb{R}_+$ for every $k > k_0$ and let $y_k := ax_{n_k}/f(x_{n_k})$ for every $k \in \mathbb{N}, k > k_0$; then, being X a topological linear space, it holds that $y_k \to ax/\alpha$, moreover $f(y_k) = (a/f(x_{n_k}))f(x_{n_k}) = a$ for every $k > k_0$; so, using *ubd-slsc* of f, we get $f(ax/\alpha) \le a$ whence $f(x) = (\alpha/a)f(ax/\alpha) \le \alpha$. □

Remark 3.6. Note that, if X is a topological linear space, if $A \subsetneq X$ is an absorbing set and if $f = g_A$, then (3.1) is true (see [5, Theorem 1.2.4 (i) and definition foregoing]).

Examples 3.7. (a)There exist a bounded function $g\colon \mathbb{R} \to \mathbb{R}$ and a point $z \in \mathbb{R}$ such that g is *ubd-slsc*, but g is not *d-slsc* at z.

(b)There exists a bounded function $h : \mathbb{R} \to \mathbb{R}$ such that h is *bd-slsc*, but h is not *ubd-slsc.*

(c) Let $W = \ell^2$ (on the field $\mathbb{F}$) endowed with its weak topology. Then there exist a function $k : W \to [0,1]$ and a point $w \in W$ such that k is *i-slsc*, but k is not *bd-slsc* at w (note that an analogue example on a topological space satisfying the first axiom of countability does not exist, in consequence of Theorem 3.4).

(a) Let

$$g(x) = \begin{cases} \operatorname{arctg} x, & \text{if } x \in]-\infty, 0], \\ -1, & \text{if } x \in]0, +\infty[, \end{cases} \tag{3.2}$$

and $z = 0$. Then g is *lsc* in every point of $\mathbb{R} \setminus \{z\}$; therefore g is *ubd-slsc*: indeed it suffices to consider $a \in\ ]-\pi/2, -1[$ and to use (b) of Remarks 3.3. Besides g is not *d-slsc* at z, because if $z_n := 1/(n+1) = z + 1/(n+1)$ for every $n \in \mathbb{N}$ we get that $z_n \to z$, $(g(z_n))_{n\in\mathbb{N}}$ is a weakly decreasing sequence, but $g(z) = 0 > -1 = \lim_{n\to+\infty} g(z_n)$.

(b) Let

$$h(x) = \begin{cases} \operatorname{arctg} x, & \text{if } -2m-1 < x \le -2m \quad (m \in \mathbb{N}), \\ 0, & \text{if } -2m-2 < x \le -2m-1 \quad (m \in \mathbb{N}), \\ 0, & \text{if } x > 0. \end{cases} \tag{3.3}$$

Then h is *lsc* in every point of $\mathbb{R} \setminus \{-2m-1 : m \in \mathbb{N}\}$; moreover it is *bd-slsc*, as for every $m \in \mathbb{N}$ it is sufficient to note that $\liminf_{x\to-2m-1} h(x) = \operatorname{arctg}(-2m-1) > -\pi/2 = \inf_{\mathbb{R}} h$ and to use (a) of Remarks 3.3. On the other hand h is not *ubd-slsc*, because for every $a > \inf_{\mathbb{R}} h = -\pi/2$ there exist $m_a \in \mathbb{N}$ and a sequence $(y_{n,a})_{n\in\mathbb{N}}$ of real numbers for which $\lim_{n\to+\infty} y_{n,a} = -2m_a - 1$ and $h(y_{n,a}) \searrow \ell_a \le a$, but $h(-2m_a - 1) = 0 > \ell_a$ (in fact it is enough to consider m_a such that $\operatorname{arctg}(-2m_a - 1) < a$ and $y_{n,a} = -2m_a - 1 + 1/(n+1)$ for every $n \in \mathbb{N}$).

(c) Let

$$k(x) = \begin{cases} \dfrac{1}{E(|\lambda|)+1} & \text{if } x = \lambda e_m \text{ for some } m \in \mathbb{N} \text{ and } \lambda \in \mathbb{F} \setminus \mathbb{Z}, \\ \dfrac{1}{|\lambda|} & \text{if } x = \lambda e_m \text{ for some } m \in \mathbb{N} \text{ and } \lambda \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if } x \in \{0\} \bigcup \left(W \setminus \bigcup_{m \in \mathbb{N}} \operatorname{sp}\{e_m\} \right), \end{cases} \tag{3.4}$$

where $e_n := (\delta_{n,m})_{m\in\mathbb{N}}$ for every $n \in \mathbb{N}$, and let $w = 0$. We get that k is *i-slsc*, as there does not exist a converging sequence $(w_n)_{n\in\mathbb{N}}$ of elements of W which satisfies

$$\lim_{n\to+\infty} k(w_n) = 0 = \inf_W k : \tag{3.5}$$

in fact, if a sequence $(w_n)_{n\in\mathbb{N}}$ verifies (3.5), then it follows that there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ there are $\lambda_n \in \mathbb{F}$ and $m_n \in \mathbb{N}$ for which $w_n = \lambda_n e_{m_n}$ and $|\lambda_n| \to +\infty$, hence $|w_n|_{\ell^2} \to +\infty$, from here the sequence is unbounded and therefore cannot converge. On the other hand k is not *bd-slsc* at w, because for every $a > \inf_W k = 0$ there exists a sequence $(z_{n,a})_{n\in\mathbb{N}}$ of elements of W for which $\lim_{n\to+\infty} z_{n,a} = 0 = w$ and $k(z_{n,a}) \searrow \ell_a \le a$, but $k(0) = 1 > \ell_a$ (indeed it is sufficient to consider $s_a \in \mathbb{N}$ such that $s_a > 1, 1/s_a < a$ and $z_{n,a} = s_a e_n$ for every $n \in \mathbb{N}$).

Example 3.8. It is possible to find an example of a Hilbert space Y and of a function that is *lsc* with respect to the topology induced on Y by its norm, but that, with respect to the weak topology on Y, is not *bd-slsc* (and so it is neither *ubd-slsc* nor *d-slsc*; indeed neither *i-slsc):* see [4, Example 4.1] and Remarks 3.2.

4. Behavior of Some Weak Concepts of Sequential Lower Semicontinuity with Respect to the Convexity

With respect to [4, Theorem 4.1], a slightly more laborious demonstration allows to get a stronger result.

Theorem 4.1. *Let X be a normed space, let $f : X \to [-\infty, +\infty]$ be a convex, i-slsc function with respect to the topology induced on X by its norm. Then f is bd-slsc with respect to the weak topology on X.*

Proof. By absurd, we suppose that f is not *bd-slsc* with respect to the weak topology on X. Then f is not $+\infty$ constantly and there exists at least a point $x \in X$ such that for every $a > \inf f$ there exists a sequence $(y_{n,a})_{n\in\mathbb{N}}$ of elements of X for which $y_{n,a} \rightharpoonup x$, $(f(y_{n,a}))_{n\in\mathbb{N}}$ is a weakly decreasing sequence, $f(y_{n,a}) \le a$ for every $n \in \mathbb{N}$ and $f(x) > \lim_{n\to+\infty} f(y_{n,a})$. Hence $f(x) > \inf f$ and we will reach an absurd if we will show that

$$\begin{gathered} \text{there exists a sequence } (x_n)_{n\in\mathbb{N}} \text{ of elements of } X \\ \text{for which } x_n \longrightarrow x \text{ and } f(x_n) \searrow \inf f, \end{gathered} \tag{4.1}$$

because, using the *i-slsc* of f at x with respect to the topology induced on X by its norm, relatively to such a sequence, it should be $f(x) = \inf f$. At first, in order to prove (4.1), we will define a sequence $(b_k)_{k\in\mathbb{N}}$ of real numbers and by induction we will define other four sequences, $(a_h)_{h\in\mathbb{N}}$ of real numbers, $(k_h)_{h\in\mathbb{N}}$ of natural numbers, $(j_h)_{h\in\mathbb{N}}$ of integer numbers and $(z_h)_{h\in\mathbb{N}}$ of elements of X such that $j_h \in \{0,\dots,h\}$ and $f(z_{j_h}) > \inf f$ if $f(z_m) > \inf f$ for some $m \in \{0,\dots,h\}(h \in \mathbb{N})$, by means of the followings:

$$b_k := \begin{cases} \inf f + \dfrac{1}{k+1} & \text{if} \inf f \in \mathbb{R} \\ -k & \text{if } \inf f = -\infty \end{cases} \quad \text{for every } k \in \mathbb{N},$$

$$k_0 := 0, \qquad a_0 := b_{k_0}, \quad z_0 \in \operatorname{co}\{y_{n,a_0} : n \in \mathbb{N}\} \quad \text{such that } |z_0 - x|_X < 1, \tag{4.2}$$

$$j_0 := \begin{cases} -1 & \text{if } f(z_0) = \inf f \\ 0 & \text{if } f(z_0) > \inf f \end{cases}$$

and, if $h \in \mathbb{N}$, defined k_p, a_p, z_p and j_p for every $p \in \{0,\dots,h\}$, we define $k_{h+1}, a_{h+1}, z_{h+1}$ and j_{h+1} in the following way:

$$k_{h+1} := \begin{cases} k_h + 1 & \text{if } f(z_0) = \cdots = f(z_h) = \inf f, \\ \min\{k \in \mathbb{N} : k > k_h,\ b_k \le f(z_{j_h})\} & \text{if } f(z_m) > \inf f \text{ for some } m \in \{0,\dots,h\}, \end{cases}$$

$$a_{h+1} := b_{k_{h+1}}, \quad z_{h+1} \in \operatorname{co}\{y_{n,a_{h+1}} : n \in \mathbb{N}\} \quad \text{such that } |z_{h+1} - x|_X < \frac{1}{h+2},$$

$$j_{h+1} := \begin{cases} -1, & \text{if } f(z_0) = \cdots = f(z_{h+1}) = \inf f, \\ \max\{j \in \{0,\dots,h+1\} : f(z_j) > \inf f\}, & \text{if } f(z_m) > \inf f \\ & \text{for some } m \in \{0,\dots,h+1\}. \end{cases} \tag{4.3}$$

(such definitions are possible, being $x \in \overline{\operatorname{co}\{y_{n,a} : n \in \mathbb{N}\}}$ for every $a > \inf f$ (as a convex closed subset of X is weakly closed too) and thanks to choice's axiom). By definition, $(b_k)_{k\in\mathbb{N}}$ is strictly decreasing, with $\lim_{k\to+\infty} b_k = \inf f$, $(k_h)_{h\in\mathbb{N}}$ is strictly increasing and so $(a_h)_{h\in\mathbb{N}}$ is a subsequence of $(b_k)_{k\in\mathbb{N}}$ and therefore is itself strictly decreasing, with $\lim_{h\to+\infty} a_h = \inf f$, $\lim_{h\to+\infty} z_h = x$ with respect to the topology induced on X by its norm; moreover $(j_h)_{h\in\mathbb{N}}$ is weakly increasing, $j(\mathbb{N}) \setminus \{-1\} = \{n \in \mathbb{N} : f(z_n) > \inf f\}$ and hence, if $j(\mathbb{N})$ is a finite set, there exists a subsequence of $(z_h)_{h\in\mathbb{N}}$ such that the values of f in the elements of such subsequence are constantly $\inf f$.

Now we will distinguish two cases:

(i) there exists a subsequence of $(z_h)_{h\in\mathbb{N}}$ such that the values of f in the elements of such subsequence are constantly $\inf f$,

(ii) there does not exist a subsequence as in (i).

In the case (i) we at once conclude, because, if $(x_n)_{n\in\mathbb{N}}$ is a subsequence of $(z_h)_{h\in\mathbb{N}}$ as in (i), then it verifies (4.1).

If we are in the case (ii), then $j(\mathbb{N})$ is an infinite set; therefore by induction we can define a new sequence $(m_n)_{n\in\mathbb{N}}$ in this way: let $m_0 := \min(j(\mathbb{N}) \setminus \{-1\})$ and, if $n \in \mathbb{N}$, defined m_p for every $p \in \{0,\ldots,n\}$, let $m_{n+1} := \min(j(\mathbb{N}) \setminus (\{-1\} \cup \{m_0,\ldots,m_n\}))$; from here, defining $x_n := z_{m_n}$ for every $n \in \mathbb{N}$ and taking into account the definition of j, it follows that $(x_n)_{n\in\mathbb{N}}$ is the subsequence of $(z_h)_{h\in\mathbb{N}}$ whose elements are exactly all the "z_h" such that $f(z_h) > \inf f$.

It will be enough to prove that $f(x_n) \searrow \inf f$, because in such way we will have proved (4.1).

For every $n \in \mathbb{N}$ it holds:

$$f(x_{n+1}) = f(z_{m_{n+1}}) \leq a_{m_{n+1}} = b_{k_{m_{n+1}}} \leq f\left(z_{j_{m_{n+1}-1}}\right) = f(z_{m_n}) = f(x_n) \tag{4.4}$$

(where for obtaining the inequality between second and third terms we used that $z_{m_{n+1}} \in \operatorname{co}\{y_{p,a_{m_{n+1}}} : p \in \mathbb{N}\}$ by definition and for deducing the equality between fifth and sixth terms we used that

$$j_{m_{n+1}-1} = \max\left\{j \in \{0,\ldots,m_{n+1}-1\} : f\left(z_j\right) > \inf f\right\} = m_n \tag{4.5}$$

by definition).

At last and analogously as seen above, it is verified:

$$f(x_n) = f(z_{m_n}) \leq a_{m_n} \quad \text{for every } n \in \mathbb{N}; \tag{4.6}$$

besides $\lim_{n\to+\infty} a_{m_n} = \inf f$, as $(a_{m_n})_{n\in\mathbb{N}}$ is a subsequence of $(b_k)_{k\in\mathbb{N}}$ and so we conclude. □

Corollary 4.2. *Let X be a normed space, let $f : X \to [-\infty,+\infty]$ be a convex function, i-slsc with respect to the weak topology on X; then f is bd-slsc with respect to the same topology (so, in the present case, by the help of hypothesis of convexity, the conclusion of Theorem 3.4 is true, although the first axiom of countability may be not fulfilled).*

Proof. From the easy observation that if τ and σ are two topologies on a set Y, with $\sigma \subseteq \tau$, and if a function verifies Definition 1.2 with respect to σ then it verifies the same condition with respect to τ too, we get that f is *i-slsc* with respect to the topology induced on X by its norm and so it is sufficient to use Theorem 4.1. □

Examples 4.3. As it will be seen below, in examples (a) and (b), there exist examples of convex functions, satisfying "the same semicontinuity conditions" of Examples 3.7 (a) and (b), but that are not upperly bounded and whose values are in $]-\infty,+\infty]$ instead of in $\mathbb{R}$; on the contrary, owing to Corollary 4.2, if W is as in (c) of Examples 3.7, an example of convex function $k : W \to [-\infty,+\infty]$ *i-slsc* but not *bd-slsc* cannot exist.

(a) For every X normed space having, as real space, algebraic dimension greater or equal to 2 and such that

$$\text{the closed unitary sphere of } X \text{ admits at least one extreme point,} \tag{4.7}$$

there exist a convex function $g : X \to [0,+\infty]$ and a point $z \in X$ such that g is *ubd-slsc*, but g is not *d-slsc* at z.

(b) For every X normed space having, as real space, algebraic dimension greater or equal to 2 there exists a convex function $h : X \to [0,+\infty]$ such that h is *bd-slsc*, but h is not *ubd-slsc*.

(a) Let $z \in \partial S_X(0,1)$ be an extreme point of $\overline{S_X(0,1)}$ (such a point exists for (4.7)) and let

$$g(x) = \begin{cases} |x|_X, & \text{if } x \in \overline{S_X\ (0,1)} \setminus \{z\}, \\ +\infty, & \text{if } x \in \{z\} \bigcup \left(X \setminus \overline{S_X(0,1)}\right) \end{cases} \tag{4.8}$$

Then g is convex, because $\overline{S_X(0,1)} \setminus \{z\}$ is a convex set (being z an extreme point of $\overline{S_X(0,1)}$); moreover g is *lsc* in every point of $X \setminus \{z\}$; therefore g is *ubd-slsc*: indeed it suffices to consider $a \in\ ]0,1[$ and to use (b) of Remarks 3.3. Besides g is not *d-slsc* at z, because if $z_n \in \partial S_X(0,1) \setminus \{z\}(n \in \mathbb{N})$ is chosen in such a way as to converge to z (this choice is possible for hypothesis, using Theorem 2.3) we get that $(g(z_n))_{n\in\mathbb{N}}$ is a weakly decreasing sequence (it is constantly 1), but $g(z) = +\infty > 1 = \lim_{n\to+\infty} g(z_n)$.

(b) Let $y \in \partial S_X(0,1)$ and let

$$h(x) = \begin{cases} |x|_X, & \text{if } x \in \{0\} \bigcup S_X(y,1), \\ +\infty, & \text{if } x \in X \setminus \left(\{0\} \bigcup S_X(y,1)\right). \end{cases} \tag{4.9}$$

Then h is convex, because $\{0\} \cup S_X(y,1)$ is a convex set; furthermore h is *lsc* in every point of $\{0\} \cup (X \setminus \partial S_X(y,1))$; moreover it is *bd-slsc*, because for every $x \in \partial S_X(y,1) \setminus \{0\}$ it is enough to note that $\liminf_{z\to x} h(z) = |x|_X > 0 = \inf_X h$ and to use (a) of Remarks 3.3. On the other hand h is not *ubd-slsc*, because for every $a > \inf_X h = 0$ there exist $x_a \in X$ and a sequence $(y_{n,a})_{n\in\mathbb{N}}$ of elements of X for which $\lim_{n\to+\infty} y_{n,a} = x_a, h(y_{n,a}) \searrow \ell_a \le a$, but $h(x_a) > \ell_a$: in fact, being $\partial S_X(y,1)$ a connected set owing to Theorem 2.3 and, chosen $0 < b_a < min\{a,2\}$, being $\partial S_X(y,1) \cap (X \setminus \overline{S_X(0,b_a)})$ a nonempty open of $\partial S_X(y,1)$ (it contains $2y$), the nonempty subset $\partial S_X(y,1) \cap \overline{S_X(0,b_a)}$ of $\partial S_X(y,1)$ (it contains 0) must have at least a not interior point x_a (with respect to $\partial S_X(y,1)$); so there exists a sequence $(z_{n,a})_{n\in\mathbb{N}}$ of elements of $\partial S_X(y,1)$ with $\lim_{n\to+\infty} z_{n,a} = x_a, |z_{n,a}|_X > b_a \ge |x_a|_X$ for every $n \in \mathbb{N}$, and it is enough to consider $y_{0,a} \in S_X(y,1)$ such that $|y_{0,a} - z_{0,a}|_X < 1$, $|y_{0,a}|_X > |x_a|_X$ and, if $n \in \mathbb{N}$, given $y_{n,a}$ (with $|y_{n,a}|_X > |x_a|_X$) to select $h_n > n$ such that $|z_{h_n,a}|_X < |y_{n,a}|_X$ and $y_{n+1,a} \in S_X(y,1)$ such that $|y_{n+1,a} - z_{h_n,a}|_X < 1/(n+2), |x_a|_X < |y_{n+1,a}|_X < |y_{n,a}|_X$; with these choices, it results that $\lim_{n\to+\infty} y_{n,a} = x_a$, $(h(y_{n,a}))_{n\in\mathbb{N}} = (|y_{n,a}|_X)_{n\in\mathbb{N}}$ is a strictly decreasing sequence with $\lim_{n\to+\infty} h(y_{n,a}) = |x_a|_X \le b_a < a$, but $h(x_a) = +\infty > |x_a|_X = \lim_{n\to+\infty} h(y_{n,a})$.

Remark 4.4. Note that hypothesis (4.7) done in (a) of Examples 4.3 is verified for example by every reflexive Banach space (see [5, Theorem 2.4.5] and use the weak topology), but there exist Banach spaces (e.g., c_0 and $L^1([a,b])(a,\ b \in \mathbb{R},\ a \neq b)$) which do not satisfy it (see [10, Examples II.8]).

Remark 4.5. For $X = \mathbb{R}$ and also if functions with values in $[-\infty, +\infty]$ are considered, examples verifying conditions of (a) or (b) of Examples 4.3 do not exist, because the following fact is true.

If $f : \mathbb{R} \to [-\infty, +\infty]$ is a convex, *i-slsc* function, then f is *d-slsc* (and so, in this case, taking into account parts (c), (d) and (e) of Remarks 3.2, under hypothesis of convexity, the four conditions of *d-slsc*, *ubd-slsc*, *bd-slsc* and *i-slsc* are each other equivalent).

Let $x,\ x_n \in \mathbb{R}$ $(n \in \mathbb{N}), x_n \to x,\ f(x_n) \searrow \ell$; then we will conclude, if we will prove that $f(x) \leq \ell$. It is not restrictive to suppose $\ell < +\infty$, $x_n \neq x$, $x_n \in \operatorname{dom} f (n \in \mathbb{N})$ and x an extreme of dom f (otherwise, if $x \in (\operatorname{dom} f)^\circ$, the function f should be continuous in x); applying Lemma 2.4 to $f_{|\operatorname{dom} f}$ it follows that $\ell = \inf\{f(y) : y \in \operatorname{dom} f \setminus \{x\}\}$; then, if by absurd $f(x) > \ell$, it should be $\ell = \inf f$, that contradicts *i-slsc* of f.

Example 4.6. For every infinite dimensional X normed space there exists a convex *bd-slsc* function $g : X \to \mathbb{R}$ that is discontinuous in every point of X (and therefore, if X is complete, is neither a *lsc* function on X (see, e.g., [5, Theorem 3.1.9])).

Indeed, we will show that such a function g can be chosen as whatever a Minkowski functional of

$$\begin{gathered}\text{an absorbing balanced convex subset } C \text{ of } X \text{ such that} \\ 0 \text{ is not an interior point for } C,\ C \subseteq \overline{S_X(0,1)}\end{gathered} \tag{4.10}$$

(for the existence of such a set, see Example 2.2). Consequently, if X is a Banach space, using Theorem 3.5, Remark 3.6 and part (c) of Remarks 3.2, such a function g neither is *ubd-slsc* nor is *d-slsc*.

Let C be as in (4.10) and let $g := g_C$. Then g is a real valued convex not continuous function (see, e.g., [8, Theorems II.12.1 and II.12.3, and foregoing Definition]); hence, using a classical result of convex analysis (see, e.g., [5, Theorem 3.1.8]) g is discontinuous in every point of X.

Owing to part (b) of Theorem 3.4, for showing the remaining condition on g, namely the *bd-slsc* of g, it suffices to prove that g is *i-slsc*.

At first, note that

$$|x|_X \leq g(x) \quad \text{for every } x \in X. \tag{4.11}$$

In fact, being $C \subseteq \overline{S_X(0,1)}$, for every $x \in X$ it holds

$$|x|_X = \inf\left\{\alpha \in \mathbb{R}_+ : \left|\frac{x}{\alpha}\right|_X \leq 1\right\} \leq \inf\left\{\alpha \in \mathbb{R}_+ : \frac{x}{\alpha} \in C\right\} = g(x). \tag{4.12}$$

Now let $x \in X$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X for which $x_n \to x$ and $\lim_{n\to+\infty} g(x_n) = 0$ $(= g(0) = \inf g)$; therefore, using (4.11), we get that $x = 0$, whence $g(x) = g(0) = 0$.

Remarks 4.7. (a) Example 4.6 shows that the classical result "every f convex *lsc* function defined on a Banach space with values in $[-\infty, +\infty]$ is continuous on the interior of dom f"

(see, e.g., [11, Theorem 2.2.20 (b) and sentences afterward Proposition 1.1.11]) fails if *lsc* is replaced by *bd-slsc* (or equivalently (see (e) of Remarks 3.2 and Theorem 3.4) by *i-slsc*).

(b) On the contrary, the above-mentioned classical result is still true if *lsc* is replaced by *d-slsc* (it is enough to use [4, Theorem 3.2], and a classical result of convex analysis (see, e.g., [5, Theorem 3.1.9]) applied to $f_{|(\mathrm{dom}\ f)^\circ}$, also considering that, if there exists a point $x_0 \in X$ such that $f(x_0) = -\infty$, then $f(x) = -\infty$ for every $x \in (\mathrm{dom}\ f)^\circ$ (see, e.g., [11, Proposition 2.1.4])), or by *ubd-slsc* (as we will show in the part (c) of Theorem 4.10).

Among other things, by means of such facts there is a different way to prove that, if X is a Banach space, the function g of Example 4.6 cannot be either *ubd-slsc* or *d-slsc* (i.e., what already claimed in the last rows of the statement of number 4.6).

(c) The above (a) and (b) show a big difference between convex, *bd-slsc* (or convex, *i-slsc*) functions on Banach spaces on the one hand and convex, *d-slsc* (or convex, *ubd-slsc*) functions on Banach spaces on the other hand; such situation in a certain way renders, in the case in which convex functions on Banach spaces are considered, more meaningful those results, as for example [4, Theorems 5.1, 5.3 and Corollary 5.1] in which the classic hypothesis of "lower semicontinuity" can be replaced by an hypothesis of "*bd-slsc*" (or of "*i-slsc*").

Working with some properties of infinite dimensional Banach spaces and with the choice of the convex set C, we will able to exhibit an example as the following one (that may be regarded in a certain way as a refinement of Example 4.6, because, also if it does not give a stronger conclusion than the one of Example 4.6, it lets to define in a constructive way a sequence of points, by means of which *ubd-slsc* is showed not to be true).

Moreover, with respect to Examples 4.3 (b), observe that in Example 4.8 we get an example of a function defined in a less general space, but having real values; hence the points by means of which we could prove that such a function is not *ubd-slsc*, unlike that in part (b) of Examples 4.3, necessarily are all at the interior of its effective domain.

Example 4.8. For every infinite dimensional X Banach space on $\mathbb{F}$, there exists a convex *bd-slsc* function $g : X \to \mathbb{R}$ that is not *ubd-slsc* (and therefore, for part (c) of Remarks 3.2, is not even a *d-slsc* function).

Indeed, we will show that such a function g can be chosen as the Minkowski functional of a suitable set C satisfying (4.10); then $\inf_X g = 0$ and we will prove that g is not *ubd-slsc*, exhibiting a sequence $(c_q)_{q\in\mathbb{Z}_+}$ of elements of X such that for every $q \in \mathbb{Z}_+$ there exists a sequence $(c_{q,k})_{k\in\mathbb{N}}$ of points of X for which $\lim_{k\to+\infty} c_{q,k} = c_q$, $(g(c_{q,k}))_{k\in\mathbb{N}}$ is a weakly decreasing sequence and $g(c_q) > 1/q = \lim_{k\to+\infty} g(c_{q,k})$ for every $q \in \mathbb{Z}_+$.

We will define the set C by means of the construction of Thorp ([6, Theorem 1]), already cited in the final part of Example 2.2, but choosing a suitable Hamel basis, formed by elements having norm less or equal to 1 and verifying other suitable conditions that we are going to introduce.

For Theorem 2.5 there exists $A \subseteq X$, A infinite, countable and linearly independent set such that

$$M, N \subseteq A,\ M \cap N = \emptyset \Longrightarrow \overline{\mathrm{sp}\, M} \cap \overline{\mathrm{sp}\, N} = \{0\}; \tag{4.13}$$

moreover let $|a|_X = 1$ for every $a \in A$ (this is not a restriction).

Consequently if $\sum_{n\in\mathbb{N}} \alpha_n a_n = 0$ with $\alpha_n \in \mathbb{F}$ and $a_n \in A$ $(n \in \mathbb{N})$, then $\alpha_n = 0$ for every $n \in \mathbb{N}$.

Let $N_q \subseteq A$ $(q \in \mathbb{Z}_+)$ be such that $A = \bigcup_{q\in\mathbb{Z}_+} N_q$, each N_q is an infinite set and $N_q \cap N_p = \emptyset$ $(q,\ p \in \mathbb{Z}_+,\ q \neq p)$. Further on, let $\lambda_n \in \mathbb{R}_+ (n \in \mathbb{N})$ be such that $\sum_{n\in\mathbb{N}} \lambda_n = 1$.

Let $b_q := \sum_{n\in\mathbb{N}} \lambda_n e_{q,n}$ for every $q \in \mathbb{Z}_+$, where $N_q = \{e_{q,n} : n \in \mathbb{N}\}$ with $e_{q,n} \neq e_{q,m}$ if $n,\ m \in \mathbb{N},\ n \neq m$ $(q \in \mathbb{Z}_+)$ (such elements b_q $(q \in \mathbb{Z}_+)$ are existing as X is a Banach space). Then $|b_q|_X \leq 1$ for every $q \in \mathbb{Z}_+$.

Hence $b_q \in \overline{\operatorname{sp} N_q} \setminus \operatorname{sp} N_q$, $\overline{\operatorname{sp} N_q} \cap \overline{\operatorname{sp} N_t} = \{0\}$ if $q, t \in \mathbb{Z}_+,\ q \neq t$ and therefore $b_q \neq b_t$ if $q,\ t \in \mathbb{Z}_+,\ q \neq t$, $A \cap \{b_q : q \in \mathbb{Z}_+\} = \emptyset$ and $A \cup \{b_q : q \in \mathbb{Z}_+\}$ is a linearly independent set of vectors.

Therefore, we can consider B a Hamel basis of X such that $B \supseteq A \cup D$, where $D := \{b_q/q : q \in \mathbb{Z}_+\}$, with $|b|_X = 1$ for every $b \in B \setminus D$, and $C := \operatorname{co}\{\alpha b : \alpha \in \mathbb{F},\ |\alpha| = 1,\ b \in B\}$.

Then C satisfies (4.10), because C is a particular case of the convex set defined, following Theorem 1 of [6], in the final part of Example 2.2.

Let $g := g_C$. Since C verifies (4.10) and g is its Minkowski functional, for the proof already seen in Example 4.6 we get that g is a real valued convex *bd-slsc* function.

At last we will prove that g is not *ubd-slsc*, exhibiting a sequence as described in the statement.

Let

$$c_q := \frac{b_q}{q}, \qquad p(q,k) \in N_q \setminus \{0, \dots, k\},$$
$$b_{q,k} := \sum_{n=0}^{k} \lambda_n e_{q,n} + \sum_{n>k} \lambda_n e_{q,p(q,k)}, \tag{4.14}$$
$$c_{q,k} := \frac{b_{q,k}}{q} \qquad (q \in \mathbb{Z}_+,\ k \in \mathbb{N});$$

then for each $q \in \mathbb{Z}_+,\ k \in \mathbb{N}$ it is $b_{q,k} \in \operatorname{co}\{e_{q,n} : n \in \mathbb{N}\} \subseteq C$, but there does not exist an $\alpha > 1$ such that $\alpha b_{q,k} \in C$ because, being B a linearly independent set of vectors, if $\alpha > 1$ the following one is the only way in which $\alpha b_{q,k}$ can be written as a linear combination of elements of B: $\alpha b_{q,k} = \sum_{n=0}^{k} \alpha\lambda_n e_{q,n} + \sum_{n>k} \alpha\lambda_n e_{q,p(q,k)}$ and, on the other hand, $\sum_{n=0}^{k} \alpha\lambda_n + \sum_{n>k} \alpha\lambda_n = \alpha > 1$; therefore $g(b_{q,k}) = 1$ and, being g positively homogeneous, $g(c_{q,k}) = 1/q$; whence, for each $q \in \mathbb{Z}_+$, the sequence $(g(c_{q,k}))_{k\in\mathbb{N}}$ is constant and therefore weakly decreasing; furthermore $g(c_q) = 1$ (for a demonstration quite similar to the above proof that $g(b_{q,k}) = 1$) and so $g(c_q) = 1 > 1/q = \lim_{k\to+\infty} g(c_{q,k})$, besides $\lim_{k\to+\infty} c_{q,k} = c_q$ and we conclude.

Lemma 4.9. *Let Y be a topological space and let X be a topological linear space. The following facts hold*

(a) *if A is a subset of Y, F is a closed subset of Y and G is an open subset of Y such that $A \cap G = F \cap G$, then $\overline{A} \cap G \subseteq F$ (and therefore $\overline{A} \cap G = F \cap G$);*

(b) *if U is an open subset of X and C is a convex subset of X such that $U \cap \overline{C} \neq \emptyset$ and $\mathring{C} \neq \emptyset$, then $U \cap \mathring{C} \neq \emptyset$;*

(c) *if A is a subset of X, C is a convex subset of X and F is a closed subset of X such that $\mathring{\overline{A}} \cap \overline{C} \neq \emptyset$, $A \cap \mathring{C} = F \cap \mathring{C}$ and $\mathring{C} \neq \emptyset$, then $\mathring{A} \neq \emptyset$.*

Proof. (a) If by absurd $\overline{A}\cap G\cap(Y\setminus F)\neq\emptyset$, then, being $G\cap(Y\setminus F)$ an open, it is $\emptyset\neq A\cap G\cap(Y\setminus F)=F\cap G\cap(Y\setminus F)=\emptyset$, that is a contradiction.

(b) Let $x\in\overset{\circ}{C}$ and $y\in U\cap\overline{C}$. If $x=y$ we have the desired result; otherwise, if $x\neq y$, from [7, demonstration inside of the proof of Theorem V.2.1] it follows that $[x,y]\setminus\{y\}\subseteq\overset{\circ}{C}$, besides from the topological linear structure of X it is $([x,y]\setminus\{y\})\cap U\neq\emptyset$ and so we can conclude.

(c) Applying (b) to $U:=\overset{\circ}{\overline{A}}$, we get $\overset{\circ}{\overline{A}}\cap\overset{\circ}{C}\neq\emptyset$. Besides, from (a) applied with $G:=\overset{\circ}{C}$, we deduce $\overset{\circ}{\overline{A}}\cap\overset{\circ}{C}\subseteq\overline{A}\cap\overset{\circ}{C}\subseteq F$ and so $\overset{\circ}{\overline{A}}\cap\overset{\circ}{C}\subseteq F\cap\overset{\circ}{C}\subseteq A$, namely, $\overset{\circ}{\overline{A}}\cap\overset{\circ}{C}$ is a not empty open set contained in A and we conclude. □

Theorem 4.10. *Let X be a topological linear space and let $f:X\to[-\infty,+\infty]$ be a convex, ubd-slsc function. Suppose that f is not $+\infty$ identically, let $a\in\mathbb{R}$ be such that $a>\inf_X f$ is relative to f as in condition (iii) of Definitions 3.1 and let $A:=\{x\in X: f(x)\leq a\}$. Then*

(a) *the set $A\cap(\mathrm{dom}\, f)^\circ$ is sequentially closed with respect to the relative topology of $(\mathrm{dom}\ f)^\circ$;*

henceforth suppose that $(\mathrm{dom}\, f)^\circ\neq\emptyset$, then also the following facts hold:

(b) *there exists a point $x_0\in\{x\in X: f(x)<a\}\cap(\mathrm{dom}\, f)^\circ$ and therefore $A-x_0$ is absorbing;*

(c) *if X is a Banach space, it results that $\overset{\circ}{A}\neq\emptyset$ and f is continuous in the points of $(\mathrm{dom}\, f)^\circ$.*

Proof. If there exists a point $z_0\in X$ such that $f(z_0)=-\infty$, then $f(x)=-\infty$ for every $x\in(\mathrm{dom}\ f)^\circ$ (see, e.g., [11, Proposition 2.1.4]) and all the parts of the desired result follow also if X is simply a topological linear space (in fact in such case $(\mathrm{dom}\, f)^\circ\subseteq\{x\in X: f(x)<a\}$).

Henceforward we can suppose that $f(z)>-\infty$ for every $z\in X$; with such a further hypothesis, we will prove all the three parts of the desired result.

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $A\cap(\mathrm{dom}\, f)^\circ$ and let $x\in(\mathrm{dom}\, f)^\circ$ be such that $x_n\to x$; let $\alpha=\liminf_{n\to+\infty}f(x_n)$. Then $\alpha\leq a$; moreover there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $f(x_{n_k})\to\alpha$ and there exists a further subsequence $(x_{n_{k_h}})_{h\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that $(f(x_{n_{k_h}}))_{h\in\mathbb{N}}$ is a weakly monotone sequence. Now, if by absurd $a<f(x)$, we get $\alpha<f(x)$ and hence, using Lemma 3.1 of [4], it is not restrictive to suppose that $(f(x_{n_{k_h}}))_{h\in\mathbb{N}}$ is a weakly decreasing sequence; so it is enough to use the *ubd-slsc* of f to obtain a contradiction.

(b) Let $y_0\in(\mathrm{dom}\ f)^\circ$ and let $z\in X$ be such that $f(z)<a$; then $z\in\mathrm{dom}\ f$ and hence, being $\mathrm{dom}\ f$ a convex set, it is $[y_0,z]\subseteq\mathrm{dom}\ f$; therefore $f_{|[y_0,z]}$ is an upper semicontinuous function (see [12, Theorem 10.2]) and so there exists a point $x_0\in[y_0,z]\setminus\{z\}$ such that $f(x_0)<a$; besides, from [7, demonstration inside of the proof of Theorem V.2.1] it follows that $[y_0,z]\setminus\{z\}\subseteq(\mathrm{dom}\ f)^\circ$; consequently $x_0\in\{x\in X: f(x)<a\}\cap(\mathrm{dom}\ f)^\circ$.

Now, if $w\in\{x\in X: f(x)<a\}\cap(\mathrm{dom}\ f)^\circ$, then w is an internal point of A, because, being an interior point of $\mathrm{dom}\ f$, for every $x\in X$ there exists a $\beta_x\in\mathbb{R}_+$ such that $[w,w+\beta_x x]\subseteq(\mathrm{dom}\ f)^\circ$; therefore $f_{|[w,w+\beta_x x]}$ is a continuous function and so, being $f(w)<a$, there exists an $\alpha_x\in\mathbb{R}_+$ such that $f(w+\lambda x)<a$ for all $\lambda\in[0,\alpha_x]$. Then $A-w$ is absorbing.

(c) Now let X be a Banach space. First of all we will prove that $\overset{\circ}{\overline{A}}\neq\emptyset$.

From (b) the set $\overline{A}-x_0$ is absorbing; therefore $X=\bigcup_{n\in\mathbb{Z}_+}n(\overline{A}-x_0)$ and from Baire's lemma there exists a $n\in\mathbb{Z}_+$ such that $(n(\overline{A}-x_0))^\circ\neq\emptyset$, hence $(\overline{A}-x_0)^\circ\neq\emptyset$, wherefore $\overset{\circ}{\overline{A}}\neq\emptyset$.

Now we will prove that part (c) of Lemma 4.9 can be applied with $C := \mathrm{dom}\ f$. Since $A \subseteq \mathrm{dom}\ f$ and $\overset{\circ}{\overline{A}} \neq \emptyset$, we get $\emptyset \neq \overset{\circ}{\overline{A}} \subseteq \overset{\circ}{\overline{A}} \cap \overline{\mathrm{dom}\ f}$. On the other hand, from (a), the set $A \cap (\mathrm{dom}\ f)^{\circ}$ is sequentially closed (and hence closed, satisfying the topology of X the first axiom of countability) with respect to the relative topology of $(\mathrm{dom}\ f)^{\circ}$ and therefore there exists F closed subset of X such that $A \cap (\mathrm{dom}\ f)^{\circ} = F \cap (\mathrm{dom}\ f)^{\circ}$. Consequently, from part (c) of Lemma 4.9, we have that $\overset{\circ}{A} \neq \emptyset$.

Then $\overset{\circ}{A}$ is a not empty open subset of dom f on which f is bounded above (from the real element a) and so we can conclude, applying Theorem 3.1.8 of [5]. □

Remark 4.11. The Example 3.1 of [4] and those cited in Remarks 3.1 of [4] give also examples of Banach spaces Y and of convex, *ubd-slsc*, with respect to the weak topology, functions defined on Y with values in $[0, +\infty]$, that are not *d-slsc* with respect to the weak topology.

In fact it suffices to use (b) of Remarks 3.3 and, using the notations of the above cited examples, to note that $\inf f = 0$ and that, relatively to whatever value of $a \in\ ]0,1[$, it is true that $D \cap \{y \in Y : g(y) \leq a\} = C \cap \{y \in Y : g(y) \leq a\}$ is convex and closed, $f_{|D\cap\{y\in Y:g(y)\leq a\}} = g_{|D\cap\{y\in Y:g(y)\leq a\}}$ and g is continuous (for Example 3.1 of [4]), $D \cap \overline{S_Y(0,a)} = C \cap \overline{S_Y(0,a)}$ is convex and closed, $f_{|D\cap\overline{S_Y(0,a)}} = |\quad|_{Y\,|D\cap\overline{S_Y(0,a)}}$ (for Example cited in (a) of Remarks 3.1 of [4]), $f_{|\overline{S_Y(0,a)}} = |\quad|_{Y\,|\overline{S_Y(0,a)}}$ (for Example cited in (b) of Remarks 3.1 of [4]).

5. Ekeland's and Caristi's Theorems

Remark 5.1. In the following theorem we will show an extension of Ekeland's variational principle (see [13, Theorem 1.1]) to the case in which the hypothesis of *lsc* is replaced by *ubd-slsc*. The proof is inspired by the demonstration of Theorem 2.1 of [1]: since we will add the result (b) (that was considered already in [13, Theorem 1.1]) and for reading convenience, here we are writing the whole proof, removing some trivial mistakes of [1].

The authors thank the referee for having pointed out to them that Theorem 2.1 of [1] can be deduced also from [14, Corollary 4 of Theorem 1] that subsequently was generalized by [15–17]: in fact it is easy to prove that the hypotheses of such corollary are verified if we assume *d-slsc* (but the same we cannot do if we assume *ubd-slsc*).

Moreover we wish here to point out another recent paper ([18]) in which other generalizations of Caristi-Kirk's Fixed Point Theorem and Ekeland's Variational Principle are given, in a different environment with respect to the one of the present paper.

Theorem 5.2. *Let (X, d) be a complete metric space and let $f : X \to\ ]-\infty, +\infty]$ be a bounded from below and not $+\infty$ identically function. Let $\varepsilon > 0$, $\lambda > 0$ and $u \in X$ be such that $f(u) \leq \inf_{x\in X} f(x) + \varepsilon$. Moreover,*

$$\begin{aligned}&\text{suppose that } f \text{ is } \textit{ubd-slsc} \text{ and that } f(u) \leq a,\\ &\text{where } a \text{ is relative to } f \text{ as in (iii) of Definitions 3.1.}\end{aligned} \tag{5.1}$$

Then there exists $v \in X$ such that
(a) $f(v) \leq f(u)$,
(b) $d(u, v) \leq \lambda$,
(c) $f(w) > f(v) - (\varepsilon/\lambda) d(w, v)$ for every $w \in X \setminus \{v\}$.

Proof. Let

$$\varepsilon_0 := a - \inf_X f. \tag{5.2}$$

Since $f(u) \leq a = \inf_X f + \varepsilon_0$ and observing that, if $\varepsilon > \varepsilon_0$, then a point $v \in X$ verifying (a), (b) and (c) with respect to ε_0, λ and u also satisfies (a), (b) and (c) with respect to ε, λ and u, it is not restrictive to suppose $\varepsilon \leq \varepsilon_0$.

For every $z \in X$ let $T_z = \{x \in X : f(x) \leq f(z) - (\varepsilon/\lambda)d(x,z)\}$; then $T_z \neq \emptyset$ and $T_z = \{z\}$ if and only if $f(x) > f(z) - (\varepsilon/\lambda)d(x,z)$ for every $x \in X \setminus \{z\}$.

If $z \in X$ and $s \in T_z$, then $T_s \subseteq T_z$, because if $x \in T_s$ we have $f(x) \leq f(s) - (\varepsilon/\lambda)d(x,s)$; besides $f(s) \leq f(z) - (\varepsilon/\lambda)d(s,z)$ as $s \in T_z$; so $f(x) \leq f(s) - (\varepsilon/\lambda)d(x,s) \leq f(z) - (\varepsilon/\lambda)d(s,z) - (\varepsilon/\lambda)d(x,s) \leq f(z) - (\varepsilon/\lambda)d(x,z)$ and hence $x \in T_z$.

Let now $\mathcal{U} := \{X\} \cup \{T_z : z \in X\}$. Then as a consequence of what was above noted,

$$T_s \subseteq U \quad \text{for every } U \in \mathcal{U},\ s \in U. \tag{5.3}$$

There exists a function $h : \{(U,s) : U \in \mathcal{U},\ s \in U\} \to X$ such that $h(U,s) \in T_s$ and $f(h(U,s)) - \inf_{x \in T_s} f(x) \leq (1/2)(f(s) - \inf_{x \in U} f(x))$ for every $U \in \mathcal{U}$ and $s \in U$: for showing such a fact, if $U \in \mathcal{U}, s \in U$ and if $f(s) = \inf_{x \in U} f(x)$ we choose $h(U,s) = s$ (in such a case we must choose $h(U,s) = s$, because $f(s) \leq f(x) \leq f(s) - (\varepsilon/\lambda)d(x,s)$ for every $x \in T_s$, being $T_s \subseteq U$ on account of (5.3), and so $T_s = \{s\}$), otherwise we use the characterization of greatest lower bound and the choice's axiom.

Then, for definition of T_s, we have that

$$f(h(U,s)) \leq f(s) \quad \text{for every } U \in \mathcal{U},\ s \in U. \tag{5.4}$$

Now we consider two recursive sequences $(x_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ defined by $x_0 = u, S_0 = X$, $x_{n+1} = h(S_n, x_n), S_{n+1} = T_{x_n}$ for every $n \in \mathbb{N}$; then, using (5.4), (5.3) and definition of h, we get

$$f(x_{n+1}) \leq f(x_n) \leq f(u) < +\infty, \qquad S_{n+1} \subseteq S_n \quad \text{for every } n \in \mathbb{N}, \tag{5.5}$$

$$f(x_{n+1}) - \inf_{x \in S_{n+1}} f(x) \leq \frac{1}{2}\Big(f(x_n) - \inf_{x \in S_n} f(x)\Big) \quad \text{for every } n \in \mathbb{N}; \tag{5.6}$$

moreover, being $x_{n+1} \in S_{n+1} = T_{x_n}$, it holds

$$\frac{\varepsilon}{\lambda} d(x_n, x_{n+1}) \leq f(x_n) - f(x_{n+1}) \quad \text{for every } n \in \mathbb{N}, \tag{5.7}$$

whence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, since there exists $\lim_{n \to +\infty} f(x_n) \in \mathbb{R}$ because of (5.5) and being f lower bounded for hypothesis.

Let $v = \lim_{n\to+\infty} x_n$; since f is a *ubd-slsc* function, $f(u) \le \inf_X f + \varepsilon_0 = a$ and using (5.5), we obtain that

$$f(v) \le \lim_{n\to+\infty} f(x_n) \le f(u) \tag{5.8}$$

and so (a) is verified.

From (5.7), using triangular inequality of distance, we deduce

$$\frac{\varepsilon}{\lambda} d(x_0, x_n) \le \sum_{k=1}^{n} \frac{\varepsilon}{\lambda} d(x_{k-1}, x_k) \le \sum_{k=1}^{n} \big(f(x_{k-1}) - f(x_k)\big) = f(x_0) - f(x_n) \tag{5.9}$$

for every $n \in \mathbb{Z}_+$; hence, considering that $x_0 = u$, $v = \lim_{n\to+\infty} x_n$, $f(x_n) \ge \inf_{x\in X} f(x)$ for every $n \in \mathbb{N}$ and by means of a limit passage for $n \to +\infty$ in the first and the last terms, we get

$$\frac{\varepsilon}{\lambda} d(u, v) \le f(u) - \lim_{n\to+\infty} f(x_n) \le \inf_{x\in X} f(x) + \varepsilon - \lim_{n\to+\infty} f(x_n) \le \varepsilon, \tag{5.10}$$

from whence (b) follows.

As an alternative, for showing (b), we can observe that $T_z \subseteq \{x \in X : d(x, z) \le \lambda\}$ for every $z \in X$ such that $f(z) \le \inf_{x\in X} f(x) + \varepsilon$: indeed, in such hypothesis on z, if $y \in T_z$ and if by absurd $d(y, z) > \lambda$ then $f(y) \le f(z) - (\varepsilon/\lambda) d(y, z) < f(z) - \varepsilon \le \inf_{x\in X} f(x)$, that gives a contradiction. Besides, using (5.5), we have that $x_n \in T_u \subseteq \{x \in X : d(x, u) \le \lambda\}$ for every $n \in \mathbb{N}$. Consequently $d(v, u) = \lim_{n\to+\infty} d(x_n, u) \le \lambda$, that is (b).

If, by absurd, (c) is not true, then

$$\text{there exists } x \in X \setminus \{v\} \quad \text{such that } f(x) \le f(v) - \frac{\varepsilon}{\lambda} d(x, v); \tag{5.11}$$

owing to (5.5) and (5.8), it is

$$f(v) \le \lim_{k\to+\infty} f(x_k) \le f(x_m) \le f(x_n) - \frac{\varepsilon}{\lambda} d(x_m, x_n) \quad \text{for every } n,\ m \in \mathbb{N}, m \ge n \tag{5.12}$$

and hence, passing to the limit for $m \to +\infty$ in the first and the fourth terms, we get $f(v) \le f(x_n) - (\varepsilon/\lambda) d(v, x_n)$ for every $n \in \mathbb{N}$; therefore, using (5.11), it holds:

$$f(x) \le f(x_n) - \frac{\varepsilon}{\lambda} d(v, x_n) - \frac{\varepsilon}{\lambda} d(x, v) \le f(x_n) - \frac{\varepsilon}{\lambda} d(x_n, x) \quad \text{for every } n \in \mathbb{N}; \tag{5.13}$$

hence $x \in S_{n+1}$ for every $n \in \mathbb{N}$, from whence

$$f(x) \ge \inf_{S_n} f \quad \text{for every } n \in \mathbb{N}; \tag{5.14}$$

besides, since from (5.5) it follows that $(\inf_{S_n} f)_{n\in\mathbb{N}}$ is a weakly increasing sequence, we can do a limit passage for $n \to +\infty$ in (5.6), obtaining that $0 \le \alpha := \lim_{n\to+\infty} (f(x_n) - \inf_{y\in S_n} f(y)) \le$

$(1/2)\alpha$, from whence $\alpha = 0$; then, using (5.11) and (5.8), we get that $f(x) < f(v) \leq \lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} \inf_{S_n} f$, that is in contradiction with (5.14). □

Example 5.3. In the hypotheses of Theorem 5.2, but without the additional hypothesis $f(u) \leq a$, it is easy to verify that conclusions (a) and (c) are still true (in fact, if $\varepsilon_0 = a - \inf_X f$ and if $f(u) > \inf_X f + \varepsilon_0$, then it is enough to consider $t \in X$ such that $f(t) \leq \inf_X f + \varepsilon_0$ and in such case a point v relative to ε_0, λ and t as in Theorem 5.2 solves the question) but it can happen that there exist ε, λ and u satisfying the remaining hypotheses, for which there is not a point v that verifies conclusions (b) and (c).

On $\mathbb{R}$ we consider the equivalence relation $\sim$ defined by $x \sim y$ if $x - y \in \mathbb{Q} (x,\ y \in \mathbb{R})$ and let $\mathcal{R} := \{A \cap [0,2] : A$ equivalence class with respect to $\sim\}$; then $\#B = \aleph_0$ for every $B \in \mathcal{R}$ and $\cup\mathcal{R} = [0,2]$, hence $\#\mathcal{R} = 2^{\aleph_0}$, so there exists a bijective function $\varphi : \mathcal{R} \to]0,1]$; moreover $\overline{B} = [0,2]$ for every $B \in \mathcal{R}$. Let now $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \operatorname{arctg} x & \text{if } x \in]-\infty, 0[\bigcup]2, +\infty[, \\ \varphi(B) & \text{if } x \in B\ (B \in \mathcal{R}). \end{cases} \tag{5.15}$$

Because f is real-valued and continuous in the points y where $f(y) < 0$, then f is *ubd-slsc*, considering $a = 0 > \inf_{\mathbb{R}} f$; moreover f is bounded from below. Let $\varepsilon = (\pi/2) + 1$, $\lambda = 1$ and $u = 1$; then $f(u) \in \varphi(\mathcal{R}) \leq 1 = \inf_{\mathbb{R}} f + \varepsilon$. If by absurd there exists $v \in \mathbb{R}$ verifying conclusions (b) and (c), then using (b) we get that $|v - 1| \leq 1$ and therefore $v \in [0,2]$; let $B \in \mathcal{R}$ such that $v \in B$. Consequently $1 \geq f(v) > 0$ and, if $\alpha \in]0, f(v)[$, we obtain that $\varphi^{-1}(\alpha) \in \mathcal{R}$, whence $\overline{\varphi^{-1}(\alpha)} = [0,2]$ and so there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\varphi^{-1}(\alpha)$ converging to v; from this, using (c), we obtain that $\alpha = \varphi(\varphi^{-1}(\alpha)) = f(x_n) > f(v) - \varepsilon|x_n - v|$ for every $n \in \mathbb{N}$ and, by means of a limit passage for $n \to +\infty$, we have $\alpha \geq f(v)$, that is a contradiction.

Example 5.4. If in Theorem 5.2 the hypothesis (5.1) is replaced by a hypothesis of *bd-slsc* on f, then, in spite of what we noted at the beginning of Example 5.3, conclusion (c) can be not true. Indeed here we will show an example of a *bd-slsc* function $f : [1/2, +\infty[\to]0,1]$ for which there does not exist $v \in [1/2, +\infty[$ verifying conclusion (c) of Theorem 5.2 with respect to $\varepsilon = \lambda = 1$ (with such a choice the hypothesis $f(u) \leq \inf_{x\in[1/2,+\infty[} f(x) + \varepsilon$ of Theorem 5.2 is satisfied by every $u \in [1/2, +\infty[$), that is for every $v \in [1/2, +\infty[$ there exists $w \in [1/2, +\infty[\setminus \{v\}$ such that $f(w) \leq f(v) - |w - v|$.

Let $B := \bigcup_{n\in\mathbb{Z}_+} (](3n^2 + 5n - 1)/(3(n+2)), n[\cup]n, (3n^2 + 7n + 1)/(3(n+2))[)$ and let

$$f(x) = \begin{cases} \dfrac{1}{n+2} + 3(n - x), & \text{if } x \in \left]\dfrac{3n^2 + 5n - 1}{3(n+2)}, n\right[\quad (n \in Z_+), \\[2ex] \dfrac{1}{n+2} + 3(x - n), & \text{if } x \in \left]n, \dfrac{3n^2 + 7n + 1}{3(n+2)}\right[\quad (n \in Z_+), \\[2ex] 1, & \text{if } x \in \left[\dfrac{1}{2}, +\infty\right[\setminus B \end{cases} \tag{5.16}$$

(i.e., if $g : [1/2,+\infty[\to]0,1]$ is the continuous piecewise affine function, with slope alternatively equal to -3 and 3 in the connected components of B and with value 1 in $[1/2,+\infty[\setminus (B \cup \mathbb{Z}_+)$, f coincides with g on $[1/2,+\infty[\setminus \mathbb{Z}_+$ and has value 1 in the points of $\mathbb{Z}_+$).

Then f is a *bd-slsc* function, by the help of (a) of Remarks 3.3.

On the other hand, now we will show that for every $v \in [1/2,+\infty[$ there exists $w \in [1/2,+\infty[\setminus \{v\}$ such that $f(w) \le f(v) - |w-v|$:

(i) if $v \in B$, it is enough to choose $w \in B$ in the same connected component of v and such that $f(w) < f(v)$, because with such a choice it holds

$$f(w) = f(v) - 3|w-v| < f(v) - |w-v|; \tag{5.17}$$

(ii) if $v \in \mathbb{Z}_+$, then a point w sufficiently close to v solves what is requested, because

$$\lim_{t\to v} f(t) < 1 = \lim_{t\to v}\left(f(v) - |t-v|\right); \tag{5.18}$$

(iii) if $v \in [1/2,+\infty[\setminus (B \cup \mathbb{Z}_+)$, then it is sufficient to consider an element $n \in \mathbb{Z}_+$ such that $|n-v| \le 1/2$ and to choose w in the interval with extremes n and v, w sufficiently close to n, because

$$\lim_{t\to n} f(t) = \frac{1}{n+2} \le \frac{1}{3} < \frac{1}{2} = 1 - \frac{1}{2} \le \lim_{t\to n}\left(f(v) - |t-v|\right). \tag{5.19}$$

Remark 5.5. In the following results we note that Caristi's fixed point theorem (see [19, Theorem (2.1)′]) and Caristi's infinite fixed points theorem can be extended to the case in which the hypothesis of *lsc* is replaced by *ubd-slsc* (see also the extensions given in the case *d-slsc* by [2, Theorem 2.1] and by [1]).

Theorem 5.6 (Caristi's fixed point theorem; see [19, Theorem (2.1)′]). *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}$ be a ubd-slsc and bounded from below function. Let $T : X \to X$ be a function such that $d(x,T(x)) \le \varphi(x) - \varphi(T(x))$ for every $x \in X$. Then there exists $x_0 \in X$ such that $T(x_0) = x_0$.*

Proof. It is enough to repeat the same demonstration of Theorem 2.2 in [1], where Theorem 5.2 has to be used instead of Theorem 2.1 in [1]. □

Theorem 5.7 (Caristi's infinite fixed points theorem). *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}$ be a ubd-slsc and bounded from below function, that does not obtain its infimum on X. Let $T : X \to X$ be a function such that $d(x,T(x)) \le \varphi(x) - \varphi(T(x))$ for every $x \in X$. Then T admits infinite fixed points in X.*

Proof. It is enough to repeat the same demonstration of Theorem 2.3 in [1], where Theorems 5.2 and 5.6 have to be used instead of Theorems 2.1 and 2.2 of [1]. □

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