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Existence of positive solutions for a singular semipositone differential system with nonlocal boundary conditions

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Abstract

In this paper we consider the existence of at least one positive solution to a class of singular semipositone coupled system of nonlocal boundary value problems. We show that the system possesses at least one positive solution by using fixed point index theory. We remark that to some extent our systems and results generalize and extend some previous works.

Keywords: positive solutions; semipositone; nonlocal nonlinear boundary condition; coupled system of boundary value

1 Introduction

In this paper, we consider the existence of at least one positive solution to the following singular semipositone coupled system of nonlocal boundary value problems:

$$\begin{cases} -x'' = f(t, y(t)) + q(t), & t \in (0,1), \\ -y'' = g(t, x(t)), & t \in (0,1), \\ x(0) = H_1(\varphi_1(y)), & x(1) = 0, \\ y(0) = H_2(\varphi_2(x)), & y(1) = 0, \end{cases} \tag{1.1}$$

where $f, g : (0,1) \times [0,+\infty) \to [0,+\infty)$ are continuous and may be singular at $t = 0, 1$, $q : (0,1) \to (-\infty,+\infty)$ is Lebesgue integrable, and $q(t)$ may have finitely many singularities in $[0,1]$, $H_i : \mathbf{R} \to \mathbf{R}$ ($\mathbf{R} = (-\infty,+\infty)$) are continuous, and $H_i([0,+\infty)) \subseteq [0,+\infty)$ and $\varphi_i : C([0,1]) \to \mathbf{R}$ ($i = 1,2$) are linear and can be realized as Stieltjes integrals with signed measures. In particular, in the Stieltjes integral representation $\varphi(y) = \int_{[0,1]} y(t)\,d\alpha(t)$ with $\alpha : [0,1] \to \mathbf{R}$ of bounded variation on $[0,1]$, we no longer assume that α is necessarily monotonically increasing. Thus, in this paper, we allow the map $y \mapsto \varphi(y)$ to be negative even if y is nonnegative.

Recently, the theory of nonlocal and nonlinear boundary value problems and singular semipositone differential systems becomes an important area of investigation because of its wide applicability in control, electrical engineering, physics, chemistry fields, and so on. Equation (1.1) is used to describe chemical reactor theory where the nonlinearity can take negative values. Many works have been done for a kind of nonlinear boundary value problems [1, 2] and nonlinear differential systems [3–6]. However, most investigators only focus on the case where the nonlinearity takes nonnegative values, that is, positone problems. For example, under conditions where $f(t,y)$ and $g(t,x)$ have no any singularities and $q(t) \equiv 0$, Agarwal and O'Regan [6], using the Leray-Schauder fixed point theorem, obtained the existence of positive solutions of the following system:

$$\begin{cases} -x'' = f(t,y(t)) + q(t), & t \in (0,1), \\ -y'' = g(t,x(t)), & t \in (0,1), \\ x(0) = x(1) = 0, \\ \alpha y(0) - \beta y'(0) = \gamma^2 y(1) + \delta y'(1). \end{cases} \tag{1.2}$$

Later, Zhang and Liu [7] obtained the existence of positive solutions of system (1.2) by the Leray-Schauder fixed point theorem under the conditions that $q(t) : (0,1) \to (-\infty,+\infty)$ is Lebesgue integrable, $q(t)$ may have finitely many singularities in $[0,1]$, and $f,g : (0,1) \times [0,+\infty) \to [0,+\infty)$ are continuous and may be singular at $t = 0,1$. The study of semipositone problems has a long history in the literature, the work of Anuradha et al. [8] being an early, classical example. More recent papers include those by Goodrich [9], Graef and Kong [10], and Infante and Webb [11]. Furthermore, recently, there have been many papers on nonlocal BVPs with nonlinear boundary conditions. For example, Anderson [12], Goodrich [9, 13–16], and Infante et al. [17–22]. In this paper, these nonlocal nonlinear boundary conditions have been investigated by Goodrich [13, 14]. For example, in [13], Goodrich investigated the existence of positive solutions of the semipositone boundary value problems with nonlocal nonlinear boundary conditions

$$\begin{cases} -y'' = f(t,y(t)), & t \in (0,1), \\ y(0) = H(\varphi(y)), & y(1) = 0, \end{cases} \tag{1.3}$$

by the fixed point index under the conditions that $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ and $H : \mathbf{R} \to \mathbf{R}$ are continuous, $H([0,+\infty)) \subseteq [0,+\infty)$, and there is a number $C \geq 0$, such that $\lim_{z\to+\infty} \frac{|H(z)-Cz|}{z} = 0$ ((H_3) in [13]). The proof of Theorem 3.1 in [13] gives a limiting condition, that is,

$$(C_2 + \varepsilon) \int_0^1 (1-t)\, d\alpha(t) + \int_0^1 \int_0^1 G(t,s) \left[\frac{1}{r_2} u(s) + v(s) \right] d\alpha(t) < 1,$$

where C_2 is a constant, $r_2 \neq 0$ satisfies some conditions, ε satisfies $0 < \varepsilon < [\int_0^1 (1-t)\, d\alpha(t)]^{-1} - C_2$, $v : [0,1] \to [0,+\infty)$ is continuous, and $u : [0,1] \to [0,+\infty)$ is not identically zero on any subinterval of $[0,1]$. Goodrich [14] investigated the existence of positive solutions of the coupled system of boundary value problems with nonlocal boundary conditions

$$\begin{cases} -x'' = f(t,y(t)), & t \in (0,1), \\ -y'' = g(t,x(t)), & t \in (0,1), \\ x(0) = H_1(\varphi_1(y)), & x(1) = 0, \\ y(0) = H_2(\varphi_2(x)), & y(1) = 0, \end{cases} \tag{1.4}$$

by the Leray-Schauder fixed point theorem under the conditions that $f,g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are $H_i : \mathbf{R} \to \mathbf{R}$ are continuous, $H_i([0,+\infty)) \subseteq [0,+\infty)$, and H_i satisfies $\lim_{z\to 0^+} \frac{H_i(z)}{z} = 0$ and $\lim_{z\to +\infty} \frac{H_i(z)}{z} = +\infty$, $i = 1,2$ ((A_3) in [14]). For example, Goodrich [16] investigated the existence of at least one positive solution of the semipositone boundary value problems with nonlocal, nonlinear boundary conditions

$$\begin{cases} -y'' = \lambda f(t,y(t)), & t \in (0,1), \\ y(0) = H(\varphi(y)), \\ y(1) = 0, \end{cases} \tag{1.5}$$

by the fixed point index, where $\lambda > 0$ is a parameter, under the conditions that $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ are $H : \mathbf{R} \to \mathbf{R}$ are continuous, $H([0,+\infty)) \subseteq [0,+\infty)$, $\lim_{z\to 0^+} \frac{H(z)}{z} = +\infty$ ((H_2) in [16]), and there is a number $C_2 \geq 0$ such that $\lim_{z\to +\infty} \frac{|H(z)-C_2 z|}{z} = 0$ ((H_3) in [16]).

Motivated by the works mentioned, in this paper, we consider the coupled system (1.1). The main features of this paper are as follows. Firstly, we have more general integral boundary conditions. Secondly, we consider coupled systems rather than a single equation. Finally, we consider f that need not have a lower bound, that is, a semipositone problem. We remark that, to some extent, our systems and results generalize some previous works.

We organize this paper as follows. In Section 2, we first approximate the singular semipositone problem to the singular positone problem by a substitution. Then we present some lemmas to be used later. In Section 3, we state our result and give its proof. In Section 4, we present an example to demonstrate an application of our main results.

2 Preliminaries and lemmas

In this section, we first approximate the singular semipositone problem to the singular positone problem by a substitution. Then we present some lemmas to be used later. We assume that there are four linear functionals $\varphi_{1,1}, \varphi_{1,2}, \varphi_{2,1}, \varphi_{2,2} : C[0,1] \to \mathbf{R}$ such that φ_1, φ_2 satisfy the decompositions

$$\varphi_1(y) = \varphi_{1,1}(y) + \varphi_{1,2}(y), \qquad \varphi_2(y) = \varphi_{2,1}(y) + \varphi_{2,2}(y). \tag{2.1}$$

Let $E = C[0,1]$, so that $(E, \|\cdot\|)$ is a Banach space with usual maximal norm $\|y\| = \max_{t\in[0,1]} |y(t)|$. Let

$$P = \big\{ y \in E : y(t) \geq 0, y(t) \geq t(1-t)\|y\|, t \in [0,1], \varphi_{1,1}(y) \geq 0, \varphi_{2,1}(y) \geq 0 \big\}. \tag{2.2}$$

Clearly, P is a cone in E. We denote $P_r := \{y \in P, \|y\| < r\}$ for any $r > 0$.

Now, for the boundary value problem

$$\begin{cases} x''(t) = 0, & t \in (0,1), \\ x(0) = 0, \\ x(1) = 0, \end{cases}$$

we denote the Green functions

$$\begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.3}$$

In the rest of the paper, we adopt the following assumptions:

(H_1) There exist constants $C_0, D_0 > 0$ such that $\varphi_{1,2}(y) \ge C_0\|y\|$ and $\varphi_{2,2}(y) \ge D_0\|y\|$ for all $y \in P$.

(H_2) The functionals described in (2.1) have the form

$$\begin{aligned} &\varphi_1(y) := \int_{[0,1]} y(t)\,d\alpha_1(t), \qquad \varphi_{1,1}(y) := \int_{[0,1]} y(t)\,d\alpha_{1,1}(t), \\ &\varphi_{1,2}(y) := \int_{[0,1]} y(t)\,d\alpha_{1,2}(t), \\ &\varphi_2(y) := \int_{[0,1]} y(t)\,d\alpha_2(t), \qquad \varphi_{2,1}(y) := \int_{[0,1]} y(t)\,d\alpha_{2,1}(t), \\ &\varphi_{2,2}(y) := \int_{[0,1]} y(t)\,d\alpha_{2,2}(t), \end{aligned}$$

where all $\alpha_i, \alpha_{i,j} : C[0,1] \to \mathbf{R}$, $i,j = 1,2$, are of bounded variation on $[0,1]$.

(H_3) We have

$$\begin{aligned} &\int_{[0,1]} G(t,s)\,d\alpha_{1,1}(t) > 0, \qquad \int_{[0,1]} G(t,s)\,d\alpha_{2,1}(t) > 0, \quad \forall s \in [0,1], \\ &\int_{[0,1]} (1-t)\,d\alpha_{1,1}(t) > 0, \qquad \int_{[0,1]} (1-t)\,d\alpha_{2,1}(t) > 0. \end{aligned}$$

(H_4) The functions $H_1, H_2 : \mathbf{R} \to \mathbf{R}$ are continuous with $H_1([0,+\infty))$, $H_2([0,+\infty)) \subseteq [0,+\infty)$.

(H_5) $f : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous, and for any $t \in (0,1)$, $f(t,y)$ is nondecreasing in y and satisfies

$$f(t,y) \le p(t)h(y), \tag{2.4}$$

where $p : (0,1) \to [0,+\infty)$ and $h : [0,+\infty) \to [0,+\infty)$ are continuous, and $\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$ for t uniformly on any closed subinterval of $(0,1)$.

(H_6) $g : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous, and $g(t,1) > 0$ for all $t \in (0,1)$. Moreover, there exist constants $\lambda_1 \ge \lambda_2 > 1$ such that, for any $t \in (0,1)$ and $x \in [0,+\infty)$,

$$c^{\lambda_1} g(t,x) \le g(t,cx) \le c^{\lambda_2} g(t,x), \quad 0 \le c \le 1, \tag{2.5}$$

with $0 < \int_0^1 G(t,t)g(t,1)\,dt < \infty$.

(H_7) $q(t) : (0,1) \to (-\infty,+\infty)$ is Lebesgue integrable such that $\int_0^1 q_-(t)\,dt > 0$, where

$$q_-(t) = \max\{-q(t), 0\}, \quad t \in (0,1).$$

Remark 2.1 Note that since both φ_1 and φ_2 are linear, there exist constants C_1 and $D_1 > 0$ such that $|\varphi_1| \le C_1\|y\|$ and $|\varphi_2| \le D_1\|y\|$ for all $y \in P$. Henceforth, C_1 and D_1 denote these constants.

To state and prove the main result of this paper, we need the following lemmas.

Lemma 2.1 ([7]) *$q : (0,1) \to (-\infty,+\infty)$ is Lebesgue integrable, and $q(t)$ may have finitely many singularities.*

Lemma 2.2 ([7]) *For any $c \ge 1$ and $(t,x) \in (0,1) \times [0,+\infty)$, we have*

$$c^{\lambda_2}g(t,x) \le g(t,cx) \le c^{\lambda_1}g(t,x). \tag{2.6}$$

Definition 2.1 If $(x,y) \in C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1)$ satisfies (1.1) and $x(t) > 0, y(t) > 0$ for any $t \in (0,1)$, then we say that (x,y) is a positive solution of system (1.1).

For $u \in E$, let us define the function $[\cdot]^*$ by

$$\begin{gathered} [u(t)]^* = \begin{cases} u(t), & u(t) \ge 0, \\ 0, & u(t) < 0, \end{cases} \\ H_i^*(z) = H_i(\max\{0,z\}), \quad i = 1,2. \end{gathered}$$

Clearly, $\omega(t) = \int_0^1 G(t,s)q_-(s)\,ds$ is a positive solution of the BVP

$$\begin{cases} -\omega''(t) = q_-(t), & t \in (0,1), \\ \omega(0) = \omega(1) = 0. \end{cases}$$

Clearly, $\omega \in P$.

In what follows, we consider the following approximately singular nonlinear system:

$$\begin{cases} -x''(t) = f(t,y(t)) + q_+(t), & t \in (0,1), \\ -y''(t) = g(t,[x(t)-\omega(t)]^*), & t \in (0,1), \\ x(0) = H_1^*(\varphi_1(y)), & x(1) = 0, \\ y(0) = H_2^*(\varphi_2(x-\omega)), & y(1) = 0, \end{cases} \tag{2.7}$$

where

$$q_+(t) = \max\{q(t),0\}, \quad t \in (0,1).$$

It is easy to check that (x,y) is a solution of (2.7) if and only if (x,y) is a solution of the following nonlinear integral equation system:

$$\begin{cases} x(t) = (1-t)H_1^*(\varphi_1(y)) + \int_0^1 G(t,s)[f(s,y(s)) + q_+(s)]\,ds, & t \in [0,1], \\ y(t) = (1-t)H_2^*(\varphi_2(x-\omega)) + \int_0^1 G(t,s)g(s,[x(s)-\omega(s)]^*)\,ds, & t \in [0,1]. \end{cases} \tag{2.8}$$

If $x \in P$ and $\omega \in P$, then $y \in P$. In fact,

$$y(t) = (1-t)H_2^*\big(\varphi_2(x-\omega)\big) + \int_0^1 G(t,s)g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds, \quad t \in [0,1]$$

and

$$y(0) = H_2^*\big(\varphi_2(x-\omega)\big), \qquad y(1) = 0.$$

If $\|y\| = y(0) = H_2^*(\varphi_2(x-\omega))$, then we have

$$\begin{aligned} y(t) &\geq (1-t)H_2^*\big(\varphi_2(x-\omega)\big) \geq t(1-t)H_2^*\big(\varphi_2(x-\omega)\big) \\ &= t(1-t)\|y\|, \quad t \in [0,1]. \end{aligned}$$

If $\|y\| > H_2^*(\varphi_2(x-\omega))$, then there exists $t_0 \in (0,1)$ such that $\|y\| = y(t_0)$. Since $\frac{G(t,s)}{G(t_0,s)} \geq t(1-t)$, $t,s \in (0,1) \times (0,1)$, we have

$$\begin{aligned} y(t) &= (1-t)H_2^*\big(\varphi_2(x-\omega)\big) + \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds \\ &\geq t(1-t)\left[(1-t_0)H_2^*\big(\varphi_2(x-\omega)\big) + \int_0^1 G(t_0,s)g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds\right] \\ &= t(1-t)y(t_0) \\ &= t(1-t)\|y\|, \quad t \in [0,1]. \end{aligned}$$

If $\|y\| < H_2^*(\varphi_2(x-\omega))$, then

$$\begin{aligned} y(t) &= (1-t)H_2^*\big(\varphi_2(x-\omega)\big) + \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds \\ &\geq (1-t)H_2^*\big(\varphi_2(x-\omega)\big) \\ &\geq t(1-t)H_2^*\big(\varphi_2(x-\omega)\big) \\ &> t(1-t)\|y\|, \quad t \in [0,1]. \end{aligned}$$

In other words, we have $x, \omega \in P$, $H_2^*(\varphi_2(x-\omega)) \geq 0$, and

$$\begin{aligned} \varphi_{i,1}(y) &= \int_{[0,1]} (1-t)H_2^*\big(\varphi_2(x-\omega)\big)\,d\alpha_{i,1}(t) \\ &\quad + \int_{[0,1]} \left(\int_0^1 G(t,s)g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds\right) d\alpha_{i,1}(t) \\ &= H_2^*\big(\varphi_2(x-\omega)\big) \int_0^1 (1-t)\,d\alpha_{i,1}(t) \\ &\quad + \int_0^1 \left[\int_{[0,1]} G(t,s)\,d\alpha_{i,1}(t)\right] g\big(s,\big[x(s)-\omega(s)\big]^*\big)\,ds \geq 0, \quad i = 1,2. \end{aligned}$$

This yields that $y \in P$.

For convenience, we have the following form:

$$\begin{aligned}\varphi_1(y) &= \int_0^1 y(t)\,d\alpha_1(t)\\ &= \int_0^1 \biggl[(1-t)H_2^*\bigl(\varphi_2(x-\omega)\bigr) + \int_0^1 G(t,s)g\bigl(s,\bigl[x(s)-\omega(s)\bigr]^*\bigr)\,ds\biggr]\,d\alpha_1(t).\end{aligned}$$

We define

$$D_x := H_1^*\bigl(\varphi_1(y)\bigr). \tag{2.9}$$

Obviously, it is a nonnegative number that only depends on x.

We list here more assumptions to be used later.

(H_8) We have

$$\frac{\frac{C_1}{C_0}\int_0^1 q_-(t)\,dt + 1}{\max_{0\le\tau\le R} h(\tau)+1} > 2\int_0^1 (1-t)\bigl[p(t)+q_+(t)\bigr]\,dt + \frac{D_{x_0}}{\max_{0\le\tau\le R} h(\tau)+1},$$

where

$$\begin{aligned}&r^* = \frac{C_1}{C_0}\int_0^1 q_-(t)\,dt + 1, \qquad \widetilde{g} = \int_0^1 G(s,s)g(s,1)\,ds,\\ &R = \max\bigl\{H_2^*\bigl(\varphi_2(x-\omega)\bigr) + \bigl(r^*+1\bigr)^{\lambda_1}\widetilde{g}, x\in\bigl[t(1-t)r^*, r^*\bigr], t\in[0,1]\bigr\},\\ &D_{x_0} = \max\bigl\{D_x, x\in\bigl[t(1-t)r^*, r^*\bigr], t\in[0,1]\bigr\}.\end{aligned}$$

(H_9) $\lim_{y\to+\infty}\frac{f(t,y)}{y} = +\infty$ for t uniformly on any closed subinterval of $(0,1)$.

As a matter of convenience, we set

$$\widetilde{x(t)} = y(t) = (1-t)H_2^*\bigl(\varphi_2(x-\omega)\bigr) + \int_0^1 G(t,s)g\bigl(s,\bigl[x(s)-\omega(s)\bigr]^*\bigr)\,ds, \quad t\in[0,1].$$

Then, clearly, the equation system (2.8) is equivalent to the equation

$$x(t) = (1-t)D_x + \int_0^1 G(t,s)\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds, \quad t\in[0,1]. \tag{2.10}$$

Next, let us define the nonlinear operator $F: P\to C([0,1])$ by

$$(Fx)(t) = (1-t)D_x + \int_0^1 G(t,s)\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds, \quad t\in[0,1]. \tag{2.11}$$

It is well known that the solutions to system (2.7) exist if and only if the solutions to equation (2.10) exist. Therefore, if $x(t)$ is a fixed point of F in P, then system (2.8) has one solution (u,v), which can be written as

$$\begin{cases} u(t) = x(t),\\ v(t) = (1-t)H_2^*(\varphi_2(x-\omega)) + \int_0^1 G(t,s)g(s,[x(s)-\omega(s)]^*)\,ds, \quad t\in[0,1].\end{cases}$$

Lemma 2.3 ([23]) *Let X be a real Banach space, P be a cone in X, Ω be a bounded open subset of X with $\theta \in \Omega$, and $A : \overline{\Omega} \cap P \to P$ be a completely continuous operator.*

(1) *Suppose that*

$$Au \neq \lambda u, \quad \forall u \in \partial\Omega \cap P, \lambda \geq 1.$$

Then $i(A, \Omega \cap P, P) = 1$.

(2) *Suppose that*

$$Au \nleq u, \quad \forall u \in \partial\Omega \cap P.$$

Then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.4 ([7]) *If $g(t,x)$ satisfies (H_6), then for any $t \in (0,1)$, $g(t,x)$ is increasing in $x \in [0,+\infty)$, and for any $[\alpha,\beta] \subset (0,1)$,*

$$\lim_{n\to+\infty} \min_{t\in[\alpha,\beta]} \frac{g(t,x)}{x} = +\infty.$$

Lemma 2.5 *Suppose that (u,v) with $u(t) \geq \omega(t)$ for any $t \in [0,1]$ is a positive solution of system* (2.7) *and $\varphi_2(u-\omega) \geq 0$, $\varphi_1(v) \geq 0$. Then $(u-\omega, v)$ is a positive solution of the singular semipositone differential system* (1.1).

Proof In fact, if (u,v) is a positive solution of (2.7) and $u(t) \geq \omega(t), \varphi_2(u-\omega) \geq 0, \varphi_1(v) \geq 0$ for any $t \in [0,1]$, then by (2.7) and the definition of $[u(t)]^*$ we have

$$\begin{cases} -u''(t) = f(t, v(t)) + q_+(t), & t \in (0,1), \\ -v''(t) = g(t, u(t) - \omega(t)), & t \in (0,1), \\ u(0) = H_1^*(\varphi_1(v)) = H_1(\varphi_1(v)), & u(1) = 0, \\ v(0) = H_2^*(\varphi_2(u-\omega)) = H_2(\varphi_2(u-\omega)), & v(1) = 0. \end{cases} \tag{2.12}$$

Let $u_1 = u - \omega$. Then $u_1'' = u'' - \omega''$, which implies that

$$u''(t) = u_1''(t) + \omega''(t) = u_1''(t) - q_-(t), \quad t \in (0,1).$$

Thus, (2.12) becomes

$$\begin{cases} -u_1''(t) = f(t, v(t)) + q_+(t) - q_-(t), & t \in (0,1), \\ -v''(t) = g(t, u_1(t)), & t \in (0,1), \\ u_1(0) = H_1(\varphi_1(v)), & u_1(1) = 0, \\ v(0) = H_2(\varphi_2(u_1)), & v(1) = 0. \end{cases} \tag{2.13}$$

Noticing that $q(t) = q_+(t) - q_-(t)$, by (2.12) we have that (u_1, v) is a positive solution of system (1.1), that is, $(u - \omega, v)$ is a positive solution of system (1.1). This completes the proof of the lemma. □

Lemma 2.6 *Assume that* (H_1)-(H_8) *hold. Then* $F : P \to P$ *is a completely continuous operator.*

Proof For convenience, the proof is divided into the following five steps.

Step 1. We show that $F : P \to P$ is well defined. For any fixed $x \in P$, choose $0 < a < 1$ such that $a\|x\| < 1$. Then $a[x(t) - \omega(t)]^* \leq a\|x\| < 1$, so by (2.4), (2.6), and Lemma 2.4 we have

$$g\bigl(t, \bigl[x(t) - \omega(t)\bigr]^*\bigr) \leq \left(\frac{1}{a}\right)^{\lambda_1} g\bigl(t, a\bigl[x(t) - \omega(t)\bigr]^*\bigr) \leq a^{\lambda_2 - \lambda_1} \|x\|^{\lambda_2} g(t, 1).$$

Then

$$\begin{aligned} &\int_0^1 G(s,\tau) g\bigl(\tau, \bigl[x(\tau) - \omega(\tau)\bigr]^*\bigr)\, d\tau \\ &\quad \leq a^{\lambda_2 - \lambda_1} \|x\|^{\lambda_2} \int_0^1 G(\tau,\tau) g(\tau, 1)\, d\tau = \widetilde{R}_1. \end{aligned} \tag{2.14}$$

Consequently, for any $t \in [0,1]$, we have

$$\begin{aligned} (Fx)(t) &= (1-t)D_x + \int_0^1 G(t,s)\bigl[f\bigl(s, \widetilde{x(s)}\bigr) + q_+(s)\bigr]\, ds \\ &\leq D_x + \int_0^1 G(s,s)\bigl[p(s)h\bigl(\widetilde{x(s)}\bigr) + q_+(s)\bigr]\, ds \\ &\leq D_x + N\int_0^1 G(s,s)\bigl[p(s) + q_+(s)\bigr]\, ds < +\infty, \end{aligned}$$

where

$$\begin{aligned} &N = \max_{0 \leq \tau \leq R_1} h(\tau) + 1, \\ &\max\bigl\{H_2^*\bigl(\varphi_2(x - \omega)\bigr), \forall x \in P\bigr\} = C < +\infty, \qquad R_1 = C + \widetilde{R}_1. \end{aligned}$$

Thus, $F : P \to P$ is well defined.

Step 2. We show that $F(P) \subset P$. For any $x \in P$, by the definition of the operator F, we obtain $(Fx)(1) = 0, (Fx)(0) = D_x$. If $\|Fx\| = D_x$, then we have

$$(Fx)(t) \geq t(1-t)D_x = t(1-t)\|Fx\|, \quad t \in [0,1].$$

Then $F(P) \subset P$. If $\|Fx\| > D_x$, then there exists $t_0 \in (0,1)$ such that $\|Fx\| = (Fx)(t_0)$. Since $\frac{G(t,s)}{G(t_0,s)} \geq t(1-t)$, $t, s \in (0,1) \times (0,1)$, we have

$$\begin{aligned} (Fx)(t) &= (1-t)D_x + \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)\bigl[f\bigl(s, \widetilde{x(s)}\bigr) + q_+(s)\bigr]\, ds \\ &\geq t(1-t)\left[(1-t_0)D_x + \int_0^1 G(t_0,s)\bigl[f\bigl(s, \widetilde{x(s)}\bigr) + q_+(s)\bigr]\, ds\right] \\ &\geq t(1-t)(Fx)(t_0) = t(1-t)\|Fx\|, \quad t \in [0,1]. \end{aligned}$$

If $\|Fx\| < D_x$, then

$$\begin{aligned}(Fx)(t) &= (1-t)D_x + \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\geq (1-t)D_x > t(1-t)\|Fx\|, \quad t \in [0,1].\end{aligned}$$

We also know that

$$\begin{aligned}\varphi_{i,1}(Fx) &= \int_{[0,1]} (1-t)D_x\,d\alpha_{i,1}(t) + \int_{[0,1]} \left(\int_0^1 G(t,s)\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds\right) d\alpha_{i,1}(t) \\ &= D_x \int_0^1 (1-t)\,d\alpha_{i,1}(t) + \int_0^1 \left[\int_{[0,1]} G(t,s)\,d\alpha_{i,1}(t)\right]\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\geq 0, \quad i = 1,2.\end{aligned}$$

Thus, $F(P) \subset P$.

Step 3. Let $B \subset P$ be any bounded set. We show that $F(B)$ is uniformly bounded. There exists a constant $L > 0$ such that $\|u\| \leq L$ for any $u \in B$. Moreover, for any $u \in B$ and $s \in [0,1]$, we have $[x(s) - \omega(s)]^* \leq x(s) \leq \|x\| \leq L < L + 1$. Then, for any $x \in B$ and $s \in [0,1]$, we have $g(s,[x(s) - \omega(s)]^*) \leq g(s, L+1) \leq (L+1)^{\lambda_1} g(s,1)$, and thus

$$\begin{aligned}(Fx)(t) &= (1-t)D_x + \int_0^1 G(t,s)\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\leq D_x + \int_0^1 G(s,s)\bigl[p(s)h\bigl(\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\leq D_x + M \int_0^1 G(s,s)\bigl[p(s) + q_+(s)\bigr]\,ds < +\infty,\end{aligned}$$

where

$$\begin{aligned}&M = \max_{0 \leq \tau \leq R_2} h(\tau) + 1, \qquad \max\bigl\{H_2^*\bigl(\varphi_2(x-\omega)\bigr), \forall x \in B \subset P\bigr\} = C_1 < +\infty, \\ &R_2 = C_1 + (L+1)^{\lambda_1} \int_0^1 G(s,s) g(s,1)\,ds.\end{aligned}$$

Therefore, $F(B)$ is uniformly bounded.

Step 4. Let $B \subset P$ be any bounded set. We show that $F(B)$ is equicontinuous on $[0,1]$. For any $(t,s) \in [0,1] \times [0,1]$, $G(t,s)$ is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists a constant $\delta = \frac{\varepsilon}{2D_x}$ such that, for any $t, t', \in [0,1]$ such that $|t - t'| < \delta$, we have

$$\bigl|G(t,s) - G\bigl(t',s\bigr)\bigr| < \frac{\varepsilon}{2M \int_0^1 [p(s) + q_+(s)]\,ds}.$$

On the other hand, for any $x \in B$, we have

$$\begin{aligned}&\bigl|(Fx)(t) - (Fx)\bigl(t'\bigr)\bigr| \\ &\quad \leq \bigl|t - t'\bigr| D_x + \int_0^1 \bigl|G(t,s) - G\bigl(t',s\bigr)\bigr|\bigl[f\bigl(s,\widetilde{x(s)}\bigr) + q_+(s)\bigr]\,ds\end{aligned}$$

$$
\begin{aligned}
&< \delta D_x + \frac{\varepsilon}{2M\int_0^1[p(s)+q_+(s)]\,ds} M\int_0^1 \bigl[p(s)+q_+(s)\bigr]\,ds \\
&= \frac{\varepsilon}{2D_x}D_x + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, $F(B)$ is equicontinuous on $[0,1]$.

Step 5. We show that $F: P \to P$ is continuous. Assume that $x_n, x_0 \in P$ and $\|x_n - x_0\| \to 0, n \to +\infty$. Then there exists a constant $L_1 > 0$ such that $\|x_n\| \leq L_1, \|x_0\| \leq L_1$ ($n = 1, 2, \ldots$). Similarly to (2.15), we have $g(s, [x(s) - \omega(s)]^*) \leq g(s, L_1 + 1) \leq (L_1 + 1)^{\lambda_1} g(s, 1)$ ($n = 1, 2, \ldots$). Then, we have

$$
\begin{aligned}
&\bigl|(Fx_n)(t) - (Fx_0)(t)\bigr| \\
&\quad\leq \bigl|(1-t)(D_{x_n} - D_{x_0})\bigr| + \int_0^1 G(s,s)\bigl|f\bigl(s,\widetilde{x_n(s)}\bigr) - f\bigl(s,\widetilde{x_0(s)}\bigr)\bigr|\,ds,
\end{aligned}
$$

where

$$
D_{x_n} = H_1^*\left(\int_0^1 \bigl[\widetilde{x_n(s)}\bigr]\,d\alpha_1(s)\right), \qquad D_{x_0} = H_1^*\left(\int_0^1 \bigl[\widetilde{x_0(s)}\bigr]\,d\alpha_1(s)\right).
$$

Set

$$
\begin{aligned}
&r_n(s) = G(s,s)\bigl|f\bigl(s,\widetilde{x_n(s)}\bigr) - f\bigl(s,\widetilde{x_0(s)}\bigr)\bigr|, \\
&F(s) = 2M_1 G(s,s)\bigl[p(s) + q_+(s)\bigr], \quad s \in (0,1),
\end{aligned}
$$

where

$$
\begin{aligned}
&M_1 = \max_{0\leq\tau\leq R_3} h(\tau) + 1, \qquad \max\bigl\{H_2^*\bigl(\varphi_2(x_n - \omega)\bigr), \forall x_n \in P\bigr\} = C_2 < +\infty, \\
&R_3' = \int_0^1 G(s,\tau) g\bigl(\tau, \bigl[x_n(\tau) - \omega(\tau)\bigr]^*\bigr)\,d\tau \leq (L_1+1)^{\lambda_1}\int_0^1 G(s,s) g(s,1)\,ds, \\
&R_3 = C_2 + R_3'.
\end{aligned}
$$

It is clear that $|r_n(s)| \leq F(s), s \in (0,1), n = 1, 2, \ldots$, and $\{r_n(s)\}$ is a sequence of measurable functions in $(0,1)$. By (H_8) we have

$$
0 \leq \int_0^1 F(s)\,ds = 2M_1 \int_0^1 G(s,s)\bigl[p(s) + q_+(s)\bigr]\,ds < +\infty. \tag{2.15}
$$

We assert that $r_n(s) \to 0$ ($n \to +\infty$) for any fixed $s \in (0,1)$. In fact, for any fixed $s \in (0,1)$, noticing the continuity of $f(s,y)$ in y, we have that $f(s,y)$ is uniformly continuous with respect to y in $[0, R_3]$; thus, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that, for any $v_1, v_2 \in [0, R_3]$ such that $|v_1 - v_2| < \delta$,

$$
\bigl|f(s,v_1) - f(s,v_2)\bigr| < \frac{\varepsilon}{G(s,s)}. \tag{2.16}
$$

On the other hand, in view of the continuity of $g(s,x)$ in x, we obtain that $g(s,x)$ is uniformly continuous in x in $[0, L_1]$, so for the above $\delta > 0$, there exists a constant $\delta_1 > 0$, such that.

for any $u_1, u_2 \in [0, L_1]$ such that $|u_1 - u_2| < \delta_1$,

$$\left|g(s,u_1) - g(s,u_2)\right| < \frac{\delta}{2G(s,s)}.$$

Since $x_n(s) \to x_0(s)$ $(n \to +\infty)$, there exists a natural number $N_0 > 0$ such that $|x_n(s) - x_0(s)| < \delta_1$ for $n > N_0$. Noting that

$$\begin{aligned}
&\left|\left[x_n(s) - \omega(s)\right]^* - \left[x_0(s) - \omega(s)\right]^*\right| \\
&\quad = \left|\frac{|x_n(s) - \omega(s)| + x_n(s) - \omega(s)}{2} - \frac{|x_0(s) - \omega(s)| + x_0(s) - \omega(s)}{2}\right| \\
&\quad = \left|\frac{|x_n(s) - \omega(s)| - |x_0(s) - \omega(s)|}{2} + \frac{x_n(s) - x_0(s)}{2}\right| \\
&\quad \le \frac{|x_n(s) - x_0(s)|}{2} + \frac{|x_n(s) - x_0(s)|}{2} \\
&\quad = \left|x_n(s) - x_0(s)\right| < \delta_1
\end{aligned}$$

and

$$\begin{aligned}
&0 \le \left[x_n(s) - \omega(s)\right]^* \le x_n(s) \le L_1, \\
&0 \le \left[x_0(s) - \omega(s)\right]^* \le x_0(s) \le L_1,
\end{aligned}$$

for $n > N_0$, we have

$$\left|g\left(s, \left[x_n(s) - \omega(s)\right]^*\right) - g\left(s, \left[x_0(s) - \omega(s)\right]^*\right)\right| < \frac{\delta}{2G(s,s)}. \tag{2.17}$$

By (2.17) we have

$$\left|\int_0^1 G(s,\tau) g\left(\tau, \left[x_n(\tau) - \omega(\tau)\right]^*\right) d\tau - \int_0^1 G(s,\tau) g\left(\tau, \left[x_0(\tau) - \omega(\tau)\right]^*\right) d\tau\right| < \frac{\delta}{2}. \tag{2.18}$$

Noting that H_2^* is continuous, for the above $\delta > 0$, there exists $\delta_2 > 0$ such that $|z_1 - z_2| < \delta_2$. Then

$$\left|H_2^*(z_1) - H_2^*(z_2)\right| < \frac{\delta}{2}.$$

Since

$$\varphi_2(x_n - \omega) := \int_{[0,1]} \left(x_n(s) - \omega(s)\right) d\alpha_2(s), \qquad \varphi_2(x_0 - \omega) := \int_{[0,1]} \left(x_0(s) - \omega(s)\right) d\alpha_2(s),$$

and $x_n \to x_0, n \to +\infty$, by the Lebesgue dominated convergence theorem we have

$$\varphi_2(x_n - \omega) \to \varphi_2(x_0 - \omega), \quad n \to +\infty.$$

For the above $\delta_2 > 0$, there exists a natural number $N_1 > 0$ such that, for any $n > N_1$, we have

$$\left|H_2^*\left(\varphi_2(x_n - \omega)\right) - H_2^*\left(\varphi_2(x_0 - \omega)\right)\right| < \frac{\delta}{2}. \tag{2.19}$$

Then it follows from (2.18) and (2.19) that

$$\begin{aligned}&\Big|(1-s)H_2^*\big(\varphi_2(x_n-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_n(\tau)-\omega(\tau)\big]^*\big)\,d\tau\\&\quad-(1-s)H_2^*\big(\varphi_2(x_0-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_0(\tau)-\omega(\tau)\big]^*\big)\,d\tau\Big|\\&<\delta,\end{aligned}$$

that is,

$$\big|\widetilde{x_n(s)}-\widetilde{x_0(s)}\big|<\delta. \tag{2.20}$$

By (2.16), choose $N=\max\{N_0,N_1\}$. For $n>N$, we have

$$\begin{aligned}&\bigg|f\bigg(s,(1-s)H_2^*\big(\varphi_2(x_n-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_n(\tau)-\omega(\tau)\big]^*\big)\,d\tau\bigg)\\&\quad-f\bigg(s,(1-s)H_2^*\big(\varphi_2(x_0-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_0(\tau)-\omega(\tau)\big]^*\big)\,d\tau\bigg)\bigg|\\&<\frac{\varepsilon}{G(s,s)},\end{aligned}$$

that is,

$$\big|f\big(s,\widetilde{x_n(s)}\big)-f\big(s,\widetilde{x_0(s)}\big)\big|<\frac{\varepsilon}{G(s,s)}.$$

Consequently, for any fixed $s\in(0,1)$ and for any $\varepsilon>0$, there exists a natural number $N_2>0$ such that, for $n>N_2$,

$$\big|r_n(s)-0\big|<\varepsilon,$$

that is, $r_n(s)\to 0$ $(n\to+\infty)$, $s\in(0,1)$.

Since H_1^* is continuous, for the above $\varepsilon>0$, there exists $\delta_3>\delta>0$ such that if $|z_1-z_2|<\delta_3$, then

$$\big|H_1^*(z_1)-H_1^*(z_2)\big|<\varepsilon.$$

So by (2.20) we have

$$\begin{aligned}&\bigg|H_1^*\bigg(\int_0^1\bigg[(1-s)H_2^*\big(\varphi_2(x_n-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_n(\tau)-\omega(\tau)\big]^*\big)\,d\tau\bigg]\,d\alpha_1(s)\bigg)\\&\quad-H_1^*\bigg(\int_0^1\bigg[(1-s)H_2^*\big(\varphi_2(x_0-\omega)\big)+\int_0^1 G(s,\tau)g\big(\tau,\big[x_0(\tau)-\omega(\tau)\big]^*\big)\,d\tau\bigg]\,d\alpha_1(s)\bigg)\bigg|\\&<\varepsilon,\end{aligned}$$

that is,

$$|D_{x_n}-D_{x_0}|<\varepsilon.$$

By the Lebesgue dominated convergence theorem we have

$$\|Fx_n - Fx_0\| < \varepsilon + \varepsilon = 2\varepsilon, \quad n \to +\infty.$$

Then

$$\|Fx_n - Fx_0\| \to 0, \quad n \to +\infty.$$

Therefore, $F : P \to P$ is continuous. Thus, $F : P \to P$ is a completely continuous operator. This completes the proof of the lemma. □

Lemma 2.7 *Assume that* (H_1)-(H_8) *hold. Then* $i(F, P_{r^*}, P) = 1$.

Proof Assume that there exist $\lambda_0 \geq 1$ and $z_0 \in \partial P_{r^*}$ such that $\lambda_0 z_0 = Fz_0$. Then $z_0 = \frac{1}{\lambda_0} Fz_0$ and $0 < \frac{1}{\lambda_0} \leq 1$. We know that $z_0(t) \geq t(1-t)\|z_0\| = t(1-t)r^*, t \in [0,1]$, and $\omega(t) = \int_0^1 G(t,s)q_-(s)\,ds \leq t(1-t)\int_0^1 q_-(s)\,ds$. Then, for any $t \in [0,1]$,

$$\begin{aligned} z_0(t) - \omega(t) &\geq z_0(t) - t(1-t)\int_0^1 q_-(s)\,ds \\ &\geq t(1-t)r^* - t(1-t)\int_0^1 q_-(s)\,ds \\ &= t(1-t)\left[r^* - \int_0^1 q_-(s)\,ds\right] \geq 0. \end{aligned}$$

Applying $z_0 = \frac{1}{\lambda_0} Fz_0$, we obtain λ_0, z_0 such that

$$\begin{cases} z_0''(t) + \frac{1}{\lambda_0}[f(t, \widetilde{z_0(t)}) + q_+(t)] = 0, \\ z_0(0) = \frac{1}{\lambda_0} H_1^*(\int_0^1 \widetilde{z_0(t)}\,dt) = \frac{1}{\lambda_0} D_{z_0}, \\ z_0(1) = 0. \end{cases} \tag{2.21}$$

Since $z_0''(t) \leq 0$ for any $t \in (0,1)$, $z_0(t)$ is a concave function on $[0,1]$. By the boundary conditions, if $\|z_0\| = z_0(0)$, then $z_0'(t) \leq 0, t \in (0,1)$, and since $z_0(0) = \frac{1}{\lambda_0} D_{z_0}$ is a nonnegative number depending only on z_0, we have $z_0'(0) = 0$. Noting that

$$\begin{aligned} \int_0^1 G(s,\tau)g\bigl(\tau, \bigl[z_0(\tau) - \omega(\tau)\bigr]^*\bigr)\,d\tau &\leq \int_0^1 G(\tau,\tau)g\bigl(\tau, \bigl[z_0(\tau) - \omega(\tau)\bigr]^*\bigr)\,d\tau \\ &= \int_0^1 G(\tau,\tau)g\bigl(\tau, \bigl[z_0(\tau) - \omega(\tau)\bigr]\bigr)\,d\tau \\ &\leq \int_0^1 G(\tau,\tau)g\bigl(\tau, z_0(\tau)\bigr)\,d\tau \\ &\leq \int_0^1 G(\tau,\tau)g\bigl(\tau, r^*\bigr)\,d\tau \\ &\leq \bigl(r^* + 1\bigr)^{\lambda_1}\int_0^1 G(\tau,\tau)g(\tau,1)\,d\tau, \end{aligned}$$

we get

$$\begin{aligned}\widetilde{z_0(s)} &= (1-s)H_2^*\big(\varphi_2(z_0-\omega)\big) + \int_0^1 G(s,\tau)g\big(\tau,\big[z_0(\tau)-\omega(\tau)\big]^*\big)\,d\tau \\ &\le H_2^*\big(\varphi_2(z_0-\omega)\big) + \big(r^*+1\big)^{\lambda_1}\int_0^1 G(s,s)g(s,1)\,ds.\end{aligned}$$

Then, choosing $t \in (0,1)$ and integrating (2.21) from 0 to t, we have

$$\begin{aligned}z_0'(t) &= \int_0^t z_0''(s)\,ds \ge -\int_0^t \big[f\big(s,\widetilde{z_0(s)}\big) + q_+(s)\big]\,ds \\ &\ge -\int_0^t \big[p(s)h\big(\widetilde{z_0(s)}\big) + q_+(s)\big]\,ds \\ &\ge -\Big[\max_{0\le\tau\le R_0} h(\tau)+1\Big]\int_0^t \big[p(s)+q_+(s)\big]\,ds,\end{aligned}$$

where

$$R_0 = H_2^*\big(\varphi_2(z_0-\omega)\big) + \big(r^*+1\big)^{\lambda_1}\int_0^1 G(s,s)g(s,1)\,ds. \tag{2.22}$$

By (H_8) we know that $R_0 \le R$. So

$$\begin{aligned}-z_0'(t) &\le \Big[\max_{0\le\tau\le R_0} h(\tau)+1\Big]\int_0^t \big[p(s)+q_+(s)\big]\,ds \\ &\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_0^t \big[p(s)+q_+(s)\big]\,ds.\end{aligned}$$

Next, integrating this inequality from 0 to 1, we obtain

$$\begin{aligned}r^* &= z_0(0) = \int_0^1 -z_0'(s)\,ds \le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_0^1 ds\int_0^s \big[p(\xi)+q_+(\xi)\big]\,d\xi \\ &\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_0^1 d\xi\int_\xi^1 \big[p(\xi)+q_+(\xi)\big]\,ds \\ &\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_0^1 (1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{r^*}{\max_{0\le\tau\le R} h(\tau)+1} &\le \int_0^1 (1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi \\ &\le 2\int_0^1 (1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi,\end{aligned}$$

which is a contradiction with (H_8). On the other hand, if $\|z_0\| > z_0(0)$, then there exists $t_0 \in (0,1)$ such that

$$\|z_0\| = z_0(t_0); \qquad z_0'(t_0) = 0, \qquad z_0'(t) \ge 0, \quad t \in (0,t_0); \qquad z_0'(t) \le 0, \quad t \in (t_0,1).$$

If $t \in (0, t_0)$, integrating (2.21) from t to t_0, we have

$$\begin{aligned} z_0'(t) &= \int_t^{t_0} -z_0''(s)\,ds \\ &\le \int_t^{t_0} \bigl[f\bigl(s, \widetilde{z_0(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\le \int_t^{t_0} \bigl[p(s)h\bigl(\widetilde{z_0(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\le \Bigl[\max_{0\le\tau\le R_0} h(\tau) + 1\Bigr] \int_t^{t_0} \bigl[p(s) + q_+(s)\bigr]\,ds, \end{aligned}$$

where R_0 is defined by (2.22), and by (H_8) we know that $R_0 \le R$. So

$$\begin{aligned} z_0'(t) &\le \Bigl[\max_{0\le\tau\le R_0} h(\tau) + 1\Bigr] \int_t^{t_0} \bigl[p(s) + q_+(s)\bigr]\,ds \\ &\le \Bigl[\max_{0\le\tau\le R} h(\tau) + 1\Bigr] \int_t^{t_0} \bigl[p(s) + q_+(s)\bigr]\,ds. \end{aligned}$$

Next, integrating this inequality from 0 to t_0, we obtain

$$\begin{aligned} r^* &= z_0(t_0) = \int_0^{t_0} z_0'(s)\,ds + z_0(0) \\ &\le \Bigl[\max_{0\le\tau\le R} h(\tau) + 1\Bigr] \int_0^{t_0} ds \int_s^{t_0} \bigl[p(\xi) + q_+(\xi)\bigr]\,d\xi + z_0(0) \\ &\le \Bigl[\max_{0\le\tau\le R} h(\tau) + 1\Bigr] \int_0^{t_0} d\xi \int_0^{\xi} \bigl[p(\xi) + q_+(\xi)\bigr]\,ds + z_0(0) \\ &\le \Bigl[\max_{0\le\tau\le R} h(\tau) + 1\Bigr] \int_0^{t_0} \xi\bigl[p(\xi) + q_+(\xi)\bigr]\,d\xi + z_0(0) \\ &\le \frac{\max_{0\le\tau\le R} h(\tau) + 1}{1 - t_0} \int_0^1 \xi(1-\xi)\bigl[p(\xi) + q_+(\xi)\bigr]\,d\xi + z_0(0). \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{r^*(1-t_0)}{\max_{0\le\tau\le R} h(\tau) + 1} \\ &\quad\le \int_0^1 \xi(1-\xi)\bigl[p(\xi) + q_+(\xi)\bigr]\,d\xi + \frac{z_0(0)(1-t_0)}{\max_{0\le\tau\le R} h(\tau) + 1}. \end{aligned}$$

For $t \in (t_0, 1)$, we have

$$\begin{aligned} z_0'(t) &= \int_{t_0}^t z_0''(s)\,ds \ge -\int_{t_0}^t \bigl[f\bigl(s, \widetilde{z_0(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\ge -\int_{t_0}^t \bigl[p(s)h\bigl(\widetilde{z_0(s)}\bigr) + q_+(s)\bigr]\,ds \\ &\ge -\Bigl[\max_{0\le\tau\le R_0} h(\tau) + 1\Bigr] \int_{t_0}^t \bigl[p(s) + q_+(s)\bigr]\,ds, \end{aligned}$$

where R_0 is defined by (2.22), and by (H_8) we know that $R_0 \le R$, so

$$\begin{aligned}
-z_0'(t) &\le \Big[\max_{0\le\tau\le R_0} h(\tau)+1\Big]\int_{t_0}^{t}\big[p(s)+q_+(s)\big]\,ds \\
&\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_{t_0}^{t}\big[p(s)+q_+(s)\big]\,ds.
\end{aligned}$$

Next, integrating this inequality from t_0 to 1, we obtain

$$\begin{aligned}
r^* &= z_0(t_0) = \int_{t_0}^{1} -z_0'(s)\,ds \\
&\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_{t_0}^{1} ds\int_{t_0}^{s}\big[p(\xi)+q_+(\xi)\big]\,d\xi \\
&\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_{t_0}^{1} d\xi\int_{\xi}^{1}\big[p(\xi)+q_+(\xi)\big]\,ds \\
&\le \Big[\max_{0\le\tau\le R} h(\tau)+1\Big]\int_{t_0}^{1} (1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi \\
&\le \frac{\max_{0\le\tau\le R} h(\tau)+1}{t_0}\int_{0}^{1} \xi(1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi.
\end{aligned}$$

Then,

$$\frac{r^* t_0}{\max_{0\le\tau\le R} h(\tau)+1} \le \int_0^1 \xi(1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi.$$

Thus,

$$\begin{aligned}
&\frac{r^*}{\max_{0\le\tau\le R} h(\tau)+1} \\
&\quad\le 2\int_0^1 \xi(1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi + \frac{z_0(0)(1-t_0)}{\max_{0\le\tau\le R} h(\tau)+1} \\
&\quad= 2\int_0^1 \xi(1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi + \frac{\frac{1}{\lambda_0}D_{z_0}(1-t_0)}{\max_{0\le\tau\le R} h(\tau)+1} \\
&\quad\le 2\int_0^1 (1-\xi)\big[p(\xi)+q_+(\xi)\big]\,d\xi + \frac{D_{z_0}}{\max_{0\le\tau\le R} h(\tau)+1},
\end{aligned}$$

which is a contradiction with (H_8). So, by Lemma 2.3, $i(F, P_{r^*}, P) = 1$. This completes the proof of the lemma. □

Lemma 2.8 *Assume that* (H_1)*-*(H_7) *and* (H_9) *hold. There exists a constant* $R^* > r^*$ *such that* $i(F, P_{R^*}, P) = 0$.

Proof We choose constants α, β, and L such that

$$[\alpha,\beta]\subset(0,1), \quad L > 2\Big[\alpha(1-\beta)\max_{0\le t\le 1}\int_\alpha^\beta G(t,s)\,ds\Big]^{-1}.$$

By (H_9) there exists $R_1^* > 2r$ such that

$$f(t,y) \geq Ly, \quad t \in [\alpha,\beta], y \in \bigl[R_1^*, +\infty\bigr). \tag{2.23}$$

On the other hand, by Lemma 2.4 there exists $R_2^* > R_1^*$ such that, for $t \in [\alpha,\beta]$ and $x \in [R_2^*, +\infty)$,

$$\frac{g(t,x)}{x} \geq \min_{t\in[\alpha,\beta]} \frac{g(t,x)}{x} \geq \frac{1}{\max_{\alpha\leq s\leq\beta}\int_\alpha^\beta G(s,\tau)\,d\tau},$$

that is,

$$g(t,x) \geq \frac{1}{\max_{\alpha\leq s\leq\beta}\int_\alpha^\beta G(s,\tau)\,d\tau} x, \quad t \in [\alpha,\beta], x \in \bigl[R_2^*, +\infty\bigr). \tag{2.24}$$

Let $R^* \geq \frac{2R_2^*}{\alpha(1-\beta)}$. Obviously, $R^* > R_2^* > R_1^* > 2r^*$. Thus, $\frac{r^*}{R^*} < \frac{1}{2}$.

Now we show that $x \ngeq Fx$, $x \in \partial P_{R^*}$. Indeed, otherwise, there exists $x_1 \in \partial P_{R^*}$ such that $x_1 \geq Fx_1$. As in the proof of Lemma 2.7, by the definition of r^*, for any $t \in [\alpha,\beta]$, we have

$$\begin{aligned}
&x_1(t) - \omega(t) \\
&\quad\geq x_1(t) - t(1-t)\int_0^1 q_-(s)\,ds \\
&\quad\geq x_1(t) - t(1-t)\biggl[\frac{C_1\int_0^1 q_-(s)\,ds}{C_0} + 1\biggr] \\
&\quad= x_1(t) - t(1-t)r^* \geq x_1(t) - \frac{x_1(t)}{\|x_1\|}r^* = x_1(t) - \frac{r^*}{R^*}x_1(t) \geq \frac{1}{2}x_1(t) \\
&\quad\geq \frac{1}{2}t(1-t)\|x_1\| \geq \frac{1}{2}R^*\alpha(1-\beta) \geq R_2^* > 0.
\end{aligned}$$

So, by (2.24), for any $s \in [\alpha,\beta]$, we have

$$\begin{aligned}
&\int_\alpha^\beta G(s,\tau)g\bigl(\tau, \bigl[x_1(\tau) - \omega(\tau)\bigr]^*\bigr)\,d\tau \\
&\quad\geq \frac{1}{\max_{\alpha\leq s\leq\beta}\int_\alpha^\beta G(s,\tau)\,d\tau}\int_\alpha^\beta G(s,\tau)\bigl[x_1(\tau) - \omega(\tau)\bigr]^*\,d\tau \\
&\quad\geq \frac{1}{2}R^*\alpha(1-\beta) \geq R_2^* > R_1^*.
\end{aligned}$$

Since f is nondecreasing in y, from the last inequality it follows that

$$\begin{aligned}
R^* &\geq x_1(t) \geq Fx_1(t) \geq \int_0^1 G(t,s)\bigl[f\bigl(s, \widetilde{x_1(s)}\bigr) + q_+(s)\bigr]\,ds \\
&\geq \int_0^1 G(t,s)f\bigl(s, \widetilde{x_1(s)}\bigr)\,ds \\
&\geq \int_\alpha^\beta G(t,s)L\widetilde{x_1(s)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_\alpha^\beta G(t,s)L\int_\alpha^\beta G(s,\tau)g\bigl(\tau,\bigl[x_1(\tau)-\omega(\tau)\bigr]^*\bigr)\,d\tau\,ds \\
&\geq \frac{1}{2}L\alpha(1-\beta)R^*\int_\alpha^\beta G(t,s)\,ds, \quad t\in[0,1].
\end{aligned}
$$

Then we have

$$
2\bigl[L\alpha(1-\beta)\bigr]^{-1}\geq \int_\alpha^\beta G(t,s)\,ds, \quad t\in[0,1].
$$

Taking the maximum in the last inequality, we get

$$
2\bigl[L\alpha(1-\beta)\bigr]^{-1}\geq \max_{0\leq t\leq 1}\int_\alpha^\beta G(t,s)\,ds.
$$

Consequently,

$$
L\leq 2\left[\alpha(1-\beta)\max_{0\leq t\leq 1}\int_\alpha^\beta G(t,s)\,ds\right]^{-1}.
$$

This contradicts to the choice of L. Thus, by Lemma 2.3, $i(F,P_{R^*},P)=0$. The proof is complete. □

3 Main results

In this section, we give our main result.

Theorem 3.1 *Suppose that* (H_1)-(H_9) *are satisfied. Then system* (1.1) *has at least one positive solution.*

Proof Applying Lemmas 2.7 and 2.8 and the definition of the fixed point index, we have $i(F,P_{R^*}\backslash\overline{P_{r^*}},P)=-1$. Thus, F has a fixed point z_0 in $P_{R^*}\backslash\overline{P_{r^*}}$ with $r^*<\|z_0\|<R^*$. Since $r^*<\|z_0\|$, we have

$$
\begin{aligned}
z_0(t)-\omega(t)&\geq t(1-t)\|z_0\|-\int_0^1 G(t,s)q_-(s)\,ds\geq t(1-t)\|z_0\|-t(1-t)\int_0^1 q_-(s)\,ds \\
&= t(1-t)\left[\|z_0\|-\int_0^1 q_-(s)\,ds\right]=kt(1-t)\geq 0, \quad t\in[0,1],
\end{aligned}
$$

where $k=\|z_0\|-\int_0^1 q_-(s)\,ds>0$.

Choosing $z_0\in P_{R^*}\backslash\overline{P_{r^*}}$, we have

$$
\varphi_i(z_0-\omega)=\varphi_i(z_0)-\varphi_i(\omega)\geq C_0\|z_0\|-\varphi_i(\omega), \quad i=1,2.
$$

Since $\varphi_i(\omega)\leq C_1\|\omega\|$ and $\omega\in P$, we have $\omega(t)\geq t(1-t)\|\omega\|$. Consequently, by the above inequalities and the definition of $\omega(t)$ we have

$$
\begin{aligned}
&0\leq t(1-t)\varphi_i(\omega)\leq C_1t(1-t)\|\omega\|\leq C_1\omega(t)=C_1\int_0^1 G(t,s)q_-(s)\,ds \\
&\quad\leq C_1t(1-t)\int_0^1 q_-(s)\,ds.
\end{aligned}
$$

Consequently, $\varphi_i(\omega) \leq C_1 \int_0^1 q_-(s)\,ds$. Then

$$\begin{aligned}\varphi_i(z_0-\omega) &\geq C_0\|z_0\| - \varphi_i(\omega) \geq C_0\left(\frac{C_1\int_0^1 q_-(s)\,ds}{C_0}+1\right) - C_1\int_0^1 q_-(s)\,ds \\ &= C_0 > 0, \quad i = 1,2.\end{aligned}$$

Then from Lemma 2.5 it follows that

$$\begin{cases} x(t) = z_0(t) - \omega(t), \\ y(t) = (1-t)H_2^*(\varphi_2(z_0-\omega)) + \int_0^1 G(t,s)g(s,(z_0(s)-\omega(s)))\,ds \end{cases}$$

is a positive solution of system (1.1). Thus, we complete the proof of Theorem 3.1. □

Remark 3.1 In comparison with [13] and [16], we consider coupled systems rather than only a single equation, the nonlinearity $f(t,x)$ may be singular at $t = 0,1$, and $q(t)$ can have finitely many singularities in $[0,1]$. Moreover, we do not assume that H satisfies merely an asymptotic condition.

Remark 3.2 In comparison with [14], we also consider the coupled system, but our system is singular semipositone. We consider f that need not have a lower bound, and we do not assume that H_i satisfy superlinearity conditions at $t = 0$ and $t = +\infty$.

Remark 3.3 In comparison with [7], we have more complex integral boundary conditions. In this paper, H_i ($i = 1,2$) are not linear, and $\varphi_i : C([0,1]) \to \mathbf{R}$ ($i = 1,2$) are linear Stieltjes integrals with signed measures. Thus, in this paper, we allow the map $y \mapsto \varphi(y)$ to be negative even if y is nonnegative. This is very different from paper [4].

4 Example

Example 4.1 Consider the singular system

$$\begin{cases} -x'' = \frac{t}{10(\pi+4)}y^2\arctan y - \frac{2}{2+3\sqrt[3]{4}}\{\frac{1}{\sqrt{t}} + \frac{1}{\sqrt[3]{(t-\frac{1}{2})^2}}\}, \quad t \in (0,1), \\ -y'' = \frac{25x^{\frac{3}{2}}}{199^2 t(1-t)}, \quad t \in (0,1), \\ x(0) = H_1(\varphi_1(y)), \qquad x(1) = 0, \\ y(0) = H_2(\varphi_2(x)), \qquad y(1) = 0, \end{cases} \tag{4.1}$$

$$\varphi_1(y) = \frac{1}{2}y\left(\frac{1}{3}\right) - \frac{1}{5}y\left(\frac{2}{3}\right) = \underbrace{\frac{1}{3}y\left(\frac{1}{3}\right) - \frac{1}{5}y\left(\frac{2}{3}\right)}_{\varphi_{1,1}(y)} + \underbrace{\frac{1}{6}y\left(\frac{1}{3}\right)}_{\varphi_{1,2}(y)}, \tag{4.2}$$

$$\varphi_2(x) = \frac{1}{3}x\left(\frac{1}{2}\right) - \frac{1}{8}x\left(\frac{7}{10}\right) = \underbrace{\frac{1}{6}x\left(\frac{1}{2}\right) - \frac{1}{8}x\left(\frac{7}{10}\right)}_{\varphi_{2,1}(x)} + \underbrace{\frac{1}{6}x\left(\frac{1}{2}\right)}_{\varphi_{2,2}(x)}. \tag{4.3}$$

By (4.2) and (4.3) we know that φ_1, φ_2 satisfy (H_5)-(H_7), and $C_0 = \frac{1}{27}, C_1 = \frac{7}{10}, D_0 = \frac{1}{24}, D_1 = \frac{11}{24}$. Define H_1, H_2 by

$$H_1(z) := z^2 + z, \qquad H_2(z) := z^{\frac{1}{3}}, \quad t \in (-\infty, +\infty).$$

We know that H_1 is not superlinear at $t = 0$ and does not satisfy an asymptotic condition and that H_2 is not superlinear at $t = 0$ and $t = +\infty$. Then system (4.1) has at least one positive solution on $C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1)$. Indeed, choose

$$
\begin{aligned}
&p(t) = \frac{t}{10(\pi + 4)}, \qquad h(y) = y^2 \arctan y, \\
&q_-(t) = \frac{2}{2 + 3\sqrt[3]{4}} \left\{ \frac{1}{\sqrt{t}} + \frac{1}{\sqrt[3]{(t - \frac{1}{2})^2}} \right\}, \\
&q_+(t) \equiv 0, \qquad g(t,x) = \frac{25x^{\frac{3}{2}}}{199^2 t(1-t)}, \qquad \lambda_1 = 2, \qquad \lambda_2 = \frac{3}{2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&r^* = \frac{C_1 \int_0^1 q_-(t)\,dt}{C_0} + 1 = \frac{194}{5}, \\
&2\int_0^1 (1-t)\bigl[p(t) + q_+(t)\bigr]\,dt = \frac{1}{30(\pi + 4)} \approx 0.0047,
\end{aligned}
\tag{4.4}
$$

$$
\bigl(r^* + 1\bigr)^{\lambda_1} \int_0^1 G(s,s)g(s,1)\,ds = 1, \tag{4.5}
$$

and thus

$$
\widetilde{g} = \int_0^1 G(s,s)g(s,1)\,ds = \left(\frac{199}{5}\right)^{-2} = \left(\frac{5}{199}\right)^2 \approx 0.0006,
$$

and $0 \le t(1-t)\|x\| \le x(t) \le \|x\|, x \in P$. If $x \in [t(1-t)r^*, r^*], t \in [0,1]$, then

$$
\varphi_2(x - \omega) \le \varphi_2(x) \le D_1\|x\| = \frac{2{,}134}{120} \approx 17.7833.
$$

Consequently,

$$
H_2^*\bigl(\varphi_2(x-\omega)\bigr) \le H_2\left(\frac{2{,}134}{120}\right) = \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} \approx 2.6102, \tag{4.6}
$$

$$
\begin{aligned}
y(t) &= (1-t)H_2^*\bigl(\varphi_2(x-\omega)\bigr) + \int_0^1 G(t,s)g\bigl(s, \bigl[x(s) - \omega(s)\bigr]^*\bigr)\,ds \\
&\le \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \int_0^1 G(t,s)g\bigl(s, x(s)\bigr)\,ds \\
&\le \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \int_0^1 G(t,s)g\bigl(s, r^*\bigr)\,ds \\
&\le \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \bigl(r^*\bigr)^{\lambda_1} \int_0^1 G(t,s)g(s,1)\,ds \\
&= \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \left(\frac{194}{5}\right)^2 \cdot \left(\frac{5}{199}\right)^2 \approx 3.5606.
\end{aligned}
$$

Then $\varphi_1(y) \le C_1\|y\| \le \frac{7}{10} \times [(\frac{2{,}134}{120})^{\frac{1}{3}} + (\frac{194}{5})^2 \cdot (\frac{5}{199})^2] \approx 2.4924$, and

$$\begin{aligned} H_1^*\bigl(\varphi_1(y)\bigr) &\le H_1\left(\frac{7}{10} \times \left[\left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \left(\frac{194}{5}\right)^2 \cdot \left(\frac{5}{199}\right)^2\right]\right) \\ &= \left\{\frac{7}{10} \times \left[\left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \left(\frac{194}{5}\right)^2 \cdot \left(\frac{5}{199}\right)^2\right]\right\}^2 \\ &\quad + \frac{7}{10} \times \left[\left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + \left(\frac{194}{5}\right)^2 \cdot \left(\frac{5}{199}\right)^2\right] \\ &\approx 8.7045. \end{aligned}$$

By (4.5) and (4.6) we have

$$\begin{aligned} &\max\bigl\{H_2^*\bigl(\varphi_2(x-\omega)\bigr) + \bigl(r^*+1\bigr)^{\lambda_1}\widetilde{g}, x \in \bigl[t(1-t)r^*, r^*\bigr], t \in [0,1]\bigr\} \\ &\quad \le \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + 1 \approx 3.6102. \end{aligned}$$

Then

$$\begin{aligned} &R = \left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + 1 \approx 3.6102, \\ &\max_{0\le\tau\le R} h(\tau) = \left(\left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + 1\right)^2 \arctan\left(\left(\frac{2{,}134}{120}\right)^{\frac{1}{3}} + 1\right) \approx 971.2290, \\ &\frac{r^*}{\max_{0\le\tau\le R} h(\tau) + 1} = \frac{\frac{194}{5}}{((\frac{2{,}134}{120})^{\frac{1}{3}} + 1)^2 \arctan((\frac{2{,}134}{120})^{\frac{1}{3}} + 1) + 1} \approx 0.0399, \qquad (4.7) \\ &\frac{D_{x_0}}{\max_{0\le\tau\le R} h(\tau) + 1} \\ &\quad \le \frac{\{\frac{7}{10} \times [(\frac{2{,}134}{120})^{\frac{1}{3}} + (\frac{194}{5})^2 \cdot (\frac{5}{199})^2]\}^2 + \frac{7}{10} \times [(\frac{2{,}134}{120})^{\frac{1}{3}} + (\frac{194}{5})^2 \cdot (\frac{5}{199})^2]}{((\frac{2{,}134}{120})^{\frac{1}{3}} + 1)^2 \arctan((\frac{2{,}134}{120})^{\frac{1}{3}} + 1) + 1} \\ &\quad \approx 0.0090. \end{aligned}$$

By (4.4), (4.7), and the last inequality we get that condition (H_8) holds.

Thus, (H_1)-(H_8) hold. Therefore, by Theorem 3.1 system (4.1) has at least one positive solution on $C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1)$.

Remark 4.1 In Example 4.1, even if we consider only one equation

$$\begin{cases} -x'' = \frac{t}{10(\pi+4)}x^2 \arctan x - \frac{2}{2+3\sqrt[3]{4}}\{\frac{1}{\sqrt{t}} + \frac{1}{\sqrt[3]{(t-\frac{1}{2})^2}}\}, & t \in (0,1), \\ x(0) = H(\varphi_1(x)), \qquad x(1) = 0 \end{cases} \qquad (4.8)$$

the function $H(z) = z^2 + z$ does not satisfy the key condition H of [13], that is, there is a number $C_2 \geq 0$ such that

$$\lim_{z\to+\infty}\frac{|H(z) - C_2 z|}{z} = 0.$$

So [13] cannot deal with the problem.

Remark 4.2 In Example 4.1, the nonlinearity term f has singularity at $t = 0$ and $t = \frac{1}{2}$. Moreover, f can tend to negative infinity as $t \to 0$ or $t \to \frac{1}{2}$, which implies that f need not have a lower bound. So, Example 4.1 well demonstrates this point. In Example 4.1, if $q(t) \equiv 0$, then $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous. Consider the system

$$\begin{cases} -x'' = \frac{t}{10(\pi+4)} y^2 \arctan y, & t \in (0,1), \\ -y'' = \frac{25x^{\frac{3}{2}}}{199^2 t(1-t)}, & t \in (0,1), \\ x(0) = H_1(\varphi_1(y)), & x(1) = 0, \\ y(0) = H_2(\varphi_2(x)), & y(1) = 0. \end{cases} \tag{4.9}$$

Let $H_1(z) := z^2 + z$ and $H_2(z) := z^{\frac{1}{3}}$. We know that H_1 is not superlinear at $t = 0$ and H_2 is not superlinear at $t = 0$ and $t = +\infty$. Then, these do not satisfy the condition for H_i ($i = 1, 2$) in [14], that is,

$$\lim_{z\to0^+}\frac{|H_i(z)|}{z} = 0, \quad i = 1, 2$$

and

$$\lim_{z\to+\infty}\frac{|H_i(z)|}{z} = +\infty.$$

So [14] cannot deal with the problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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References

1. Zhang, Y: Positive solutions of singular Enden-Fowler boundary value problems. J. Math. Anal. Appl. **185**, 215-222 (1994)
2. Zhang, X, Liu, L, Wu, C: Nontrivial solution of third order nonlinear eigenvalue problems. Appl. Math. Comput. **176**, 714-721 (2006)
3. Fink, AM, Gatica, JA: Positive solutions of second order systems of boundary value problems. J. Math. Anal. Appl. **180**, 93-108 (1993)

4. Ma, R: Multiple nonnegative solutions of second order systems of boundary value problems. Nonlinear Anal. **42**, 1003-1010 (2000)
5. João, M, Sebastián, L, Pedro, U: Local superlinearity for elliptic systems involving parameters. J. Differ. Equ. **211**, 1-19 (2005)
6. Agarwal, RP, O'Regan, D: Multiple solutions for a coupled systems of boundary value problems. Dyn. Contin. Discrete Impuls. Syst. **7**, 97-106 (2000)
7. Zhang, X, Liu, L: Existence of positive solutions for a singular semipositone differential system. Math. Comput. Model. **47**, 115-126 (2008)
8. Anuradha, V, Hai, DD, Shivaji, R: Existence results for superlinear semipositone BVPs. Proc. Am. Math. Soc. **124**, 757-763 (1996)
9. Goodrich, CS: On semipositone discrete fractional boundary value problems with nonlocal boundary conditions. J. Differ. Equ. Appl. **19**, 1758-1780 (2013)
10. Greaf, J, Kong, L: Positive solutions for third order semipositone boundary value problems. Appl. Math. Lett. **22**, 1154-1160 (2009)
11. Webb, JRL, Infante, G: Semipositone nonlocal boundary value problems of arbitrary order. Commun. Pure Appl. Anal. **9**, 563-581 (2010)
12. Anderson, DR: Existence of three solutions for a first-order problem with nonlinear nonlocal boundary conditions. J. Math. Anal. Appl. **408**, 318-323 (2013)
13. Goodrich, CS: A note on semipositone boundary value problems with nonlocal, nonlinear boundary conditions. Arch. Math. (Basel) **103**, 177-187 (2014)
14. Goodrich, CS: Coupled systems of boundary value problems with nonlocal boundary conditions. Appl. Math. Lett. **41**, 17-22 (2015)
15. Goodrich, CS: On a nonlocal BVP with nonlinear boundary conditions. Results Math. **63**, 1351-1364 (2013)
16. Goodrich, CS: Semipositone boundary value problems with nonlocal, nonlinear boundary conditions. Adv. Differ. Equ. **20**, 1/2 (2015)
17. Infante, G: Nonlocal boundary value problems with two nonlinear boundary conditions. Commun. Appl. Anal. **12**, 279-288 (2008)
18. Infante, G, Pietramala, P: Existence and multiplicity of nonnegative solutions for system of perturbed Hammerstein integral equations. Nonlinear Anal. **71**, 1301-1310 (2009)
19. Infante, G, Pietramala, P: Multiple nonnegative solutions of systems with coupled nonlinear boundary conditions. Math. Methods Appl. Sci. **67**, 2080-2090 (2014)
20. Cianciaruso, F, Infante, G, Pietramala, P: Existence and multiplicity of nonnegative solutions of perturbed Hammerstein integral equations with applications. Nonlinear Anal., Real World Appl. **33**, 317-347 (2017)
21. Liu, L, Hao, X, Wu, Y: Positive solutions for singular secong order differential equations with integral boundary conditions. Math. Comput. Model. **57**, 836-847 (2013)
22. Cui, Y, Liu, L, Zhang, X: Uniqueness and existence of positive solutions for singular differential systems with coupled integral boundary value problems. Abstr. Appl. Anal. **2013** Article ID 340487 (2013)
23. Guo, DJ, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, San Diego (1988)