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$(\psi,\varphi,\epsilon,\lambda)$-Contraction theorems in probabilistic metric spaces for single valued case

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Abstract

In this article, we prove some fixed-point theorems for $(\psi,\varphi,\epsilon,\lambda)$-contraction in probabilistic metric spaces for single valued case. We will generalize the definition of $(\psi,\varphi,\epsilon,\lambda)$-contraction and present fixed-point theorem in the generalized $(\psi,\varphi,\epsilon,\lambda)$-contraction.

Keywords: probabilistic metric spaces; $(\psi,\varphi,\epsilon,\lambda)$-contraction; fixed-point

1 Introduction

The probabilistic metric space was introduced by Menger [1]. Mihet presented the class of $(\psi,\varphi,\epsilon,\lambda)$-contraction for a single valued case in fuzzy metric spaces [2, 3]. This class is a generalization of the (ϵ,λ)-contraction which was introduced in [4]. We defined the class of $(\psi,\varphi,\epsilon,\lambda)$-contraction for the multi-valued case in a probabilistic metric space before [5]. Now, we obtain two fixed-point theorems of $(\psi,\varphi,\epsilon,\lambda)$-contraction for single valued case. Also, we extend the concept of $(\psi,\varphi,\epsilon,\lambda)$-contraction to the generalized $(\psi,\varphi,\epsilon,\lambda)$-contraction.

The structure of this paper is as follows: Section 2 is a review of some concepts in probabilistic metric spaces and probabilistic contractions. In Section 3, we will show two theorems for $(\psi,\varphi,\epsilon,\lambda)$-contraction in the single-valued case and explain the generalized $(\psi,\varphi,\epsilon,\lambda)$-contraction.

2 Preliminary notes

We recall some concepts from probabilistic metric space, convergence and contraction. For more details, we refer the reader to [6–8].

Let D_+ be the set of all distribution of functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0,1]$ such that $\lim_{x\to\infty} F(x) = 1$).

The ordered pair (S,F) is said to be a probabilistic metric space if S is a nonempty set and $F : S \times S \to D_+$ ($F(p,q)$ written by F_{pq} for every $(p,q) \in S \times S$) satisfies the following conditions:

(1) $F_{uv}(x) = 1$ for every $x > 0 \Leftrightarrow u = v$ $(u,v \in S)$,
(2) $F_{uv} = F_{vu}$ for every $u,v \in S$,
(3) $F_{uv}(x) = 1$ and $F_{vw}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for every $u,v,w \in S$, and every $x,y \in \mathbb{R}^+$.

A Menger space is a triple (S, F, T) where (S, F) is a probabilistic metric space, T is a triangular norm (abbreviated t-norm) and the following inequality holds:

$$F_{uv}(x+y) \geq T\big(F_{uw}(x), F_{wv}(y)\big) \quad \text{for every } u, v, w \in S, \text{ and every } x, y \in \mathbb{R}^+.$$

Recall the mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied: $T(a,1) = a$ for every $a \in [0,1]$; $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$; $a \geq b, c \geq d \Rightarrow T(a,c) \geq T(b,d)$, $a,b,c,d \in [0,1]$; $T(T(a,b),c) = T(a,T(b,c))$, $a,b,c \in [0,1]$. Basic examples of t-norms are T_L (Lukasiewicz t-norm), T_P and T_M, defined by $T_L(a,b) = \max\{a+b-1, 0\}$, $T_P(a,b) = ab$ and $T_M(a,b) = \min\{a,b\}$. If T is a t-norm and $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ $(n \in \mathbb{N}^*)$, one can define recurrently $\top_{i=1}^n x_i = T(\top_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. One can also extend T to a countable infinitary operation by defining $\top_{i=1}^\infty x_i$ for any sequence $(x_i)_{i \in \mathbb{N}^*}$ as $\lim_{n\to\infty} \top_{i=1}^n x_i$.

If $q \in (0,1)$ is given, we say that the t-norm T is q-convergent if $\lim_{n\to\infty} \top_{i=n}^\infty (1-q^i) = 1$. We remark that if T is q-convergent, then

$$\forall \lambda \in (0,1)\ \exists s = s(\lambda) \in \mathrm{N}\ \top_{i=1}^n \big(1 - q^{s+i}\big) > 1 - \lambda, \quad \forall n \in \mathrm{N}.$$

Also, note that if the t-norm T is q-convergent, then $\sup_{0 \leq t < 1} T(t,t) = 1$.

Proposition 2.1 *Let (S, F, T) be a Menger space. If $\sup_{0 \leq t < 1} T(t,t) = 1$, then the family $\{U_\epsilon\}_{\epsilon > 0}$, where*

$$U_\epsilon = \big\{(x,y) \in S \times S \mid F_{x,y}(\epsilon) > 1 - \epsilon\big\}$$

is a base for a metrizable uniformity on S, called the F-uniformity [6–8]. *The F-uniformity naturally determines a metrizable topology on S, called the strong topology or F-topology* [9], *a subset O of S is F-open if for every $p \in O$ there exists $t > 0$ such that $N_p = \{q \in S | F_{pq}(t) > 1 - t\} \subset O$.*

Definition 2.1 [6] A sequence $(x_n)_{n \in \mathrm{N}}$ is called an F-convergent sequence to $x \in S$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $N = N(\epsilon, \lambda) \in \mathrm{N}$ such that $F_{x_n x}(\epsilon) > 1 - \lambda$, $\forall n \geq N$.

Definition 2.2 [6] Let $\varphi : (0,1) \to (0,1)$ be a mapping, we say that the t-norm T is φ-convergent if

$$\forall \delta \in (0,1), \forall \lambda \in (0,1)\ \exists s = s(\delta, \lambda) \in \mathrm{N}\ \top_{i=1}^n \big(1 - \varphi^{s+i}(\delta)\big) > 1 - \lambda, \quad \forall n \geq 1.$$

Definition 2.3 [6] A sequence $(x_n)_{n \in \mathrm{N}}$ is called a convergent sequence to $x \in S$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $N = N(\epsilon, \lambda) \in \mathrm{N}$ such that $F_{x_n x}(\epsilon) > 1 - \lambda$, $\forall n \geq N$.

Definition 2.4 [6] A sequence $(x_n)_{n \in \mathrm{N}}$ is called a Cauchy sequence if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $N = N(\epsilon, \lambda) \in \mathrm{N}$ such that $F_{x_n x_{n+m}}(\epsilon) > 1 - \lambda$, $\forall n \geq N$, $\forall m \in \mathbb{N}$.

We also have

$$x_n \to^F x \quad \Leftrightarrow \quad F_{x_n x}(t) \to 1 \quad \forall t > 0.$$

A probabilistic metric space (S, F, T) is called sequentially complete if every Cauchy sequence is convergent.

The concept of $(\psi, \varphi, \epsilon, \lambda)$-contraction has been introduced by Mihet [3].

We will consider comparison functions from the class ϕ of all mapping $\varphi : (0,1) \to (0,1)$ with the properties:

(1) φ is an increasing bijection;

(2) $\varphi(\lambda) < \lambda$ $\forall \lambda \in (0,1)$.

Since every such a comparison mapping is continuous, if $\varphi \in \phi$, then $\lim_{n\to\infty} \varphi^n(\lambda) = 0$ $\forall \lambda \in (0,1)$.

Definition 2.5 [3] Let (S, F) be a probabilistic space, $\varphi \in \phi$ and ψ be a map from $(0,\infty)$ to $(0,\infty)$. A mapping $f : S \to S$ is called a $(\psi, \varphi, \epsilon, \lambda)$-contraction on S if it satisfies in the following condition:

$$x, y \in S, \epsilon > 0, \lambda \in (0,1), \quad F_{x,y}(\epsilon) > 1 - \lambda \quad \Rightarrow \quad F_{f(x),f(y)}\bigl(\psi(\epsilon)\bigr) > 1 - \varphi(\lambda).$$

In the rest of paper we suppose that ψ is increasing bijection.

Example 2.1 Let $S = \{0, 1, 2, \dots\}$ and (for $x \neq y$)

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \le 2^{-\min(x,y)}, \\ 1 - 2^{-\min(x,y)} & \text{if } 2^{-\min(x,y)} < t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Suppose that $f : S \to S, f(r) = r + 1$.

Then (S, F, T_L) is a probabilistic metric space [10].

Let x, y, ϵ, λ be such that $F_{x,y}(\epsilon) > 1 - \lambda$.

(i) If $2^{-\min(x,y) < \epsilon \le 1}$, then $1 - 2^{-\min(x,y)} > 1 - \lambda$.

This implies $1 - 2^{-\min(x+1,y+1)} > 1 - \frac{1}{2}\lambda$, that is,

$$F_{fx,fy}(\epsilon) > 1 - \frac{1}{2}\lambda.$$

(ii) If $\epsilon > 1$ then $F_{fx,fy}(\epsilon) = 1$, hence again $F_{fx,fy}(\epsilon) > 1 - \frac{1}{2}\lambda$. Thus, the mapping f is a $(\psi, \varphi, \epsilon, \lambda)$-contraction on S with $\psi(\epsilon) = \epsilon$ and $\varphi(\lambda) = \frac{1}{2}\lambda$.

3 Main results

In this section, we will show $(\psi, \varphi, \epsilon, \lambda)$-contraction is continuous. By using this assumption, we will also prove two theorems.

Definition 3.1 Let F be a probabilistic distance on S. A mapping $f : S \to S$ is called continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$F_{u,v}(\delta) > 1 - \delta \quad \Rightarrow \quad F_{fu,fv}(\epsilon) > 1 - \epsilon.$$

Before we start to present the theorems, we will explain the following lemma.

Lemma 3.1 *Every $(\psi, \varphi, \epsilon, \lambda)$-contraction is continuous.*

Proof Suppose that $\epsilon > 0$ be given and $\delta \in (0,1)$ be such that $\delta < \min\{\epsilon, \psi^{-1}(\epsilon)\}$ and since ψ is increasing bijection then $\psi(\delta) < \epsilon$. If $F_{x,y}(\delta) > 1 - \delta$ then, by $(\psi, \varphi, \epsilon, \lambda)$-contraction we have $F_{fx,fy}(\psi(\delta)) > 1 - \varphi(\delta)$, from where we obtain that $F_{fx,fy}(\epsilon) > F_{fx,fy}(\psi(\delta)) > 1 - \varphi(\delta) > 1 - \delta > 1 - \epsilon$. So f is continuous. □

Theorem 3.1 *Let (S, F, T) be a complete Menger space and T a t-norm satisfies in $\sup_{0 \leq a < 1} T(a,a) = 1$. Also, $f : S \to S$ a $(\psi, \varphi, \epsilon, \lambda)$-contraction where $\lim_{n\to\infty} \psi^n(\delta) = 0$ for every $\delta \in (0, \infty)$. If $\lim_{t\to\infty} F_{x_0, f^m x_0}(t) = 1$ for some $x_0 \in S$ and all $m \in N$, then there exists a unique fixed point x of the mapping f and $x = \lim_{n\to\infty} f^n(x_0)$.*

Proof Let $x_n = f^n x_0$, $n \in N$. We shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let $n, m \in N$, $\epsilon > 0$, $\lambda \in (0,1)$. Since $\lim_{t\to\infty} F_{x_0, f^m x_0(t)}(t) = 1$, it follows that for every $\xi \in (0,1)$ there exists $\eta > 0$ such that $F_{x_0, f^m(x_0)}(\eta) > 1 - \xi$ and by induction $F_{f^m x_0, f^{n+m} x_0}(\psi^n(\eta)) > 1 - \varphi^n(\xi)$ for all $n \in \mathbb{N}$. By choosing n such that $\psi^n(\eta) < \epsilon$ and $\varphi^n(\xi) < \lambda$, we obtain

$$F_{x_n, x_{n+m}}(\epsilon) > 1 - \lambda.$$

Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and since S is complete, it follows the existence of $x \in S$ such that $x = \lim_{n\to\infty} x_n$. By continuity of f and $x_{n+1} = f x_n$ for every $n \in \mathbb{N}$, when $n \to \infty$, we obtain that $x = fx$. □

Example 3.1 Let (S, F, T) be a complete Menger space where $S = \{x_1, x_2, x_3, x_4\}$, $T(a,b) = \min\{a,b\}$ and $F_{xy}(t)$ is defined as

$$F_{x_1,x_2}(t) = F_{x_2,x_1}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 0.9 & \text{if } 0 < t \leq 3, \\ 1 & \text{if } t > 3 \end{cases}$$

and

$$\begin{aligned} F_{x_1,x_3}(t) &= F_{x_3,x_1}(t) = F_{x_1,x_4}(t) = F_{x_4,x_1}(t) = F_{x_2,x_3}(t) = F_{x_3,x_2}(t) = F_{x_2,x_4}(t) \\ &= F_{x_4,x_2}(t) = F_{x_3,x_4}(t) = F_{x_4,x_3}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 0.7 & \text{if } 0 < t < 6, \\ 1 & \text{if } 6 \leq t \end{cases} \end{aligned}$$

$f : S \to S$ is given by $f(x_1) = f(x_2) = x_2$ and $f(x_3) = f(x_4) = x_1$. If we take $\varphi(\lambda) = \frac{\lambda}{2}$, $\psi(\epsilon) = \frac{\epsilon}{2}$, then f is a $(\psi, \varphi, \epsilon, \lambda)$-contraction where $\lim_{n\to\infty} \psi^n(\delta) = \lim_{n\to\infty} \frac{\delta}{2^n} = 0$ for every $\delta \in (0, \infty)$ and if we set $x_0 = x_2$, then for all $m \in N$, we have $f^m x_0 = f^m x_2 = x_2$ and $\lim_{t\to\infty} F_{x_2 x_2}(t) = 1$, so x_2 is the unique fixed point for f.

Theorem 3.2 *Let (S, F, T) be a complete Menger space, T be a t-norm such that $\sup_{0 \leq a < 1} T(a,a) = 1$ and $f : S \to S$ a $(\psi, \varphi, \epsilon, \lambda)$-contraction where the series $\sum_i \psi^i(\delta)$ is convergent for all $\delta > 0$ and suppose that for some $p \in S$ and $j > 0$*

$$\sup_{x>0} x^j \bigl(1 - F_{p,fp}(x)\bigr) < \infty.$$

If t-norm T is φ-convergent, then there exist a unique fixed point z of mapping f and $z = \lim_{l\to\infty} f^l p$.

Proof Choose $\epsilon > 0$ and $\lambda \in (0,1)$. Let $z_l = f^l p$, $l \in N$. We shall prove that $(z_l)_{l\in N}$ is a Cauchy sequence. It means we prove that there exists $n_0(\epsilon,\lambda) \in N$ such that

$$F_{f^l p, f^{l+m} p}(\epsilon) > 1 - \lambda \quad \text{for every } l \geq n_0(\epsilon,\lambda) \text{ and every } m \in N.$$

Suppose that $\mu \in (0,1)$, $M > 0$ are such that

$$x^j\big(1 - F_{p,fp}(x)\big) \leq M \quad \forall x > 0. \tag{1}$$

Let n_1 be such that

$$1 - M\big(\mu^j\big)^{n_1} \in [0,1).$$

From (1), it follows that

$$F_{p,fp}\left(\frac{1}{\mu^n}\right) > 1 - M\big(\mu^j\big)^n \quad \forall n \in N \text{ specially for } n = n_1.$$

Since f is $(\psi,\varphi,\epsilon,\lambda)$-contraction, we derived by induction $F_{f^l p, f^{l+1} p}(\psi^l(\frac{1}{\mu^{n_1}})) > 1 - \varphi^l(1 - M(\mu^j)^{n_1})$ $\forall l > 1$. Since the series $\sum_{i=1}^{\infty} \psi^i(\delta)$ is convergent, there exists $n_2 = n_2(\epsilon) \in N$ such that $\sum_{i=l}^{\infty} \psi^i(\delta) \leq \epsilon$ $\forall l \geq n_2$. We know $\sum_{i=l}^{\infty} \psi^i(\frac{1}{\mu^{n_1}}) \leq \epsilon$ for every $l > \max\{n_1, n_2\}$.

Now

$$\begin{aligned}
F_{f^l p, f^{l+m} p}(\epsilon) &\geq F_{f^l p, f^{l+m} p}\left(\sum_{i=l}^{\infty} \psi^i\left(\frac{1}{\mu^{n_1}}\right)\right) \geq F_{f^l p, f^{l+m} p}\left(\sum_{i=l}^{l+m-1} \psi^i\left(\frac{1}{\mu^{n_1}}\right)\right) \\
&\geq T\left(T\left(\cdots T\left(F_{f^l p, f^{l+1} p}\left(\psi^l\left(\frac{1}{\mu^{n_1}}\right)\right), F_{f^{l+1} p, f^{l+2} p}\left(\psi^{l+1}\left(\frac{1}{\mu^{n_1}}\right)\right)\right), \ldots,\right.\right. \\
&\qquad \left.\left. F_{f^{l+m-1} p, f^{l+m} p}\left(\psi^{l+m-1}\left(\frac{1}{\mu^{n_1}}\right)\right)\right)\right) \\
&\geq T\big(T\big(\cdots T\big(\big(1 - \varphi^l\big(1 - M\big(\mu^j\big)^{n_1}\big), \big(1 - \varphi^{l+1}\big(1 - M\big(\mu^j\big)^{n_1}\big)\big), \ldots, \\
&\qquad \big(1 - \varphi^{l+m-1}\big(1 - M\big(\mu^j\big)^{n_1}\big)\big)\big)\big)\big) \\
&\geq \top_{i=l}^{\infty}\big(1 - \varphi^i\big(1 - M\big(\mu^j\big)^{n_1}\big)\big).
\end{aligned}$$

Since T is φ-convergent, we conclude that $(f^l p)_{l\in\mathbb{N}}$ is a Cauchy sequence. On the other hand, S is complete, therefore, there is a $z \in S$ such that $z = \lim_{l\to\infty} f^l p$. By the continuity of the mapping f and $z_{l+1} = f z_l$ when $l \to +\infty$, it follows that $fz = z$. □

Example 3.2 Let (S,F,T) and the mappings f, ψ and φ be the same as in Example 3.1. Since $\sum_i \psi^i(\delta) = \sum_i \frac{\delta}{2^i} = \delta$ for all $\delta > 0$ and if we set $p = x_2 \in S$, $j > 0$ then $t^j(1 - F_{x_2 x_2}(t)) = 0$ for every $t > 0$ or $\sup_{t>0} t^j(1 - F_{x_2 x_2}(t)) < \infty$, so x_2 is the unique fixed point for f.

Mihet in [3] showed, if $f : S \to S$ is a $(\psi,\varphi,\epsilon,\lambda)$-contraction and (S, M, T) is a complete fuzzy metric space, then f has an unique fixed point. Now we present a generalization of the $(\psi,\varphi,\epsilon,\lambda)$-contraction. First, we define the class of functions $\aleph$ as follows.

Let $\aleph$ be the family of all the mappings $m : \overline{R} \to \overline{R}$ such that the following conditions are satisfied:

(1) $\forall t, s \geq 0 : m(t+s) \geq m(t) + m(s)$;
(2) $m(t) = 0 \Leftrightarrow t = 0$;
(3) m is continuous.

Definition 3.2 Let (S, F) be a probabilistic metric space and $f : S \to S$. The mapping f is a generalized $(\psi, \varphi, \epsilon, \lambda)$-contraction if there exist a continuous, decreasing function $h : [0,1] \to [0,\infty]$ such that $h(1) = 0$, $m_1, m_2 \in \aleph$, and $\lambda \in (0,1)$ such that the following implication holds for every $p, q \in S$ and for every $\epsilon > 0$:

$$hoF_{p,q}\bigl(m_2(\epsilon)\bigr) < m_1(\lambda) \quad \Rightarrow \quad hoF_{f(p),f(q)}\bigl(m_2\bigl(\psi(\epsilon)\bigr)\bigr) < m_1\bigl(\varphi(\lambda)\bigr).$$

If $m_1(a) = m_2(a) = a$, and $h(a) = 1 - a$ for every $a \in [0,1]$, we obtain the Mihet definition.

Theorem 3.3 *Let (S, F, T) be a complete Menger space with t-norm T such that $\sup_{0 \leq a < 1} T(a,a) = 1$ and $f : S \to S$ be a generalized $(\psi, \varphi, \epsilon, \lambda)$-contraction such that ψ is continuous on $(0,\infty)$ and $\lim_{n\to\infty} \psi^n(\delta) = 0$ for every $\delta \in (0,\infty)$. Suppose that there exists $\lambda \in (0,1)$ such that $h(0) < m_1(\lambda)$ and φ, ψ satisfy $\varphi(0) = \psi(0) = 0$. Then $x = \lim_{n\to\infty} f^n(p)$ is the unique fixed point of the mapping f for an arbitrary $p \in S$.*

Proof First we shall prove that f is uniformly continuous. Let $\zeta > 0$ and $\eta \in (0,1)$. We have to prove that there exists $N(\overline{\zeta}, \overline{\eta}) = \{(p,q) | (p,q) \in S \times S, F_{p,q}(\overline{\zeta}) > 1 - \overline{\eta}\}$ such that

$$(p,q) \in N(\overline{\zeta}, \overline{\eta}) \quad \Rightarrow \quad F_{f(p),f(q)}(\zeta) > 1 - \eta.$$

Let ϵ be such that $m_2(\psi(\epsilon)) < \zeta$ and $\lambda \in (0,1)$ such that

$$m_1\bigl(\varphi(\lambda)\bigr) < h(1 - \eta). \tag{2}$$

Since m_1 and m_2 are continuous at zero, and $m_1(0) = m_2(0) = 0$ such numbers ϵ and λ exist. We prove that $\overline{\zeta} = m_2(\epsilon)$, $\overline{\eta} = 1 - h^{-1}(m_1(\lambda))$. If $(p,q) \in N(\overline{\zeta}, \overline{\eta})$, we have

$$\begin{aligned} F_{p,q}\bigl(m_2(\epsilon)\bigr) &> 1 - \bigl(1 - h^{-1}\bigl(m_1(\lambda)\bigr)\bigr) \\ &= h^{-1}\bigl(m_1(\lambda)\bigr). \end{aligned}$$

Since h is decreasing, it follows that $hoF_{p,q}(m_2(\epsilon)) < m_1(\lambda)$. Hence,

$$hoF_{f(p),f(q)}\bigl(m_2\bigl(\psi(\epsilon)\bigr)\bigr) < m_1\bigl(\varphi(\lambda)\bigr).$$

Using (2), we conclude that

$$hoF_{f(p),f(q)}\bigl(m_2\bigl(\psi(\epsilon)\bigr)\bigr) < h(1 - \eta)$$

and since h is decreasing we have

$$F_{f(p),f(q)}(\zeta) \geq F_{f(p),f(q)}\bigl(m_2\bigl(\psi(\epsilon)\bigr)\bigr) > 1 - \eta.$$

Therefore, $(f(p), f(q)) \in N(\zeta, \eta)$ if $(p,q) \in N(\overline{\zeta}, \overline{\eta})$. We prove that for every $\zeta > 0$ and $\eta \in (0,1)$ there exists $n_0(\zeta, \eta) \in N$ such that for every $p, q \in S$

$$n > n_0(\zeta, \eta) \quad \Rightarrow \quad F_{f^n(p), f^n(q)}(\zeta) > 1 - \eta. \tag{3}$$

By assumption, there is a $\lambda \in (0,1)$ such that $h(0) < m_1(\lambda)$. From $F_{p,q}(m_2(\epsilon)) \geq 0$, it follows that

$$hoF_{p,q}\big(m_2(\epsilon)\big) \leq h(0) < m_1(\lambda)$$

which implies that $hoF_{f(p),f(q)}(m_2(\psi(\epsilon))) < m_1(\varphi(\lambda))$, and continuing in this way we obtain that for every $n \in N$

$$hoF_{f^n(p), f^n(q)}\big(m_2\big(\psi^n(\epsilon)\big)\big) < m_1\big(\varphi^n(\lambda)\big).$$

Let $n_0(\zeta, \eta)$ be a natural number such that $m_2(\psi^n(\epsilon)) < \zeta$ and $m_1(\varphi^n(\lambda)) < h(1 - \eta)$, for every $n \geq n_0(\zeta, \eta)$. Then $n > n_0(\zeta, \eta)$ implies that

$$F_{f^n(p), f^n(q)}(\zeta) \geq F_{f^n(p), f^n(q)}\big(m_2\big(\psi^n(\epsilon)\big)\big) > 1 - \eta.$$

If $q = f^m(p)$, from (3) we obtain that

$$F_{f^n(p), f^{n+m}(p)}(\zeta) > 1 - \eta \quad \text{for every } n > n_0(\zeta, \eta) \text{ and every } m \in N. \tag{4}$$

Relation (4) means that $(f^n(p))_{n \in N}$ is a Cauchy sequence, and since S is complete there exists $x = \lim_{n \to \infty} f^n(p)$, which is obviously a fixed point of f since f is continuous.

For every $p \in S$ and $q \in S$ such that $f(p) = p$ and $f(q) = q$ we have for every $n \in N$ that $f^n(p) = p$, $f^n(q) = q$ and, therefore, from (3) we have $F_{p,q}(\zeta) > 1 - \eta$ for every $\eta \in (0,1)$ and $\zeta > 0$. This implies that $F_{p,q}(\zeta) = 1$ for every $\zeta > 0$ and, therefore, $p = q$. □

Example 3.3 Let (S, F, T) and the mappings f, ψ and φ be the same as in Example 3.1. Set $h(a) = e^{-a} - e^{-1}$ for every $a \in [0,1]$ and $m_1(a) = m_2(a) = a$. The mapping f is generalized $(\psi, \varphi, \epsilon, \lambda)$-contraction and $\lim_{n \to \infty} \psi^n(\delta) = \lim_{n \to \infty} \frac{\delta}{2^n} = 0$ for every $\delta \in (0, \infty)$. On the other hand, there exists $\lambda \in (0,1)$ such that $h(0) = 1 - \frac{1}{e} < \lambda$ and $\psi(0) = \varphi(0) = 0$. So x_2 is the unique fixed point for f.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

PA defined the definitions and wrote the Introduction, preliminaries and abstract. AB proved the theorems. AB has approved the final manuscript. Also, PA has verified the final manuscript.

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