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Homoclinic orbits for second-order discrete Hamiltonian systems with subquadratic potential

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Abstract

Under the assumptions that $W(n,x)$ is indefinite sign and subquadratic as $|x| \to +\infty$ and $L(n)$ satisfies

$$\liminf_{|n|\to+\infty}\Big[|n|^{\nu-2}\inf_{|x|=1}\big(L(n)x,x\big)\Big] > 0$$

for some constant $\nu < 2$, we establish a theorem on the existence of infinitely many homoclinic solutions for the second-order self-adjoint discrete Hamiltonian system

$$\triangle\big[p(n)\triangle u(n-1)\big] - L(n)u(n) + \nabla W\big(n,u(n)\big) = 0,$$

where $p(n)$ and $L(n)$ are $\mathcal{N}\times\mathcal{N}$ real symmetric matrices for all $n \in \mathbb{Z}$, and $p(n)$ is always positive definite.

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1 Introduction

Consider the second-order self-adjoint discrete Hamiltonian system

$$\triangle\big[p(n)\triangle u(n-1)\big] - L(n)u(n) + \nabla W\big(n,u(n)\big) = 0, \tag{1.1}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}^{\mathcal{N}}$, $\Delta u(n) = u(n+1) - u(n)$ is the forward difference, $p, L : \mathbb{Z} \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ and $W : \mathbb{Z}\times\mathbb{R}^{\mathcal{N}} \to \mathbb{R}$, $W(n,x)$ is continuously differentiable in x for every $n \in \mathbb{Z}$.

As usual, we say that a solution $u(n)$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

The existence and multiplicity of nontrivial homoclinic solutions for problem (1.1) have been extensively investigated in the literature with the aid of critical point theory and variational methods; see, for example, [1–13]. Most of them treat the case where $W(n,x)$ is superquadratic as $|x| \to \infty$.

Compared to the superquadratic case, as far as the author is aware, there are a few papers [10, 12, 13] concerning the case where $W(n,x)$ has subquadratic growth at infinity.

Specifically, [12] and [10] dealt with the existence and multiplicity of homoclinic solutions for (1.1) under the following assumptions on L:

(L_*) *$L(n)$ is an $\mathcal{N}\times\mathcal{N}$ real symmetric positive definite matrix for all $n\in\mathbb{Z}$ and there exists a constant $\beta>0$ such that*

$$\big(L(n)x,x\big)\geq\beta|x|^2,\quad \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}};$$

(L_ν) *$L(n)$ is an $\mathcal{N}\times\mathcal{N}$ real symmetric positive definite matrix for all $n\in\mathbb{Z}$ and there exists a constant $\nu<2$ such that*

$$\liminf_{|n|\to+\infty}\Big[|n|^{\nu-2}\inf_{|x|=1}\big(L(n)x,x\big)\Big]>0,$$

respectively. In the above two cases, since $L(n)$ is positive definite, the variational functional associated with system (1.1) is bounded from below, techniques based on the genus properties have been well applied. In particular, Clark's theorem is an efficacious tool to prove the existence and multiplicity of homoclinic solutions for system (1.1). However, if $L(n)$ is not global positive definite on $\mathbb{Z}$, the problem is far more difficult as 0 is a saddle point rather than a local minimum of the variational functional, which is strongly indefinite and it is not easy to obtain the boundedness of the Palais-Smale sequence. In a recent paper [13], based on a new direct sum decomposition of the 'work space', Tang and Lin proved the following theorem by using a linking theorem which was developed in [14].

Theorem 1.1 [13] *Assume that $p(n)$ is an $\mathcal{N}\times\mathcal{N}$ real symmetric positive definite matrix for all $n\in\mathbb{Z}$, L and W satisfy the following assumptions*:

(L'_ν) *$L(n)$ is an $\mathcal{N}\times\mathcal{N}$ real symmetric matrix for all $n\in\mathbb{Z}$ and there exists a constant $\nu<2$ such that*

$$\liminf_{|n|\to+\infty}\Big[|n|^{\nu-2}\inf_{|x|=1}\big(L(n)x,x\big)\Big]>0;$$

(W1) *there exist constants* $\max\{1,2/(3-\nu)\}<\gamma_1<\gamma_2<2$ *and* $a_1,a_2\geq 0$ *such that*

$$\big|W(n,x)\big|\leq a_1|x|^{\gamma_1}+a_2|x|^{\gamma_2},\quad \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}};$$

(W2) *there exists a function* $\varphi\in C([0,+\infty),[0,+\infty))$ *such that*

$$\big|\nabla W(n,x)\big|\leq\varphi\big(|x|\big),\quad \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}},$$

where $\varphi(s)=O(s^{\gamma_3-1})$ *as* $s\to 0^+$, $\max\{1,2/(3-\nu)\}<\gamma_3<2$;

(W3) *there exist constants* $b_1>0$, $b_2,b_3\geq 0$ *and* $\max\{1,2/(3-\nu)\}<\gamma_6<\gamma_5<\gamma_4<2$ *such that*

$$2W(n,x)-\nabla W(n,x)\geq b_1|x|^{\gamma_4}-b_2|x|^{\gamma_5}-b_3|x|^{\gamma_6},\quad \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{N};$$

(W4) *there exist constants* $b_4>0$, $b_5,b_6\geq 0$ *and* $\max\{1,2/(3-\nu)\}<\gamma_7<\gamma_8<\gamma_9<2$ *such that*

$$W(n,x)\geq b_4|x|^{\gamma_7}-b_5|x|^{\gamma_8}-b_6|x|^{\gamma_9},\quad \forall(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}};$$

(W5) $W(n,-x) = W(n,x)$, $\forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$.

Then system (1.1) *possesses infinitely many nontrivial homoclinic solutions*.

We remark that the condition 'positive definite' is removed in (L'_ν), *i.e.*, $L(n)$ is not required to be global positive definite on $\mathbb{Z}$. The main goal of this paper is to weaken conditions (W1), (W2), (W3) and (W4) of Theorem 1.1 under assumption (L'_ν).

To state our result, we first introduce the following assumptions:

(W1′) *there exist constants* $\sigma_i \in [0, 2-\nu)$, $a_i \geq 0$ *and* $\max\{1, 2(1+\sigma_i)/(3-\nu)\} < \gamma_i < 2$ *with* $i = 1, 2$ *such that*

$$\left|W(n,x)\right| \leq \sum_{i=1}^{2} a_i\left(1 + |n|^{\sigma_i}\right)|x|^{\gamma_i}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}};$$

(W2′) *there exist two constants* $\max\{1, 2(1+\sigma_i)/(3-\nu)\} < \gamma_{i+2} < 2$, $i = 1, 2$ *and two functions* $\varphi_1, \varphi_2 \in C([0,+\infty), [0,+\infty))$ *such that*

$$\left|\nabla W(n,x)\right| \leq \sum_{i=1}^{2}\left(1 + |n|^{\sigma_i}\right)\varphi_i\left(|x|\right), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}},$$

where $\varphi_i(s) = O(s^{\gamma_{i+2}-1})$ *as* $s \to 0^+$, $i = 1, 2$;

(W3′) *there exist constants* $b_1 > 0$, $b_2 \geq 0$ *and* $1 < \gamma_6 < \gamma_5 < 2$ *such that*

$$2W(n,x) - \nabla W(n,x) \geq b_1|x|^{\gamma_5} - b_2|x|^{\gamma_6}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{N};$$

(W4′) *there exist constants* $b_3 > 0$, $b_4 \geq 0$ *and* $1 < \gamma_7 < \gamma_8 < 2$ *such that*

$$W(n,x) \geq b_3|x|^{\gamma_7} - b_4|x|^{\gamma_8}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}.$$

We are now in a position to state the main result of this paper.

Theorem 1.2 *Assume that* $p(n)$ *is an* $\mathcal{N} \times \mathcal{N}$ *real symmetric positive definite matrix for all* $n \in \mathbb{Z}$, L *and* W *satisfy* (L'_ν), (W1′), (W2′), (W3′), (W4′) *and* (W5). *Then system* (1.1) *possesses infinitely many nontrivial homoclinic solutions*.

2 Preliminaries

In what follows, we always assume that $p(n)$ is a real symmetric positive definite matrix for all $n \in \mathbb{Z}$. As done in [13], we define

$$l(n) = \inf_{x \in \mathbb{R}^{\mathcal{N}}, |x|=1}\left(L(n)x, x\right) \tag{2.1}$$

and

$$\mathbb{Z}^1 = \left\{n \in \mathbb{Z} : l(n) \leq 0\right\}, \qquad \mathbb{Z}^2 = \left\{n \in \mathbb{Z} : l(n) > 0\right\}. \tag{2.2}$$

Then by (L'_ν), $l(n)$ is bounded from below and so $\mathbb{Z}^1$ is a finite set and

$$l_* := \min\left\{l(n) : n \in \mathbb{Z}^2\right\} > 0. \tag{2.3}$$

Define

$$\tilde{L}(n) = \begin{cases} l_* I_{\mathcal{N}}, & n \in \mathbb{Z}^1, \\ L(n), & n \in \mathbb{Z}^2; \end{cases} \qquad \tilde{l}(n) = \begin{cases} l_*, & n \in \mathbb{Z}^1, \\ l(n), & n \in \mathbb{Z}^2. \end{cases} \tag{2.4}$$

Then, it follows from (2.1), (2.2), (2.3) and (2.4) that

$$\big(\tilde{L}(n)x, x\big) \ge \tilde{l}(n)|x|^2 \ge l_*|x|^2, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}. \tag{2.5}$$

Let

$$S = \big\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^{\mathcal{N}}, n \in \mathbb{Z}\big\},$$
$$E = \left\{u \in S : \sum_{n\in\mathbb{Z}} \big[\big(p(n+1)\triangle u(n), \triangle u(n)\big) + \big(\tilde{L}(n)u(n), u(n)\big)\big] < +\infty\right\},$$

and for $u, v \in E$, let

$$(u,v) = \sum_{n\in\mathbb{Z}} \big[\big(p(n+1)\triangle u(n), \triangle v(n)\big) + \big(\tilde{L}(n)u(n), v(n)\big)\big].$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| = \left\{\sum_{n\in\mathbb{Z}} \big[\big(p(n+1)\triangle u(n), \triangle u(n)\big) + \big(\tilde{L}(n)u(n), u(n)\big)\big]\right\}^{1/2}, \quad u \in E.$$

As usual, for $1 \le q < +\infty$, set

$$l^q\big(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}\big) = \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^{\mathcal{N}}, n \in \mathbb{Z}, \sum_{n\in\mathbb{Z}} \big|u(n)\big|^q < +\infty\right\}$$

and

$$l^\infty\big(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}\big) = \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^{\mathcal{N}}, n \in \mathbb{Z}, \sup_{n\in\mathbb{Z}} \big|u(n)\big| < +\infty\right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}} \big|u(n)\big|^q\right)^{1/q}, \quad \forall u \in l^q\big(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}\big);$$
$$\|u\|_\infty = \sup_{n\in\mathbb{Z}} \big|u(n)\big|, \quad \forall u \in l^\infty\big(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}\big),$$

respectively.

Lemma 2.1 [9, Lemma 2.2] *For $u \in E$, one has*

$$\|u\|_\infty \le \frac{1}{\sqrt[4]{(l_* + 4\alpha)l_*}} \|u\|, \tag{2.6}$$

where $\alpha = \inf\{(p(n)x, x) : n \in \mathbb{Z}, x \in \mathbb{R}^{\mathcal{N}}, |x| = 1\}$.

Set

$$b(u,v) = \sum_{n\in\mathbb{Z}}\big[\big(p(n+1)\triangle u(n), \triangle v(n)\big) + \big(L(n)u(n), v(n)\big)\big], \quad \forall u,v \in E. \tag{2.7}$$

Lemma 2.2 [13, Lemma 2.3] *Suppose that L satisfies* (L'_ν). *Then*

(i) *$b(u,v)$ is a bilinear function on E, and there exists a constant $C_0 > 0$ such that*

$$\big|b(u,v)\big| \le C_0\|u\|\|v\|, \quad \forall u,v \in E; \tag{2.8}$$

(ii)

$$b(u,u) = \|u\|^2 - \sum_{n\in\mathbb{Z}^1}\big(\big(\tilde{L}(n) - L(n)\big)u(n), u(n)\big), \quad \forall u \in E. \tag{2.9}$$

By (L'_ν), there exist an integer $N_0 > \max\{|n| : n \in \mathbb{Z}^1\}$ and $M_0 > 0$ such that

$$|n|^{\nu-2}\inf_{|x|=1}\big(L(n)x,x\big) \ge M_0, \quad |n| \ge N_0,$$

which implies

$$|n|^{\nu-2}\big(L(n)x,x\big) \ge M_0|x|^2, \quad |n| \ge N_0, x \in \mathbb{R}^{\mathcal{N}}. \tag{2.10}$$

Lemma 2.3 *Suppose that L satisfies* (L'_ν). *Then, for $\sigma \in [0, 2-\nu)$ and $1 \le q \in (2(1+\sigma)/(3-\nu), 2)$, E is compactly embedded in $l^q(\mathbb{Z}, \mathbb{R}^{\mathcal{N}})$; moreover,*

$$\sum_{|n|>N}\big(1+|n|^\sigma\big)\big|u(n)\big|^q \le \frac{K(\sigma,q)}{N^\kappa}\|u\|^q, \quad \forall u \in E, N \ge N_0 \tag{2.11}$$

and

$$\sum_{n\in\mathbb{Z}}\big(1+|n|^\sigma\big)\big|u(n)\big|^q \le \left[\left(\sum_{|n|\le N}\big(1+|n|^\sigma\big)^{2/(2-q)}\big[\tilde{l}(n)\big]^{-q/(2-q)}\right)^{1-\frac{q}{2}} + \frac{K(\sigma,q)}{N^\kappa}\right]\|u\|^q,$$
$$\forall u \in E, N \ge N_0, \tag{2.12}$$

where

$$\kappa = \frac{(3-\nu)q - 2(1+\sigma)}{2} > 0, \qquad K(\sigma,q) = 2\left[\frac{2(2-q)}{(3-\nu)q - 2(1+\sigma)}\right]^{1-\frac{q}{2}} M_0^{-q/2}. \tag{2.13}$$

Proof Let $r = [(3-\nu)q - 2(1+\sigma)]/(2-q)$. Then $r > 0$. For $u \in E$ and $N \ge N_0$, it follows from (2.10), (2.13) and the Hölder inequality that

$$\begin{aligned}\sum_{|n|>N}\big(1+|n|^\sigma\big)\big|u(n)\big|^q &\le 2\left(\sum_{|n|>N}|n|^{-[(2-\nu)q-2\sigma]/(2-q)}\right)^{1-\frac{q}{2}}\left(\sum_{|n|>N}|n|^{2-\nu}\big|u(n)\big|^2\right)^{\frac{q}{2}} \\ &= 2\left(\sum_{|n|>N}|n|^{-(r+1)}\right)^{1-\frac{q}{2}}\left(\sum_{|n|>N}|n|^{2-\nu}\big|u(n)\big|^2\right)^{\frac{q}{2}}\end{aligned}$$

$$\begin{aligned}
&\leq 2\left(\frac{2}{rN^r}\right)^{1-\frac{q}{2}}\left[\frac{1}{M_0}\sum_{|n|>N}\big(L(n)u(n),u(n)\big)\right]^{\frac{q}{2}} \\
&\leq \frac{2^{1+(2-q)/2}}{M_0^{q/2}r^{(2-q)/2}N^{\kappa}}\|u\|^q \\
&= \frac{K(\sigma,q)}{N^{\kappa}}\|u\|^q.
\end{aligned}$$

This shows that (2.11) holds. Hence, from (2.5), (2.11) and the Hölder inequality, one has

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\big(1+|n|^{\sigma}\big)\big|u(n)\big|^q \\
&\quad= \sum_{|n|\leq N}\big(1+|n|^{\sigma}\big)\big|u(n)\big|^q + \sum_{|n|>N}\big(1+|n|^{\sigma}\big)\big|u(n)\big|^q \\
&\quad\leq \left(\sum_{|n|\leq N}\big(1+|n|^{\sigma}\big)^{2/(2-q)}\big[\tilde{l}(n)\big]^{-q/(2-q)}\right)^{1-\frac{q}{2}}\left(\sum_{|n|\leq N}\tilde{l}(n)\big|u(n)\big|^2\right)^{\frac{q}{2}} + \frac{K(\sigma,q)}{N^{\kappa}}\|u\|^q \\
&\quad\leq \left(\sum_{|n|\leq N}\big(1+|n|^{\sigma}\big)^{2/(2-q)}\big[\tilde{l}(n)\big]^{-q/(2-q)}\right)^{1-\frac{q}{2}}\|u\|^q + \frac{K(\sigma,q)}{N^{\kappa}}\|u\|^q.
\end{aligned}$$

This shows that (2.12) holds.

Finally, we prove that E is compactly embedded in $l^q(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$. Let $\{u_k\}\subset E$ be a bounded sequence. Then by (2.6), there exists a constant $\Lambda>0$ such that

$$\|u_k\|_{\infty} \leq \frac{1}{\sqrt[4]{(l^*+4\alpha)l^*}}\|u_k\| \leq \Lambda, \quad k\in\mathbb{N}. \tag{2.14}$$

Since E is reflexive, $\{u_k\}$ possesses a weakly convergent subsequence in E. Passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. It is easy to verify that

$$\lim_{k\to\infty} u_k(n) = u_0(n), \quad \forall n\in\mathbb{Z}. \tag{2.15}$$

For any given number $\varepsilon>0$, we can choose $N_{\varepsilon}>0$ such that

$$\frac{2^{q-1}K(\sigma,q)}{N_{\varepsilon}^{\kappa}}\left\{\left[\sqrt[4]{\big(l^*+4\alpha\big)l^*}\Lambda\right]^q + \|u_0\|^q\right\} < \varepsilon. \tag{2.16}$$

It follows from (2.15) that there exists $k_0\in\mathbb{N}$ such that

$$\sum_{|n|\leq N_{\varepsilon}}\big|u_k(n)-u_0(n)\big|^q < \varepsilon \quad \text{for } k\geq k_0. \tag{2.17}$$

On the other hand, it follows from (2.11), (2.14) and (2.16) that

$$\begin{aligned}
\sum_{|n|>N_{\varepsilon}}\big|u_k(n)-u_0(n)\big|^q &\leq 2^{q-1}\sum_{|n|>N_{\varepsilon}}\big(\big|u_k(n)\big|^q + \big|u_0(n)\big|^q\big) \\
&\leq \frac{2^{q-1}K(\sigma,q)}{N_{\varepsilon}^{\kappa}}\big(\|u_k\|^q + \|u_0\|^q\big)
\end{aligned}$$

$$\begin{aligned} &\leq \frac{2^{q-1}K(\sigma,q)}{N_\varepsilon^\kappa}\left\{\left[\sqrt[4]{(l^*+4\alpha)l^*}\Lambda\right]^q+\|u_0\|^q\right\} \\ &\leq \varepsilon, \quad k\in\mathbb{N}. \end{aligned} \tag{2.18}$$

Since ε is arbitrary, combining (2.17) with (2.18), we get

$$\|u_k-u_0\|_q^q=\sum_{n\in\mathbb{Z}}\left|u_k(n)-u_0(n)\right|^q\to 0 \quad \text{as } k\to\infty.$$

This shows that $\{u_k\}$ possesses a convergent subsequence in $l^q(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$. Therefore, E is compactly embedded in $l^q(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$ for $1\leq q\in(2(1+\sigma)/(3-\nu),2)$. □

Lemma 2.4 *Suppose that L and W satisfy* (L'_ν) *and* $(\mathrm{W1}')$. *Then, for $u\in E$,*

$$\sum_{n\in\mathbb{Z}}\left|W\big(n,u(n)\big)\right|\leq\phi_1(N)\|u\|^{\gamma_1}+\phi_2(N)\|u\|^{\gamma_2}, \quad N\geq N_0, \tag{2.19}$$

where

$$\kappa_1=\frac{(3-\nu)\gamma_1-2(1+\sigma_1)}{2}, \qquad \kappa_2=\frac{(3-\nu)\gamma_2-2(1+\sigma_2)}{2}; \tag{2.20}$$

$$\phi_1(N)=a_1\left[\left(\sum_{|n|\leq N}\left(1+|n|^{\sigma_1}\right)^{2/(2-\gamma_1)}\left[\tilde{l}(n)\right]^{-\gamma_1/(2-\gamma_1)}\right)^{1-\frac{\gamma_1}{2}}+\frac{K(\sigma_1,\gamma_1)}{N^{\kappa_1}}\right], \tag{2.21}$$

$$\phi_2(N)=a_2\left[\left(\sum_{|n|\leq N}\left(1+|n|^{\sigma_2}\right)^{2/(2-\gamma_2)}\left[\tilde{l}(n)\right]^{-\gamma_2/(2-\gamma_2)}\right)^{1-\frac{\gamma_2}{2}}+\frac{K(\sigma_2,\gamma_2)}{N^{\kappa_2}}\right]. \tag{2.22}$$

Proof For $N\geq N_0$, it follows from $(\mathrm{W1}')$, (2.12), (2.20), (2.21) and (2.22) that

$$\begin{aligned} \sum_{n\in\mathbb{Z}}\left|W\big(n,u(n)\big)\right| &\leq \sum_{i=1}^2 a_i\sum_{n\in\mathbb{Z}}\left(1+|n|^{\sigma_i}\right)\left|u(n)\right|^{\gamma_i} \\ &\leq \sum_{i=1}^2 a_i\left[\left(\sum_{|n|\leq N}\left(1+|n|^{\sigma_i}\right)^{2/(2-\gamma_i)}\left[\tilde{l}(n)\right]^{-\gamma_i/(2-\gamma_i)}\right)^{1-\frac{\gamma_i}{2}}+\frac{K(\sigma_i,\gamma_i)}{N^{\kappa_i}}\right]\|u\|^{\gamma_i} \\ &= \phi_1(N)\|u\|^{\gamma_1}+\phi_2(N)\|u\|^{\gamma_2}. \end{aligned}$$

This shows that (2.19) holds. □

Lemma 2.5 *Assume that L and W satisfy* (L'_ν), $(\mathrm{W1}')$ *and* $(\mathrm{W2}')$. *Then the functional $f:E\to\mathbb{R}$ defined by*

$$f(u)=\frac{1}{2}b(u,u)-\sum_{n\in\mathbb{Z}}W\big(n,u(n)\big), \quad \forall u\in E \tag{2.23}$$

is well defined and of class $C^1(E,\mathbb{R})$ and

$$\left\langle f'(u),v\right\rangle=b(u,v)-\sum_{n\in\mathbb{Z}}\big(\nabla W\big(n,u(n)\big),v(n)\big), \quad \forall u,v\in E. \tag{2.24}$$

Furthermore, the critical points of f in E are the solutions of system (1.1) *with* $u(\pm\infty)=0$.

Proof Lemmas 2.2 and 2.4 imply that f defined by (2.23) is well defined on E. Next, we prove that (2.24) holds. By (W2′), there exist $M_1, M_2 > 0$ such that

$$\varphi_i\big(|x|\big) \le M_i|x|^{\gamma_{2+i}-1}, \quad \forall x \in \mathbb{R}^{\mathcal{N}}, |x| \le 1, i = 1, 2. \tag{2.25}$$

For any $u, v \in E$, there exists an integer $N_1 > N_0$ such that $|u(n)| + |v(n)| < 1$ for $|n| > N_1$. Then, for any sequence $\{\theta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ with $|\theta_n| < 1$ for $n \in \mathbb{Z}$ and any number $h \in (0,1)$, by (W2′), (2.11) and (2.25), we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \max_{h\in[0,1]} \big|\big(\nabla W\big(n, u(n) + \theta_n h v(n)\big), v(n)\big)\big| \\
&\quad\le \sum_{|n|\le N_1} \max_{h\in[0,1]} \big|\nabla W\big(n, u(n) + \theta_n h v(n)\big)\big|\big|v(n)\big| \\
&\qquad + \sum_{|n|>N_1} \max_{h\in[0,1]} \big|\nabla W\big(n, u(n) + \theta_n h v(n)\big)\big|\big|v(n)\big| \\
&\quad\le \sum_{|n|\le N_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} \big|\nabla W(n,x)\big|\big|v(n)\big| \\
&\qquad + \sum_{i=1}^{2} M_i \sum_{|n|>N_1} \big(1 + |n|^{\sigma_i}\big)\big(\big|u(n)\big| + \big|v(n)\big|\big)^{\gamma_{2+i}-1}\big|v(n)\big| \\
&\quad\le \sum_{|n|\le N_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} \big|\nabla W(n,x)\big|\big|v(n)\big| + \sum_{i=1}^{2} M_i \sum_{|n|>N_1} \big(1 + |n|^{\sigma_i}\big)\big|v(n)\big|^{\gamma_{2+i}} \\
&\qquad + \sum_{i=1}^{2} M_i \bigg(\sum_{|n|>N_1} \big(1 + |n|^{\sigma_i}\big)\big|u(n)\big|^{\gamma_{2+i}}\bigg)^{1-\frac{1}{\gamma_{2+i}}} \\
&\qquad \times \bigg(\sum_{|n|>N_1} \big(1 + |n|^{\sigma_i}\big)\big|v(n)\big|^{\gamma_{2+i}}\bigg)^{\frac{1}{\gamma_{2+i}}} \\
&\quad\le \sum_{|n|\le N_1} \max_{|x|\le\|u\|_\infty+\|v\|_\infty} \big|\nabla W(n,x)\big|\big|v(n)\big| \\
&\qquad + \sum_{i=1}^{2} \frac{M_i K(\sigma_i, \gamma_{2+i})}{N_1^{\kappa_{2+i}}} \big(\|u\|^{\gamma_{2+i}-1} + \|v\|^{\gamma_{2+i}-1}\big)\|v\| < +\infty,
\end{aligned} \tag{2.26}$$

where $\kappa_{2+i} = [\gamma_{2+i}(3 - \nu) - 2(1 + \sigma_i)]/2 > 0$, $i = 1, 2$. Then by (2.23), (2.26) and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\big\langle f'(u), v\big\rangle &= \lim_{h\to 0^+} \frac{f(u + hv) - f(u)}{h} \\
&= \lim_{h\to 0^+} \bigg[b(u, v) + \frac{hb(v, v)}{2} - \sum_{n\in\mathbb{Z}} \big(\nabla W\big(n, u(n) + \theta_n h v(n)\big), v(n)\big)\bigg] \\
&= b(u, v) - \sum_{n\in\mathbb{Z}} \big(\nabla W\big(n, u(n)\big), v(n)\big).
\end{aligned}$$

This shows that (2.24) holds. In view of the proof of [13, Lemma 2.6], the critical points of f in E are the solutions of system (1.1) with $u(\pm\infty) = 0$. □

Let us prove now that f' is continuous. Let $u_k \to u$ in E. Then there exists a constant $\delta > 0$ such that

$$\|u\| \le \sqrt[4]{(l^* + 4\alpha)l^*}\delta, \qquad \|u_k\| \le \sqrt[4]{(l^* + 4\alpha)l^*}\delta, \quad k = 1, 2, \ldots. \tag{2.27}$$

It follows from (2.6) that

$$\|u\|_\infty \le \delta, \qquad \|u_k\|_\infty \le \delta, \quad k = 1, 2, \ldots. \tag{2.28}$$

By (W2′), there exist $M_3, M_4 > 0$ such that

$$\varphi_i(|x|) \le M_{2+i}|x|^{\gamma_{2+i}-1}, \quad \forall x \in \mathbb{R}^{\mathcal{N}}, |x| \le \delta, i = 1, 2. \tag{2.29}$$

From (2.11), (2.24), (2.27), (2.28), (2.29), (W2′) and the Hölder inequality, we have

$$\begin{aligned}
&\bigl|\langle f'(u_k) - f'(u), v\rangle\bigr| \\
&\quad\le \bigl|b(u_k - u, v)\bigr| + \sum_{n\in\mathbb{Z}}\bigl|\bigl(\nabla W\bigl(n, u_k(n)\bigr) - \nabla W\bigl(n, u(n)\bigr), v(n)\bigr)\bigr| \\
&\quad\le C_0\|u_k - u\|\|v\| + \sum_{|n|\le N}\bigl|\nabla W\bigl(n, u_k(n)\bigr) - \nabla W\bigl(n, u(n)\bigr)\bigr|\bigl|v(n)\bigr| \\
&\qquad + \sum_{|n|>N}\bigl(\bigl|\nabla W\bigl(n, u_k(n)\bigr)\bigr| + \bigl|\nabla W\bigl(n, u(n)\bigr)\bigr|\bigr)\bigl|v(n)\bigr| \\
&\quad\le o(1) + \sum_{i=1}^{2} M_{2+i}\sum_{|n|>N}\bigl(1 + |n|^{\sigma_i}\bigr)\bigl(\bigl|u_k(n)\bigr|^{\gamma_{2+i}-1} + \bigl|u(n)\bigr|^{\gamma_{2+i}-1}\bigr)\bigl|v(n)\bigr| \\
&\quad\le o(1) + \sum_{i=1}^{2} M_{2+i}\Bigl(\sum_{|n|>N}\bigl(1 + |n|^{\sigma_i}\bigr)\bigl|u_k(n)\bigr|^{\gamma_{2+i}}\Bigr)^{1-\frac{1}{\gamma_{2+i}}}\Bigl(\sum_{|n|>N}\bigl(1 + |n|^{\sigma_i}\bigr)\bigl|v(n)\bigr|^{\gamma_{2+i}}\Bigr)^{\frac{1}{\gamma_{2+i}}} \\
&\qquad + \sum_{i=1}^{2} M_{2+i}\Bigl(\sum_{|n|>N}\bigl(1 + |n|^{\sigma_i}\bigr)\bigl|u(n)\bigr|^{\gamma_{2+i}}\Bigr)^{1-\frac{1}{\gamma_{2+i}}}\Bigl(\sum_{|n|>N}\bigl(1 + |n|^{\sigma_i}\bigr)\bigl|v(n)\bigr|^{\gamma_{2+i}}\Bigr)^{\frac{1}{\gamma_{2+i}}} \\
&\quad\le o(1) + \sum_{i=1}^{2}\frac{M_{2+i}K(\sigma_i, \gamma_{2+i})}{N^{\kappa_{2+i}}}\bigl(\|u_k\|^{\gamma_{2+i}-1} + \|u\|^{\gamma_{2+i}-1}\bigr)\|v\| \\
&\quad= o(1), \quad k \to +\infty, N \to +\infty, \forall v \in E,
\end{aligned}$$

which implies the continuity of f'. The proof is complete. □

Lemma 2.6 [14] *Let X be an infinite dimensional Banach space and let $f \in C^1(X, \mathbb{R})$ be even, satisfy the (PS)-condition, and $f(0) = 0$. If $X = X_1 \oplus X_2$ (direct sum), where X_1 is finite dimensional, and f satisfies*

(i) *f is bounded from below on X_2;*

(ii) *for each finite dimensional subspace $\tilde{X} \subset X$, there are positive constants $\rho = \rho(\tilde{X})$ and $\sigma = \sigma(\tilde{X})$ such that $f|_{B_\rho \cap \tilde{X}} \le 0$ and $f|_{\partial B_\rho \cap \tilde{X}} \le -\sigma$, where $B_\rho = \{x \in X : \|x\| = \rho\}$.*

Then f possesses infinitely many nontrivial critical points.

3 Proof of the theorem

Proof of Theorem 1.2 For $u \in E$, we define two functions as follows:

$$u^-(n) = \begin{cases} u(n), & n \in \mathbb{Z}^1, \\ 0, & n \in \mathbb{Z}^2; \end{cases} \qquad u^+(n) = \begin{cases} 0, & n \in \mathbb{Z}^1, \\ u(n), & n \in \mathbb{Z}^2. \end{cases} \tag{3.1}$$

Set

$$X_1 = \{u^- : u \in E\}, \qquad X_2 = \{u^+ : u \in E\}. \tag{3.2}$$

Then $X := E = X_1 \oplus X_2$ (direct sum) and $\dim(X_1) < +\infty$. Obviously, (W1′) and (W5) imply $f(0) = 0$ and f is even. In view of Lemma 2.5, $f \in C^1(E, \mathbb{R})$. In what follows, we first prove that f satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (PS)-sequence: $\{f(u_k)\}_{k\in\mathbb{N}}$ is bounded and $\|f'(u_k)\| \to 0$ as $k \to +\infty$. From (2.23), (2.24) and (W3′), we have

$$\begin{aligned} \langle f'(u_k), u_k\rangle - 2f(u_k) &= \sum_{n\in\mathbb{Z}} \bigl[2W\bigl(n, u_k(n)\bigr) - \bigl(\nabla W\bigl(n, u_k(n)\bigr), u_k(n)\bigr)\bigr] \\ &\geq b_1 \sum_{n\in\mathbb{Z}} \bigl|u_k(n)\bigr|^{\gamma_5} - b_2 \sum_{n\in\mathbb{Z}} \bigl|u_k(n)\bigr|^{\gamma_6} \\ &= b_1 \|u_k\|_{\gamma_5}^{\gamma_5} - b_2 \|u_k\|_{\gamma_6}^{\gamma_6}. \end{aligned}$$

It follows that there exists a constant $C_1 > 0$ such that

$$b_1 \|u_k\|_{\gamma_5}^{\gamma_5} - b_2 \|u_k\|_{\gamma_6}^{\gamma_6} \leq C_1\bigl(1 + \|u_k\|\bigr). \tag{3.3}$$

Since $\dim(X_1) < +\infty$, it follows that there exists a constant $C_2 > 0$ such that

$$\bigl\|u_k^-\bigr\|_2^2 = \bigl(u_k^-, u_k\bigr)_{l^2} \leq \bigl\|u_k^-\bigr\|_{\gamma_5'} \|u_k\|_{\gamma_5} \leq C_2 \bigl\|u_k^-\bigr\|_2 \|u_k\|_{\gamma_5}, \tag{3.4}$$

where $\gamma_5' = \gamma_5/(\gamma_5 - 1)$. Combining (3.3) with (3.4), one has

$$\begin{aligned} \sum_{n\in\mathbb{Z}^1} \bigl(\bigl(\tilde{L}(n) - L(n)\bigr)u_k(n), u_k(n)\bigr) &= \sum_{n\in\mathbb{Z}^1} \bigl(\bigl(\tilde{L}(n) - L(n)\bigr)u_k^-(n), u_k^-(n)\bigr) \\ &\leq C_3 \bigl\|u_k^-\bigr\|_2^2 \\ &\leq C_4\bigl(1 + \|u_k\|^{2/\gamma_5} + \|u_k\|^{2\gamma_6/\gamma_5}\bigr). \end{aligned} \tag{3.5}$$

From (2.19), (2.23) and (3.5), we obtain

$$\begin{aligned} \|u_k\|^2 &= \sum_{n\in\mathbb{Z}} \bigl[\bigl(p(n+1)\triangle u_k(n), \triangle u_k(n)\bigr) + \bigl(\tilde{L}(n)u_k(n), u_k(n)\bigr)\bigr] \\ &= b(u_k, u_k) + \sum_{n\in\mathbb{Z}^1} \bigl(\bigl(\tilde{L}(n) - L(n)\bigr)u_k(n), u_k(n)\bigr) \\ &= \sum_{n\in\mathbb{Z}^1} \bigl(\bigl(\tilde{L}(n) - L(n)\bigr)u_k(n), u_k(n)\bigr) + 2f(u_k) + 2\sum_{n\in\mathbb{Z}} W\bigl(n, u_k(n)\bigr) \end{aligned}$$

$$\begin{aligned}&\le C_5\bigl(1+\|u_k\|^{2/\gamma_5}+\|u_k\|^{2\gamma_6/\gamma_5}\bigr)\\&\quad+2\phi_1(N_0)\|u_k\|^{\gamma_1}+2\phi_2(N_0)\|u_k\|^{\gamma_2}\\&\le C_6\bigl(1+\|u_k\|^{\gamma_1}+\|u_k\|^{\gamma_2}+\|u_k\|^{2/\gamma_5}+\|u_k\|^{2\gamma_6/\gamma_5}\bigr).\end{aligned}\tag{3.6}$$

Since $1<\gamma_1<\gamma_2<2$, $1<\gamma_6<\gamma_5<2$, it follows from (3.6) that $\{\|u_k\|\}$ is bounded. Let $A>0$ such that

$$\|u_k\|_\infty\le\frac{1}{\sqrt[4]{(l_*+4\alpha)l_*}}\|u_k\|\le A,\quad k\in\mathbb{N}.\tag{3.7}$$

So, passing to a subsequence if necessary, it can be assumed that $u_k\rightharpoonup u_0$ in E. It is easy to verify that

$$\lim_{k\to\infty}u_k(n)=u_0(n),\quad\forall n\in\mathbb{Z}.\tag{3.8}$$

By (W2′), there exist $M_5,M_6>0$ such that

$$\varphi_i\bigl(|x|\bigr)\le M_{4+i}|x|^{\gamma_{2+i}-1},\quad\forall x\in\mathbb{R}^{\mathcal{N}},|x|\le A,i=1,2.\tag{3.9}$$

For any given number $\varepsilon>0$, we can choose an integer $N_3>N_0$ such that

$$\frac{M_{4+i}K(\sigma_i,\gamma_{2+i})}{N_3^{\kappa_{2+i}}}\bigl\{\bigl[\sqrt[4]{(l_*+4\alpha)l_*}A\bigr]^{\gamma_{2+i}}+\|u_0\|^{\gamma_{2+i}}\bigr\}<\varepsilon,\quad i=1,2.\tag{3.10}$$

It follows from (3.8) and the continuity of $\nabla W(n,x)$ on x that there exists $k_0\in\mathbb{N}$ such that

$$\sum_{n=-N_2}^{N_2}\bigl|\nabla W\bigl(n,u_k(n)\bigr)-\nabla W\bigl(n,u_0(n)\bigr)\bigr|\bigl|u_k(n)-u_0(n)\bigr|<\varepsilon\quad\text{for }k\ge k_0.\tag{3.11}$$

On the other hand, it follows from (2.11), (3.7), (3.9), (3.10) and (W2′) that

$$\begin{aligned}&\sum_{|n|>N_2}\bigl|\nabla W\bigl(n,u_k(n)\bigr)-\nabla W\bigl(n,u_0(n)\bigr)\bigr|\bigl|u_k(n)-u_0(n)\bigr|\\&\quad\le\sum_{|n|>N_2}\bigl[\bigl|\nabla W\bigl(n,u_k(n)\bigr)\bigr|+\bigl|\nabla W\bigl(n,u_0(n)\bigr)\bigr|\bigr]\bigl(\bigl|u_k(n)\bigr|+\bigl|u_0(n)\bigr|\bigr)\\&\quad\le\sum_{i=1}^{2}\sum_{|n|>N_2}\bigl(1+|n|^{\sigma_i}\bigr)\bigl[\varphi_i\bigl(\bigl|u_k(n)\bigr|\bigr)+\varphi_i\bigl(\bigl|u_0(n)\bigr|\bigr)\bigr]\bigl(\bigl|u_k(n)\bigr|+\bigl|u_0(n)\bigr|\bigr)\\&\quad\le\sum_{i=1}^{2}M_{4+i}\sum_{|n|>N_2}\bigl(1+|n|^{\sigma_i}\bigr)\bigl(\bigl|u_k(n)\bigr|^{\gamma_{2+i}-1}+\bigl|u_0(n)\bigr|^{\gamma_{2+i}-1}\bigr)\bigl(\bigl|u_k(n)\bigr|+\bigl|u_0(n)\bigr|\bigr)\\&\quad\le 2\sum_{i=1}^{2}M_{4+i}\sum_{|n|>N_2}\bigl(1+|n|^{\sigma_i}\bigr)\bigl(\bigl|u_k(n)\bigr|^{\gamma_{2+i}}+\bigl|u_0(n)\bigr|^{\gamma_{2+i}}\bigr)\\&\quad\le\sum_{i=1}^{2}\frac{2M_{4+i}K(\sigma_i,\gamma_{2+i})}{N_2^{\kappa_{2+i}}}\bigl(\|u_k\|^{\gamma_{2+i}}+\|u_0\|^{\gamma_{2+i}}\bigr)\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{2} \frac{2M_{4+i}K(\sigma_i, \gamma_{2+i})}{N_2^{\kappa_{2+i}}} \bigl\{\bigl[\sqrt[4]{(l_* + 4\alpha)l_*}A\bigr]^{\gamma_{2+i}} + \|u_0\|^{\gamma_{2+i}}\bigr\} \\
&\le 4\varepsilon, \quad k \in \mathbb{N}.
\end{aligned} \tag{3.12}$$

Since ε is arbitrary, combining (3.11) with (3.12), we get

$$\sum_{n\in\mathbb{Z}} \bigl(\nabla W\bigl(n, u_k(n)\bigr) - \nabla W\bigl(n, u_0(n)\bigr), u_k(n) - u_0(n)\bigr) \to 0 \quad \text{as } k \to \infty. \tag{3.13}$$

It follows from (2.24) that

$$\begin{aligned}
&\bigl\langle f'(u_k) - f'(u_0), u_k - u_0\bigr\rangle \\
&\quad = b(u_k - u_0, u_k - u_0) - \sum_{n\in\mathbb{Z}} \bigl(\nabla W\bigl(n, u_k(n)\bigr) - \nabla W\bigl(n, u_0(n)\bigr), u_k(n) - u_0(n)\bigr) \\
&\quad = \|u_k - u_0\|^2 - \sum_{n\in\mathbb{Z}^1} \bigl(\bigl(\tilde{L}(n) - L(n)\bigr)(u_k - u_0), u_k - u_0\bigr) \\
&\qquad - \sum_{n\in\mathbb{Z}} \bigl(\nabla W\bigl(n, u_k(n)\bigr) - \nabla W\bigl(n, u_0(n)\bigr), u_k(n) - u_0(n)\bigr).
\end{aligned} \tag{3.14}$$

Since $\langle f'(u_k) - f'(u_0), u_k - u_0\rangle \to 0$, it follows from (3.8), (3.13) and (3.14) that $u_k \to u_0$ in E. Hence, f satisfies the (PS)-condition.

Next, for $u \in X_2$, it follows from (2.9), (2.19) and (2.23) that

$$\begin{aligned}
f(u) &= \frac{1}{2} b(u, u) - \sum_{n\in\mathbb{Z}} W\bigl(n, u(n)\bigr) = \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W\bigl(n, u(n)\bigr) \\
&\ge \frac{1}{2}\|u\|^2 - \phi_1(N_0)\|u\|^{\gamma_1} - \phi_2(N_0)\|u\|^{\gamma_2} \to +\infty
\end{aligned} \tag{3.15}$$

as $\|u\| \to +\infty$ and $u \in X_2$, since $1 < \gamma_1 < \gamma_2 < 2$.

Finally, we prove that assumption (ii) in Lemma 2.6 holds. Let $\tilde{X} \subset X$ be any finite dimensional subspace. Then there exist constants $c_0 = c(\tilde{X}) > 0$ and $c_* = c(\tilde{X}) > 0$ such that

$$c_0\|u\| \le \|u\|_{\gamma_i} \le c_*\|u\|, \quad \forall i = 7, 8, u \in \tilde{X}. \tag{3.16}$$

From (2.9), (2.23), (3.16) and (W4′), one has

$$\begin{aligned}
f(u) &= \frac{1}{2} b(u, u) - \sum_{n\in\mathbb{Z}} W\bigl(n, u(n)\bigr) \\
&\le \frac{1}{2}\|u\|^2 - b_3 \sum_{n\in\mathbb{Z}} \bigl|u(n)\bigr|^{\gamma_7} + b_4 \sum_{n\in\mathbb{Z}} \bigl|u(n)\bigr|^{\gamma_8} \\
&= \frac{1}{2}\|u\|^2 - b_3\|u\|_{\gamma_7}^{\gamma_7} + b_4\|u\|_{\gamma_8}^{\gamma_8} \\
&\le \frac{1}{2}\|u\|^2 - b_3 c_0^{\gamma_7}\|u\|^{\gamma_7} + b_4 c_*^{\gamma_8}\|u\|^{\gamma_8}, \quad \forall u \in \tilde{X}.
\end{aligned}$$

Since $1 < \gamma_7 < \gamma_8 < 2$, the above estimation implies that there exist $\rho = \rho(b_3, b_4, c_0, c_*) = \rho(\tilde{X}) > 0$ and $\sigma = \sigma(b_3, b_4, c_0, c_*) = \sigma(\tilde{X}) > 0$ such that

$$f(u) \leq 0, \quad \forall u \in B_\rho \cap \tilde{X}; \qquad f(u) \leq -\sigma, \quad \forall u \in \partial B_\rho \cap \tilde{X}.$$

This shows that assumption (ii) in Lemma 2.6 holds. By Lemma 2.6, f has infinitely many critical points which are homoclinic solutions for system (1.1). □

4 Example

In this section, we give an example to illustrate our result.

Example 4.1 In system (1.1), let $p(n)$ be an $\mathcal{N} \times \mathcal{N}$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$, $L(n) = (1 + \sin^2 n)(|n|^{4/5} - 6)I_{\mathcal{N}}$, and let

$$W(n,x) = \left(1 + \sin^2 n\right)\left[\left(1 + |n|^{1/9}\right)|x|^{5/4} - 3|x|^{3/2} + \left(1 + |n|^{1/2}\right)|x|^{7/4}\right]. \tag{4.1}$$

Then L satisfies (L'_ν) with $\nu = 6/5$, and

$$\nabla W(n,x) = \left(1 + \sin^2 n\right)\left[\frac{5}{4}\left(1 + |n|^{1/9}\right)|x|^{-3/4}x - \frac{9}{2}|x|^{-1/2}x + \frac{7}{4}\left(1 + |n|^{1/2}\right)|x|^{-1/4}x\right],$$

$$\left|W(n,x)\right| \leq 5\left(1 + |n|^{1/9}\right)|x|^{5/4} + 5\left(1 + |n|^{1/2}\right)|x|^{7/4}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N,$$

$$\left|\nabla W(n,x)\right| \leq 7\left(1 + |n|^{1/9}\right)|x|^{1/4} + 8\left(1 + |n|^{1/2}\right)|x|^{3/4}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N,$$

$$2W(n,x) - \nabla W(n,x) \geq \frac{1}{4}|x|^{7/4} - 3|x|^{3/2}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N$$

and

$$W(n,x) \geq |x|^{5/4} - 6|x|^{3/2}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Thus all the conditions of Theorem 1.2 are satisfied with

$$\frac{5}{4} = \gamma_1 = \gamma_3 = \gamma_7 < \gamma_6 = \gamma_8 = \frac{3}{2} < \gamma_5 = \gamma_4 = \gamma_2 = \frac{7}{4};$$

$$a_1 = a_2 = 5; \qquad b_1 = \frac{1}{4}, \qquad b_2 = 3, \qquad b_3 = 1, \qquad b_4 = 6;$$

$$\sigma_1 = \frac{1}{9}, \qquad \sigma_2 = \frac{1}{2}; \qquad \varphi_1(s) = 7s^{1/4}, \qquad \varphi_2(s) = 8s^{3/4}.$$

Hence, by Theorem 1.2, system (1.1) has infinitely many nontrivial homoclinic solutions. However, one can see that $W(n,x)$ defined by (4.1) does not satisfy (W1) and (W2).

Competing interests

The author declares that they have no competing interests.

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