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An application on the second-order generalized difference equations

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Abstract
In this paper, we study the solutions of the second-order generalized difference equation having the form of

$$\Delta_\ell^2 u(k) + f(k, u(k)) = 0, \quad k \in [a, \infty), a > 0, \ell \in (0, \infty), \tag{1}$$

where $\Delta_\ell u(k) = u(k+\ell) - u(k)$. Then we provide applications on $\ell_{2(\ell)}$ and $c_{0(\ell)}$.

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1 Introduction and preliminaries

The basic theory of difference equations is based on the difference operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, which allows the recursive computation of solutions. Later, the following definition was suggested for Δ_ℓ by [1–3] and [4]:

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \quad k \in \mathbb{R}, \ell \in \mathbb{R} - \{0\}; \tag{2}$$

however, no significant progress took place on this line. Recently, equation (2) was reconsidered and its inverse was defined by Δ_ℓ^{-1}, and many interesting results in applications such as in number theory as well as in fluid dynamics were obtained; see, for example, [5]. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were studied for the solutions of difference equations involving Δ_ℓ. The ℓ_2 and c_0 solutions of the second-order difference equation of (1) when $\ell = 1$ were discussed in [6] and further generalized in [7]. In this paper, we discuss some applications of Δ_ℓ in the finite and infinite series of number theory.

In this section, we present some of the preliminary definitions and results which will be useful for future discussion. The following definitions were held in [5] and [8], respectively.

Definition 1.1 Let $u(k)$, $k \in [0, \infty)$, be a real- or complex-valued function and $\ell \in (0, \infty)$. Then the generalized difference operator Δ_ℓ is defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{3}$$

Then the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows: If

$$\Delta_\ell v(k) = u(k), \quad \text{then } v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{4}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - [\frac{k}{\ell}]\ell$. If $\lim_{k\to\infty} u(k) = 0$, then we can take $c_j = 0$. Further, the generalized polynomial factorial for $\ell > 0$ is defined as

$$k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots\bigl(k-(n-1)\ell\bigr). \tag{5}$$

The following lemmas were proved in [9] and [10], respectively.

Lemma 1.2 (Product formula) *Let $u(k)$ and $v(k)$, $k \in [0,\infty)$, be any two real-valued functions. Then*

$$\begin{aligned}\Delta_\ell\bigl\{u(k)v(k)\bigr\} &= u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k)\\ &= v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k).\end{aligned} \tag{6}$$

Lemma 1.3 *Let $\ell > 0$, $n \in \mathbb{N}(2)$, $k \in (\ell,\infty)$ and $k_\ell^{(n)} \neq 0$. Then*

$$\Delta_\ell^{-1}\frac{1}{k_\ell^{(n)}} = \frac{-1}{(n-1)\ell(k-\ell)_\ell^{(n-1)}} + c_j. \tag{7}$$

Definition 1.4 A function $u(k)$, $k \in [a,\infty)$, is said to be in the $\ell_{2(\ell)}$-space if

$$\sum_{\gamma=0}^{\infty}\bigl|u(a+j+\gamma\ell)\bigr|^2 < \infty \quad \text{for all } j \in [0,\ell). \tag{8}$$

If $\lim_{r\to\infty} |u(a+j+r\ell)| = 0$ for all $j \in [0,\ell)$, then $u(k)$ is said to be in the $c_{0(\ell)}$-space.

In what follows, we have the summation formula for finite and infinite series.

Lemma 1.5 *If a real-valued function $u(k)$ is defined for all $k \in [0,\infty)$, then*

$$\Delta_\ell^{-1}u(k) = \sum_{r=1}^{[\frac{k}{\ell}]} u(k-r\ell) + c_j, \tag{9}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - [\frac{k}{\ell}]\ell$. Since $[0,\infty) = \bigcup_{0\le j<\ell} N_\ell(j)$, each complex number c_j $(0 \le j < \ell)$ is called an initial value of $k \in N_\ell(j)$. Usually, each initial value c_j is taken from any one of the values $u(j)$, $u(j+\ell)$, $u(j+2\ell)$, etc. Further, if $\lim_{k\to\infty} u(k) = 0$ and $\ell > 0$, then

$$\Delta_\ell^{-1}u(k) = -\sum_{r=0}^{\infty} u(k+r\ell). \tag{10}$$

Proof Assume $z(k) = \sum_{r=0}^{\infty} u(k+r\ell)$. Then

$$\Delta_\ell z(k) = \sum_{r=0}^{\infty} u(k+\ell+r\ell) - \sum_{r=0}^{\infty} u(k+r\ell) = -u(k).$$

Now, the proof follows from $\lim_{k\to\infty} u(k) = 0$ and Definition 1.1. $\square$

The next lemma is an expansion of Lemma 1.5 and its proof is straightforward.

Lemma 1.6 *If* $\lim_{k\to\infty} u(k) = 0$ *and* $\ell > 0$, *then*

$$\Delta_\ell^{-2} u(k) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} u(k + r_1\ell + r_2\ell). \tag{11}$$

Theorem 1.7 *Let* $n \in \mathbb{N}(2)$, $k \in (0,\infty)$ *such that* $k_\ell^{(n)} \neq 0$. *Then*

$$\sum_{r=0}^{\infty} \frac{1}{(k + r\ell)_\ell^{(n)}} = \frac{1}{(n-1)\ell(k-\ell)_\ell^{(n-1)}}. \tag{12}$$

Proof The proof follows from Lemma 1.3 and Lemma 1.5 by taking $u(k) = \frac{1}{k_\ell^{(n)}}$ and $c_j = 0$. □

Corollary 1.8 *Let* $k \in (\ell,\infty)$ *and* $\ell \in (0,\infty)$. *Then*

$$\sum_{r=0}^{\infty} \frac{1}{(k + r\ell)(k + r\ell - \ell)} = \frac{1}{\ell(k-\ell)}. \tag{13}$$

Proof Since $\Delta_\ell^{-1} \frac{1}{k(k-\ell)} = \frac{-1}{\ell(k-\ell)}$, the proof follows from Theorem 1.7 by taking $n = 2$. □

2 Applications of Δ_ℓ in number theory

In this section, we present some formulae and examples to find the values of finite and infinite series in number theory as an application of Δ_ℓ. The following theorem and example were given in [7]. In fact, the example is to illustrate Theorem 2.1.

Theorem 2.1 *Let* $k \in [\ell,\infty)$ *and* $\ell \in (0,\infty)$. *Then*

$$\sum_{r=1}^{[\frac{k}{\ell}]+s} \frac{(k - r\ell + 2\ell)^2 - 3\ell^2}{\ell^r (k - r\ell + 4\ell)_\ell^{(2)} (k - r\ell + \ell)_\ell^{(\lceil \frac{k-r\ell+\ell}{\ell} \rceil)}} = \frac{c_j}{\ell^{\lceil \frac{k}{\ell} \rceil}} - \frac{1}{(k + 3\ell) k_\ell^{(\lceil \frac{k}{\ell} \rceil)}}, \tag{14}$$

where $s = -1$ *for* $k \in \mathbb{N}_\ell(\ell)$, $s = 0$ *for* $k \notin \mathbb{N}_\ell(\ell)$ *and each* c_j *is a constant for all* $k \in \mathbb{N}_\ell(j)$, $j = k - [\frac{k}{\ell}]\ell$. *In particular* c_j *is obtained from* (14) *by substituting* $k = \ell + j$.

Example 2.2 By taking $\ell = 1.7$, $k = 2$ and $j = 0.3$ in (14), we get $c_j = \frac{85}{81}$ and hence (14) becomes

$$\begin{aligned} &\sum_{r=1}^{[\frac{k}{\ell}]} \frac{(k - 1.7r + 2(1.7))^2 - 3(1.7)^2}{1.7^r (k - 1.7r + 4(1.7))_{1.7}^{(2)} (k - 1.7r + 1.7)_{1.7}^{(\lceil \frac{k-1.7r+1.7}{1.7} \rceil)}} \\ &\quad = \frac{85}{81(1.7)^{\lceil \frac{k}{1.7} \rceil}} - \frac{1}{(k + 3(1.7)) k_{1.7}^{(\lceil \frac{k}{1.7} \rceil)}}, \quad k = 2, 3.7, 5.4, \ldots. \end{aligned}$$

Theorem 2.3 *Let* $k \in [\ell,\infty)$ *and* $\ell \in (0,\infty)$. *Then*

$$\sum_{r=0}^{\infty} \frac{k + r\ell + \ell}{3^{\frac{k+r\ell}{\ell}} (2(k + r\ell) + \ell)_{2\ell}^{(2)}} = \frac{1}{4(3)^{\frac{k}{\ell}-1}(2k-\ell)}. \tag{15}$$

Proof By Definition 1.1, we find

$$\Delta_\ell^{-1}\left(\frac{(k+\ell)}{3^{\frac{k}{\ell}}(2k+\ell)_{2\ell}^{(2)}k}\right)=\frac{-1}{4(3)^{\frac{k}{\ell}-1}(2k-\ell)}$$

and (15) follows by Lemma 1.5 and $c_j = 0$ as $k \to \infty$. □

The following is the illustration for Theorem 2.3.

Example 2.4 Taking $\ell = 0.2$ in (15), we arrive at

$$\frac{k+0.2}{3^{\frac{k}{0.2}}(2k+0.2)_{0.4}^{(2)}}+\frac{k+0.4}{3^{\frac{k+0.2}{0.2}}(2k+0.6)_{0.4}^{(2)}}+\frac{(k+0.6)}{3^{\frac{k+0.4}{0.2}}(2k+1)_{0.4}^{(2)}}+\cdots=\frac{1}{4(3)^{\frac{k}{0.2}-1}(2k-0.2)}$$

and one can take any value $k \in [\ell, \infty)$.

Theorem 2.5 *For $k \in [\ell, \infty)$ and $\ell \in (0, \infty)$, then*

$$\sum_{r=0}^{\infty}\frac{(k+r\ell)^3-\ell^3}{\ell^r((k+r\ell)^2-2\ell^2)_\ell^{(2)}(k+r\ell+\ell)_\ell^{(\lceil\frac{k+r\ell+\ell}{\ell}\rceil)}}=\frac{1}{((k-\ell)^2-2\ell^2)k_\ell^{(\lceil\frac{k}{\ell}\rceil)}}. \tag{16}$$

Proof By Definition 1.1, we find

$$\Delta_\ell^{-1}\frac{(k^3-\ell^3)\ell^{\lceil\frac{k}{\ell}\rceil}}{(k^2-2\ell^2)_\ell^{(2)}(k+\ell)_\ell^{(\lceil\frac{k+\ell}{\ell}\rceil)}}=\frac{-\ell^{\lceil\frac{k}{\ell}\rceil}}{((k-\ell)^2-2\ell^2)k_\ell^{(\lceil\frac{k}{\ell}\rceil)}}$$

and (16) follows by (10) and $c_j = 0$ as $k \to \infty$. □

The following theorem generates the formula to find the sum of first partial sums of an infinite series.

Theorem 2.6 *For the positive integer $n \in \mathbb{N}(3)$, $k \in [2\ell, \infty)$ and $\ell \in (0, \infty)$,*

$$\sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty}\frac{1}{(k+r_2\ell+r_1\ell)_\ell^{(n)}}=\frac{1}{(n-1)(n-2)\ell^2(k-2\ell)_\ell^{(n-2)}}. \tag{17}$$

Proof Using Definition 1.1 and operating Δ_ℓ^{-1} on (7), we find

$$\Delta_\ell^{-2}\frac{1}{k_\ell^{(n)}}=\frac{1}{(n-1)(n-2)\ell^2(k-2\ell)_\ell^{(n-2)}}$$

and hence (17) follows by Lemma 1.6 as $c_j = 0$ when $k \to \infty$. □

The following example illustrates Theorem 2.6.

Example 2.7 Substituting $\ell = 0.5$, $n = 4$ in (17), we obtain

$$\frac{1}{(k)_{0.5}^{(4)}}+\frac{2}{(k+0.5)_{0.5}^{(4)}}+\frac{3}{(k+1)_{0.5}^{(4)}}+\cdots=\frac{1}{6(0.5)^2(k-1)_{0.5}^{(2)}}.$$

In particular, when $k = 2$, the above series becomes

$$\frac{1}{2 \times 1.5 \times 1 \times 0.5} + \frac{2}{2.5 \times 2 \times 1.5 \times 1} + \frac{3}{3 \times 2.5 \times 2 \times 1.5} + \cdots = \frac{1}{6 \times 0.5^3}.$$

Similarly, one can take any value for $k \in [2\ell, \infty)$ and $\ell \in (0, \infty)$.

The following example shows that $\frac{1}{k_\ell^{(n)}} \in c_{0(\ell)}$ and $\ell_{2(\ell)}$ when $k_\ell^{(n)} \neq 0$.

Example 2.8 Let $n \in \mathbb{N}(2)$, $\ell \in (0, \infty)$, $j \in [0, \ell)$ and $a \geq n\ell$. Replacing k by $a + j$, in (12), we get

$$\sum_{r=0}^{\infty} \frac{1}{(a + j + r\ell)_\ell^{(n)}} = \frac{1}{(n-1)\ell(a + j - \ell)_\ell^{(n-1)}}. \tag{18}$$

Since

$$\left|\frac{1}{(a + j + r\ell)_\ell^{(n)}}\right|^2 < \frac{1}{(a + j + r\ell)_\ell^{(n)}},$$

thus from (18) it follows that

$$\sum_{r=0}^{\infty} \left|\frac{1}{(a + j + r\ell)_\ell^{(n)}}\right|^2 < \sum_{r=0}^{\infty} \frac{1}{(a + j + r\ell)_\ell^{(n)}} = \frac{1}{(n-1)\ell(a + j - \ell)_\ell^{(n-1)}} < \infty.$$

The function $\frac{1}{k_\ell^{(n)}} \in \ell_{2(\ell)}$ follows from Definition 1.4 by taking

$$u(a + j + r\ell) = \frac{1}{(a + j + r\ell)_\ell^{(n)}}.$$

Since $\lim_{r\to\infty} \frac{1}{(a+j+r\ell)_\ell^{(n)}} = 0$ for all $j \in [0, \ell)$, Definition 1.4 yields $\frac{1}{k_\ell^{(n)}} \in c_{0(\ell)}$.

3 Concluding remarks

In the present work, we provide an application on $\ell_{2(\ell)}$ and $c_{0(\ell)}$ and solutions of the second-order some generalized difference equation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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