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Convergence of algorithms for fixed points of generalized asymptotically quasi-φ-nonexpansive mappings with applications

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Abstract

In this article, strong convergence of Krasnoselski-Mann iterative sequences and Halpern iterative sequences are investigated based on hybrid projection methods. Strong convergence theorems for common fixed points of a family of generalized asymptotically quasi-φ-nonexpansive mappings are established in the framework of Banach spaces.

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1.Introduction

Fixed point theory as an important branch of nonlinear analysis theory has been applied in the study of nonlinear phenomena. During the four decades, many famous existence theorems of fixed points were established; see, for example, [1-5]. However, from the standpoint of real world applications it is not only to know the existence of fixed points of nonlinear mappings, but also to be able to construct an iterative process to approximate their fixed points. The computation of fixed points is important in the study of many real world problems, including inverse problems; for instance, it is not hard to show that the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain operators, respectively (see [6,7] for more details and the references therein).

Recently, the study of the convergence of various iterative processes for solving various nonlinear mathematical models forms the major part of numerical mathematics. Among these iterative processes, Krasnoselski-Mann iterative process and Halpern iterative process are popular and hot. Let C be a nonempty, closed, and convex subset of a underlying space X, and $T : C \to C$ a mapping. Halpern iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.1}$$

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where x_0 is an initial and u is a fixed element in C. Krasnoselski-Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_0 \in C, \quad x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n, \quad \forall n \geq 0, \tag{1.2}$$

It is known that Algorithm (1.2) only has weak convergence even for nonexpansive mappings in infinite-dimensional Hilbert spaces (see [8] for more details and the reference therein). In many disciplines, including economics [9], image recovery [10], quantum physics [11], and control theory [12], problems arises in infinite dimension spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. The important of strong convergence is also underlined in [13], where a convex function f is minimized via the proximal-point algorithm: it is shown that the rate of convergence of the value sequence $\{f(x_n)\}$ is better when $\{x_n\}$ converges strongly that it converges weakly. Such properties have a direct impact when the process is executed directly in the underlying infinite dimensional space. To improve the weak convergence of Krasnoselski-Mann iterative process, so called hybrid projections have been considered (see [14-25] for more details and the references therein).

Algorithm (1.1) was initially introduced in [26]; for more details see the references therein. In [26], Halpern showed that the following conditions

(C1) $\lim_{n\to\infty}, \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

are necessary in the sense that if Algorithm (1.1) is strongly convergent for all nonempty, closed, and convex subsets of a Hilbert space H and all nonexpansive mappings on C, then the sequence $\{x_n\}$ must satisfy conditions (C1), and (C2). Due to the restriction of (C2), Algorithm (1.1) is widely believed to have slow convergence though the rate of convergence has not be determined. Thus to improve the rate of convergence of algorithm (1.1), one can not rely only on the process itself; instead, some additional step of iteration should be taken; see [27-30] and the references therein. One of the purposes of this article is to show algorithm (1.1) is strong convergence under (C1) only with the help of projections.

The purposes of this article is to study Algorithms (1.1) and (1.2) with the help of additional metric projections for the new mapping. The organization of this article is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, Algorithms (1.1) and (1.2) are studied with the help of projections. Two main strong convergence theorems are established in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. In Section 4, applications of the main results are provided.

2.Preliminaries

Let H be a real Hilbert space, C a nonempty subset of H, and $T : C \to C$ a mapping. The symbol $F(T)$ stands for the fixed point set of T. Recall the following. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$, and

$$\|p - Ty\| \leq \|p - y\|, \quad \forall p \in F(T),\ y \in C.$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\|, \quad \forall x, y \in C,\ n \geq 1.$$

It is easy to see that a nonexpansive mapping is an asymptotically nonexpansive mapping with the sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2]. Since 1972, a host of authors have studied the convergence of iterative algorithms for such a class of mappings.

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|p - T^n y\| \leq (1 + \mu_n)\|p - y\|, \quad \forall p \in F(T),\ y \in C,\ n \geq 1.$$

It is easy to see that a quasi-nonexpansive mapping is an asymptotically quasi-nonexpansive mapping with the sequence $\{1\}$.

T is said to be *generalized asymptotically nonexpansive* if there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + \xi_n, \quad \forall x, y \in C, \quad n \geq 1.$$

T is said to be *generalized asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0,\infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq (1 + \mu_n)\|x - y\| + \xi_n, \quad \forall x \in C, \quad y \in F(T),\ n \geq 1.$$

The class of generalized asymptotically (quasi)-nonexpansive has been considered by Shahzad and Zegeye [31] (see also Agarwal et al. [32]). It is easy to see that the class of generalized asymptotically (quasi)-nonexpansive include the class of asymptotically (quasi)-nonexpansive as a special case.

In what follows, we always assume that E is a Banach space with the dual space E^*. Let C be a nonempty, closed, and convex subset of E. We use the symbol J to stand for the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\right\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E, and E^*. It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive, and smooth, then J is single valued, and demicontinuous (see [33] for more details and the references therein).

It is also well known that if D is a nonempty, closed, and convex subset of a Hilbert space H, and $P_C : H \to D$ is the metric projection from H onto D, then P_D is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [34] introduced a generalized projection operator in Banach spaces which is an analogue of the metric projection in Hilbert spaces.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in U_E$ with $x \neq y$. It is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be *smooth* provided $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U_E$. It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U_E$.

E is said to enjoy *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on Kadec-Klee property, the readers can refer to [35] and the references therein. It is well known that if E is a uniformly convex Banach spaces, then E enjoys Kadec-Klee property.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

Notice that, in a Hilbert space H, (2.1) is reduced to $\varphi(x, y) = \|x-y\|^2$ for all $x,y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\varphi(x, y)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \min_{y\in C}\phi(y, x).$$

The existence, and uniqueness of the operator Π_C follow from the properties of the functional $\varphi(x, y)$, and the strict monotonicity of the mapping J (see, for example, [33,36]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function φ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E, \tag{2.2}$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.3}$$

Remark 2.1. If E is a reflexive, strictly convex, and smooth Banach space, then, for all $x,y \in E$, $\varphi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\varphi(x, y) = 0$, then $x = y$. From (2.2), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we see that $Jx = Jy$. It follows that $x = y$; see [33,36] for more details.

Next, we recall the following.

(1) A point p in C is said to be an *asymptotic fixed point* of T [37] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\tilde{F}(T)$.

(2) T is said to be *relatively nonexpansive* if

$$\tilde{F}(T) = F(T) \neq \emptyset, \quad \text{and } \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T).$$

(3) T is said to be *relatively asymptotically nonexpansive* if

$$\tilde{F}(T) = F(T) \neq \emptyset, \quad \text{and } \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C,\ p \in F(T),\ n \geq 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ *as* $n \to \infty$.

Remark 2.2. The class of relatively asymptotically nonexpansive mappings was first considered in Su and Qin [38] (see also, Agarwal et al. [39], and Qin et al. [40]).

(4) T is said to be *quasi-φ-nonexpansive* if

$$F(T) \neq \emptyset, \ \text{and } \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T).$$

(5) T is said to be *asymptotically quasi-φ-nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \ \text{and } \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x), \quad \forall x \in C,\ p \in F(T),\ n \geq 1.$$

Remark 2.3. The class of quasi-φ-nonexpansive mappings and the class of asymptotically quasi-φ-nonexpansive mappings were first considered in Zhou et al. [24] (see also Qin and Agarwal [18], Qin et al. [20], Qin et al. [21], Qin et al. [41]).

Remark 2.4. The class of quasi-φ-nonexpansive mappings and the class of asymptotically quasi-φ-nonexpansive mappings are more general than the class of relatively nonex-pansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-φ-nonexpansive mappings and asymptotically quasi-φ-nonexpansive do not require $F(T) = \tilde{F}(T)$.

Remark 2.5. The class of quasi-φ-nonexpansive mappings and the class of asymptotically quasi-φ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

In this article, we introduce and consider the following new nonlinear mapping: generalized asymptotically quasi-φ-nonexpansive mappings.

(6) T is said to be an *generalized asymptotically quasi-φ-nonexpansive mapping* if $F(T) \neq \emptyset$, and there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$, and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + \xi_n, \quad \forall x \in C,\ p \in F(T),\ n \geq 1.$$

Remark 2.6. The class of generalized asymptotically quasi-φ-nonexpansive mappings is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces.

(7) T is said to be *asymptotically regular* on C if, for any bounded subset K of C,

$$\lim_{n \to \infty} \sup_{x \in K} \left\{ \left\| T^{n+1}x - T^n x \right\| \right\} = 0.$$

In order to prove our main results, we also need the following lemmas:

Lemma 2.1. [34]*Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and* $x \in E$. *Then* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2. [34]*Let E be a reflexive, strictly, convex, and smooth Banach space, C a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi\left(y, \Pi_C x\right) + \phi\left(\Pi_C x, x\right) \le \phi(y, x), \quad \forall y \in C.$$

3.Main results

Theorem 3.1. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let Δ be an index set, and $T_i : C \to C$ a closed, asymptotically regular, and generalized asymptotically quasi-φ-nonexpansive mapping with the sequences $\{\mu_{n,i}\}$, and $\{\xi_{n,i}\}$, for every $i \in \Delta$. Assume that $\cap_{i\in\Delta} F(T_i)$ is nonempty, and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ y_{n,i} = J^{-1}\left(\alpha_{n,i} J\left(T_i^n x_n\right) + (1-\alpha_{n,i}) J x_n\right), \quad n \ge 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \le \phi(u, x_n) + \mu_{n,i} M_n + \xi_{n,i}\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases} \qquad (\lt ?\text{show[CSFchar} =)$$

where $M_n = \sup\{\varphi(z, x_n) : z \in \cap_{i\in\Delta} F(T_i)\}$, and $\{\alpha_{n,i}\}$ are sequences in (0,1] *such that* $\liminf_{n\to\infty} \alpha_{n,i} > 0$. *Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta} F(T_i)} x_0$, where $\Pi_{\cap_{i\in\Delta} F(T_i)}$stands for the generalized projection from E onto $\cap_{i\in\Delta} F(T_i)$.*

Proof. The proof is split into seven steps.

Step 1. Show, for every $i \in \Delta$, that $F(T_i)$ is closed, and convex. This proves that $\Pi_{\cap_{i\in\Delta} F(T_i)} x_0$ is well defined, for every $x_0 \in E$. On the closedness of $\cap_{i\in\Delta} F(T_i)$, we can easily conclude from the closedness of T_i the desired conclusion. We only prove that $\cap_{i\in\Delta} F(T_i)$ is convex. Let $p_{1,i}, p_{2,i} \in F(T_i)$, and $p_i = t_i p_{1,i} + (1 - t_i) p_{2,i}$, where $t_i \in (0,1)$, for every $i \in \Delta$. We see that $p_i = T_i p_i$. Indeed, we see from the definition of T_i that

$$\phi\left(p_{1,i}, T_i^n p_i\right) \le (1 + \mu_{n,i}) \phi(p_{1,i}, p_i) + \xi_{n,i}, \tag{3.1}$$

and

$$\phi\left(p_{2,i}, T_i^n p_i\right) \le (1 + \mu_{n,i}) \phi(p_{2,i}, p_i) + \xi_{n,i}. \tag{3.2}$$

In view of (2.3), we obtain that

$$\phi\left(p_{1,i}, T_i^n p_i\right) = \phi\left(p_{1,i}, p_i\right) + \phi\left(p_i, T_i^n p_i\right) + 2\left\langle p_{1,i} - p_i, J p_i - J T_i^n p_i\right\rangle, \tag{3.3}$$

and

$$\phi\left(p_{2,i}, T_i^n p_i\right) = \phi\left(p_{2,i}, p_i\right) + \phi\left(p_i, T_i^n p_i\right) + 2\left\langle p_{2,i} - p_i, J p_i - J T_i^n p_i\right\rangle. \tag{3.4}$$

It follows from (3.1), (3.2), (3.3), and (3.4) that

$$\phi\left(p_i, T_i^n p_i\right) \le 2\left\langle p_i - p_{1,i}, J p_i - J\left(T_i^n p_i\right)\right\rangle + \mu_{n,i} \phi(p_{1,i}, p_i) + \xi_{n,i}, \tag{3.5}$$

and

$$\phi\left(p_i, T_i^n p_i\right) \le 2\left\langle p_i - p_{2,i}, J p_i - J\left(T_i^n p_i\right)\right\rangle + \mu_{n,i} \phi(p_{2,i}, p_i) + \xi_{n,i}. \tag{3.6}$$

Multiplying t_i and $(1 - t_i)$ on the both sides of (3.5) and (3.6), respectively yields that

$$\phi\left(p_i, T_i^n p_i\right) \le t_i \mu_{n,i} \phi(p_{1,i}, p_i) + (1 - t_i)\mu_{n,i}\phi(p_{2,i}, p_i) + \xi_{n,i}.$$

It follows that

$$\lim_{n\to\infty} \phi\left(p_i, T_i^n p_i\right) = 0.$$

In light of (2.2), we arrive at

$$\lim_{n\to\infty} \left\|T_i^n p_i\right\| = \left\|p_i\right\|. \tag{3.7}$$

It follows that

$$\lim_{n\to\infty} \left\|J\left(T_i^n p_i\right)\right\| = \left\|Jp_i\right\|. \tag{3.8}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J\left(T_i^n p_i\right) \rightharpoonup e^{*,i} \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e^i \in E$ such that $Je^i = e^{*i}$. It follows that

$$\begin{aligned}\phi\left(p_i T_i^n p_i\right) &= \left\|p_i\right\|^2 - 2\left\langle p_i, J\left(T_i^n p_i\right)\right\rangle + \left\|T_i^n p_i\right\|^2 \\ &= \left\|p_i\right\|^2 - 2\left\langle p_i, J\left(T_i^n p_i\right)\right\rangle + \left\|J\left(T_i^n p_i\right)\right\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned}0 &\ge \left\|p_i\right\|^2 - 2\left\langle p_i, e^{*,i}\right\rangle + \left\|e^{*,i}\right\|^2 \\ &= \left\|p_i\right\|^2 - 2\left\langle p_i, Je^i\right\rangle + \left\|Je^i\right\|^2 \\ &= \left\|p_i\right\|^2 - 2\left\langle p_i, Je^i\right\rangle + \left\|e^i\right\|^2 \\ &= \phi(p_i, e^i).\end{aligned}$$

This implies that $p_i = e^i$, that is, $Jp_i = e^{*i}$. It follows that $J\left(T_i^n p_i\right) \rightharpoonup Jp_i \in E^*$. In view of Kadec-Klee property of E^*, we obtain from (3.8) that

$$\lim_{n\to\infty} \left\|J\left(T_i^n p_i\right) - Jp_i\right\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T_i^n p_i \rightharpoonup p_i$. By virtue of Kadec-Klee property of E, we see from (3.7) that $T_i^n p_i \to p_i$ as $n\to\infty$. Hence

$$T_i T_i^n p_i = T_i^{n+1} p_i \to p_i,$$

as $n\to\infty$. In view of the closedness of T_i, we can obtain that $p_i \in F(T_i)$, for every $i \in \Delta$. This shows, for every $i \in \Delta$, that $F(T_i)$ is convex. This proves that $\cap_{i\in\Delta} F(T_i)$ is convex. This completes the proof of Step 1.

Step 2. Show that C_n is closed, and convex for all $n \ge 1$. It suffices to show, for any fixed but arbitrary $i \in \Delta$, that $C_{n,i}$ is closed, and convex, for every $n \ge 1$. This can be proved by induction on n. It is obvious that $C_{1,i} = C$ is closed, and convex. Assume that $C_{h,i}$ is closed, and convex for some $h \ge 1$. We next prove that $C_{h+1,i}$ is closed, and convex for the same h. This completes the proof that C_n is closed, and convex. The closedness of $C_{h+1,i}$ is clear. We only prove the convexness. Indeed, $\forall a, b \in C_{h+1,i}$, we see that $a,b \in C_{h,i}$, and

$$\phi\left(a, y_{h,i}\right) \leq \phi(a, x_h) + \mu_{h,i} M_h + \xi_{h,i}, \tag{3.9}$$

and

$$\phi\left(b, y_{h,i}\right) \leq \phi(b, x_h) + \mu_{h,i} M_h + \xi_{h,i}. \tag{3.10}$$

Notice that (3.9), and (3.10) are equivalent to the following inequalities, respectively.

$$2\left\langle a, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \mu_{h,i} M_h + \xi_{h,i},$$

and

$$2\left\langle b, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \mu_{h,i} M_h + \xi_{h,i}.$$

These imply that

$$2\left\langle ta + (1-t)b, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \mu_{h,i} M_h + \xi_{h,i}, \quad \forall t \in (0,1). \tag{3.11}$$

Since $C_{h,i}$ is convex, we see that $ta + (1 - t)b \in C_{h,i}$. Notice that (3.11) is equivalent to

$$\phi\left(ta + (1-t)b, y_{h,i}\right) \leq \phi(ta + (1-t)b, x_h) + \mu_{h,i} M_h + \xi_{h,i}.$$

This proves that $C_{h+1,i}$ is convex. This completes the proof of Step 2.

Step 3. Show that $\cap_{i\in\Delta} F(T_i) \subset C_n$, for every $n \geq 1$. It suffices to claim that $\cap_{i\in\Delta} F(T_i) \subset C_{n,i}$, for every $n \geq 1$, and for every $i \geq \Delta$. Note that $\cap_{i\in\Delta} F(T_i) \subset C_{1,i} = C$. Suppose that $\cap_{i\in\Delta} F(T_i) \subset C_{h,i}$ for some h, and for every $i \in \Delta$. Then, for all $w \in \cap_{i\in\Delta} F(T_i) \subset C_{h,i}$, we have

$$\begin{aligned}
&\phi(w, y_{h,i}) \\
&= \phi\left(w, J^{-1}\left(\alpha_{h,i} J\left(T_i^h x_h\right) + (1-\alpha_{h,i}) Jx_h\right)\right) \\
&= \|w\|^2 - 2\left\langle w, \alpha_{h,i} J\left(T_i^h x_h\right) + (1-\alpha_{h,i}) Jx_h\right\rangle \\
&\quad + \left\|\alpha_{h,i} J\left(T_i^h x_h\right) + (1-\alpha_{h,i}) Jx_h\right\|^2 \\
&\leq \|w\|^2 - 2\alpha_{h,i}\left\langle w, J\left(T_i^h x_h\right)\right\rangle - 2(1-\alpha_{h,i})\langle w, Jx_h\rangle + \alpha_{h,i}\left\|T_i^h x_h\right\|^2 \\
&\quad + (1-\alpha_{h,i})\|x_h\|^2 \\
&= \alpha_{h,i}\phi\left(w, T_i^h x_h\right) + (1-\alpha_{h,i})\phi(w, x_h) \\
&\leq \alpha_{h,i}\mu_{h,i}\phi(w, x_h) + \alpha_{h,i}\xi_{h,i} + \phi(w, x_h) \\
&\leq \phi(w, x_h) + \mu_{h,i} M_h + \xi_{h,i},
\end{aligned}$$

where $M_h = \sup_{z\in\cap_{i\in\Delta} F(T_i)} \{\phi(z, x_h)\}$.This shows that $w \in C_{h+1,i}$.This implies that $\cap_{i\in\Delta} F(T_i) \subset C_n$, for every $n \geq 1$. This completes the proof of Step 3.

Step 4. Show that $\{x_n\}$ is bounded. In view of $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n.$$

Since $\cap_{i\in\Delta} F(T_i) \subset C_n$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in \cap_{i\in\Delta} F(T_i). \tag{3.12}$$

It follows from Lemma 2.2 that

$$\begin{aligned}\phi(x_n, x_0) &= \phi\left(\Pi_{C_n} x_0, x_0\right)\\ &\le \phi\left(\Pi_{\cap_{i\in\Delta}F(T_i)}x_0, x_0\right) - \phi\left(\Pi_{\cap_{i\in\Delta}F(T_i)}x_0, x_n\right)\\ &\le \phi\left(\Pi_{\cap_{i\in\Delta}F(T_i)}x_0, x_0\right).\end{aligned}$$

This implies that the sequence $\{\varphi(x_n, x_0)\}$ is bounded. It follows from (2.2) that the sequence $\{x_n\}$ is also bounded. This completes the proof of Step 4.

Step 5. Show that $x_n \to \bar{x}$, where $\bar{x}$ is some point in C as $n\to\infty$. Since $\{x_n\}$ is bounded, and the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed, and convex, we see that $\bar{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi\left(\bar{x}, x_0\right) &= \|\bar{x}\|^2 - 2\left\langle \bar{x}, Jx_0\right\rangle + \|x_0\|^2\\ &\le \liminf_{n\to\infty}\left(\|x_n\|^2 - 2\left\langle x_n, Jx_0\right\rangle + \|x_0\|^2\right)\\ &= \liminf_{n\to\infty}\phi(x_n, x_0)\\ &\le \limsup_{n\to\infty}\phi(x_n, x_0)\\ &\le \phi(\bar{x}, x_0),\end{aligned}$$

which implies that $\phi\left(x_n, x_0\right) \to \left(\bar{x}, x_0\right)$ as $n\to\infty$. Hence, $\|x_n\| \to \|\bar{x}\|$ as $n\to\infty$. In view of Kadec-Klee property of E, we see that $x_n \to \bar{x}$ as $n\to\infty$. This completes the proof of Step 5.

Step 6. Show that $\bar{x} \in \cap_{i\in\Delta}F(T_i)$. In view of construction of $x_{n+1} = \Pi_{\cap_{i\in\Delta}F(T_i)}x_0 \in C_{n+1} \subset C_n$, we arrive at

$$\begin{aligned}\phi\left(x_{n+1}, x_n\right) &= \phi\left(x_{n+1}, \Pi_{C_n}x_0\right)\\ &\le \phi\left(x_{n+1}, x_0\right) - \phi\left(\Pi_{C_n}x_0, x_0\right)\\ &= \phi\left(x_{n+1}, x_0\right) - \phi\left(x_n, x_0\right).\end{aligned}$$

Since $x_n = \Pi_{C_n}x_0$, and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we arrive at $\varphi(x_n, x_0) \le \varphi(x_{n+1},x_0)$, $\forall n \ge 1$. This shows that $\{\varphi(x_n, x_0)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} \varphi(x, x_0)$ exists. It follows that

$$\lim_{n\to\infty}\phi\left(x_{n+1}, x_n\right) = 0. \tag{3.13}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we arrive at

$$\phi\left(x_{n+1}, y_{n,i}\right) \le \phi\left(x_{n+1}, x_n\right) + \mu_{n,i}M_n + \xi_{n,i}.$$

This in turn implies from (3.13) that

$$\lim_{n\to\infty}\phi\left(x_{n+1}, y_{n,i}\right) = 0. \tag{3.14}$$

In view of (2.2), we see that

$$\lim_{n\to\infty}\left(\|x_{n+1}\| - \left\|y_{n,i}\right\|\right) = 0.$$

This in turn implies that

$$\lim_{n\to\infty}\left\|y_{n,i}\right\| = \|\bar{x}\|.$$

It follows that

$$\lim_{n\to\infty} \left\| Jy_{n,i} \right\| = \|J\bar{x}\| . \tag{3.15}$$

This implies that $\{Jy_{n,i}\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Jy_{n,i} \rightharpoonup y^{*i} \in E^*$, for every $i \in \Delta$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{*i}$. It follows that

$$\begin{aligned}\phi\left(x_{n+1}, y_{n,i}\right) &= \|x_{n+1}\|^2 - 2\left\langle x_{n+1}, Jy_{n,i}\right\rangle + \left\|y_{n,i}\right\|^2 \\ &= \|x_{n+1}\|^2 - 2\left\langle x_{n+1}, Jy_{n,i}\right\rangle + \left\|Jy_{n,i}\right\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\left\langle \bar{x}, y^{*,i}\right\rangle + \left\|y^{*,i}\right\|^2 \\ &= \|\bar{x}\|^2 - 2\left\langle \bar{x}, Jy^i\right\rangle + \left\|Jy^i\right\|^2 \\ &= \|\bar{x}\|^2 - 2\left\langle \bar{x}, Jy^i\right\rangle + \left\|y^i\right\|^2 \\ &= \phi(\bar{x}, y^i).\end{aligned}$$

That is, $\bar{x} = y^i$, which in turn implies that $y^{*,i} = J\bar{x}$, for every $i \in \Delta$. It follows that $Jy_{n,i} \rightharpoonup J\bar{x} \in E^*$, for every $i \in \Delta$. Since E^* enjoys Kadec-Klee property, we obtain from (3.15) that

$$\lim_{n\to\infty} Jy_{n,i} = J\bar{x}.$$

Notice that

$$\left\|Jx_n - Jy_{n,i}\right\| \leq \|Jx_n - J\bar{x}\| + \left\|J\bar{x} - Jy_{n,i}\right\| .$$

It follows that

$$\lim_{n\to\infty} \left\|Jx_n - Jy_{n,i}\right\| = 0. \tag{3.16}$$

Notice from (Υ) that

$$Jx_n - Jy_{n,i} = \alpha_{n,i}\left(J\left(T_i^n x_n\right) - Jx_n\right).$$

In view of the assumption that $\liminf_{n\to\infty} \alpha_{n,i} > 0$, we arrive at

$$\lim_{n\to\infty} \left\|J\left(T_i^n x_n\right) - Jx_n\right\| = 0. \tag{3.17}$$

Notice that

$$\left\|J\left(T_i^n x_n\right) - J\bar{x}\right\| \leq \left\|J\left(T_i^n x_n\right) - Jx_n\right\| + \|Jx_n - J\bar{x}\| .$$

This implies from (3.17) that

$$\lim_{n\to\infty} \left\|J\left(T_i^n x_n\right) - J\bar{x}\right\| = 0. \tag{3.18}$$

The demi-continuity of $J^{-1} : E^* \to E$ implies that $T_i^n x_n \rightharpoonup \bar{x}$, for every $i \in \Delta$. Note that

$$\left|\left\|T_i^n x_n\right\| - \|\bar{x}\|\right| = \left|\left\|J\left(T_i^n x_n\right)\right\| - \|J\bar{x}\|\right| \leq \left\|J\left(T_i^n x_n\right) - J\bar{x}\right\| .$$

In view of (3.18), we see that $\left\|T_i^n x_n\right\| \to \|\bar{x}\|$, for every $i \in \Delta$ as $n \to \infty$. Since E enjoy Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} \left\|T_i^n x_n - \bar{x}\right\| = 0. \tag{3.19}$$

Notice that

$$\left\|T_i^{n+1} x_n - \bar{x}\right\| \le \left\|T_i^{n+1} x_n - T_i^n x_n\right\| + \left\|T_i^n x_n - \bar{x}\right\|.$$

It follows from the asymptotic regularity of T_i, and (3.19) that

$$\lim_{n\to\infty} \left\|T_i^{n+1} x_n - \bar{x}\right\| = 0,$$

that is, $T_i T_i^n x_n - \bar{x} \to 0$ as $n \to \infty$. It follows from the closedness of T_i that $T_i \bar{x} = \bar{x}$, for every $i \in \Delta$. This completes the proof of Step 6.

Step 7. Show that $\bar{x} = \Pi_{\cap_{i\in\Delta} F(T_i)} x_0$. Letting $n \to \infty$ in (3.12), we arrive at

$$\langle \bar{x} - w, Jx_0 - J\bar{x}\rangle \ge 0, \quad \forall w \in \cap_{i\in\Delta} F(T_i).$$

It follows from Lemma 2.1 that $\bar{x} = \Pi_{\cap_{i\in\Delta} F(T_i)} x_0$. This completes the proof of Step 7. The proof of Theorem 3.1 is completed.

Remark 3.2. Comparing Theorem 3.1 with Theorem 2.1 in Qin et al. [21], we have the following:

(a) extend the mapping from the class of asymptotically quasi-φ-nonexpansive mappings to the class of generalized asymptotically quasi-φ-nonexpansive mappings;
(b) extend the mapping from a single mapping to a family of mappings;
(c) extend the space from a uniformly smooth, and strictly convex Banach space which also enjoys the Kadec-Klee property to a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property.

Remark 3.3. Strictly convex, reflexive, and smooth Musielak-Orlicz spaces satisfy the restrictions imposed on the framework of the spaces [35], while, in general, these spaces need not to be uniformly convex or uniformly smooth.

For a single mapping, we can easily conclude the following.

Corollary 3.4. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a closed, asymptotically regular, and generalized asymptotically quasi-φ-nonexpansive mapping with the sequences $\{\mu_n\}$, and $\{\xi_n\}$. Assume that F(T) is nonempty, and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\left(\alpha_n J(T^n x_n) + (1-\alpha_n) Jx_n\right), \quad n \ge 1, \\ C_{n+1} = \left\{u \in C_n : \phi(u, y_n) \le \phi(u, x_n) + \mu_n M_n + \xi_n\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1. \end{cases}$$

where $M_n = \sup\{\varphi(z, x_n) : z \in F(T)\}$, *and* $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *such that* $\liminf_{n\to\infty} \alpha_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_0$, *where* $\Pi_{F(T)}$ *stands for the generalized projection from E onto* $F(T)$.

If $\alpha_n = 1$, then Theorem 3.1 is reduced to the following.

Corollary 3.5. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let* Δ *be an index set, and* $T_i : C \to C$ *a closed, asymptotically regular, and generalized asymptotically quasi-φ-nonexpansive mapping with the sequences* $\{\mu_{n,i}\}$, *and* $\{\xi_{n,i}\}$, *for every* $i \in \Delta$. *Assume that* $\cap_{i\in\Delta} F(T_i)$ *is nonempty, and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi\left(u, T_i^n x_n\right) \le \phi\left(u, x_n\right) + \mu_{n,i} M_n + \xi_{n,i}\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where $M_n = \sup\{\varphi(z, x_n) : z \in \cap_{i\in\Delta} F(T_i)\}$, *and* $\{\alpha_{n,i}\}$ *are sequences in* $(0,1]$ *such that* $\liminf_{n\to\infty} \alpha_{n,i} > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Delta} F(T_i)}x_0$, *where* $\Pi_{\cap_{i\in\Delta} F(T_i)}$ *stands for the generalized projection from E onto* $\cap_{i\in\Delta} F(T_i)$.

In the framework of Hilbert spaces, Theorem 3.1 is reduced to the following.

Corollary 3.6. *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let* Δ *be an index set, and* $T_i : C \to C$ *a closed, asymptotically regular, and generalized asymptotically quasi-nonexpansive mapping with the sequences* $\{\mu_{n,i}\}$, *and* $\{\xi_{n,i}\}$, *for every* $i \in \Delta$. *Assume that* $\cap_{i\in\Delta} F(T_i)$ *is nonempty, and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = P_{C_1} x_0, \\ y_{n,i} = \alpha_{n,i} T_i^n x_n + (1 - \alpha_{n,i}) x_n, \quad n \ge 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \left\|u - y_{n,i}\right\|^2 \le \|u - x_n\|^2 + \mu_{n,i} M_n + \xi_{n,i}\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : z \in \cap_{i\in\Delta} F(T_i)\}$, *and* $\{\alpha_{n,i}\}$ *are sequences in* $(0,1]$ *such that* $\liminf_{n\to\infty} \alpha_{n,i} > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\cap_{i\in\Delta} F(T_i)}x_0$, *where* $P_{\cap_{i\in\Delta} F(T_i)}$ *stands for the metric projection from E onto* $\cap_{i\in\Delta} F(T_i)$.

For a single mapping, we can easily conclude the following.

Corollary 3.7. *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let* $T : C \to C$ *be a closed, asymptotically regular, and generalized asymptotically quasi-nonexpansive mapping with the sequences* $\{\mu_n\}$, *and* $\{\xi_n\}$. *Assume that* $F(T)$ *is nonempty, and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_{n,i} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n \geq 1, \\ C_{n+1} = \left\{u \in C_n : \left\|u - y_n\right\|^2 \leq \|u - x_n\|^2 + \mu_n M_n + \xi_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : z \in \cap_{i\in\Delta} F(T_i)\}$, *and* $\{\alpha_n\}$ *is a sequence in* $(0,1]$ *such that* $\liminf_{n\to\infty} \alpha_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$, *where* $P_{F(T)}$ *stands for the metric projection from E onto F(T).*

Next, we turn our attention to Algorithm (1.1).

Theorem 3.8. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let* Δ *be an index set, and* $T_i : C \to C$ *a closed, asymptotically regular, and generalized asymptotically quasi-φ-nonexpansive mapping with the sequences* $\{\mu_{n,i}\}$, *and* $\{\xi_{n,i}\}$, *for every* $i \in \Delta$. *Assume that* $\cap_{i\in\Delta} F(T_i)$ *is nonempty, and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1}x_0, \\ y_{n,i} = J^{-1}\left(\alpha_{n,i}Jx_1 + \left(1-\alpha_{n,i}\right)J\left(T_i^n x_n\right)\right), \quad n \geq 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi\left(u, y_{n,i}\right) \leq \phi\left(u, x_n\right) + \alpha_{n,i}M + \xi_{n,i}\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases}$$

where $M = \sup_{z\in\cap_{i\in\Delta}F(T_i)}\{\phi(z, x_1)\}$, *and* $\{\alpha_{n,i}\}$ *are sequences in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_{n,i} = 0$. *Assume that* $\mu_{n,i} \leq \frac{\alpha_{n,i}}{1-\alpha_{n,i}}$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Delta}F(T_i)}x_1$, *where* $\Pi_{\cap_{i\in\Delta}F(T_i)}$*stands for the generalized projection from E onto* $\cap_{i\in\Delta} F(T_i)$.

Proof. In view of the proof of Theorem 3.1, we show the difference only. From the proof of Step 1 of Theorem 3.1, we see that $\cap_{i\in\Delta} F(T_i)$ is closed, and convex.

Next, we show that C_n is closed, and convex for all $n \geq 1$. It suffices to show, for any fixed but arbitrary $i \in \Delta$, that $C_{n,i}$ is closed, and convex, for every $n \geq 1$. This can be proved by induction on n. It is obvious that $C_{1,i} = C$ is closed, and convex. Assume that $C_{h,i}$ is closed, and convex for some $h \geq 1$. We next prove that $C_{h+1,i}$ is closed, and convex for the same h. This completes the proof that C_n is closed, and convex. The closedness of $C_{h+1,i}$ is clear. We only prove the convexness. Indeed, $\forall a, b \in C_{h+1,i}$, we see that $a,b \in C_{h,i}$, and

$$\phi\left(a, y_{h,i}\right) \leq \phi\left(a, x_h\right) + \alpha_{h,i}M + \xi_{h,i}, \tag{3.20}$$

and

$$\phi\left(b, y_{h,i}\right) \leq \phi\left(b, x_h\right) + \alpha_{h,i}M + \xi_{h,i}. \tag{3.21}$$

Notice that (3.20), and (3.21) are equivalent to the following inequalities, respectively.

$$2\left\langle a, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \alpha_{h,i}M + \xi_{h,i},$$

and

$$2\left\langle b, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \alpha_{h,i}M + \xi_{h,i}.$$

These imply that

$$2\left\langle ta + (1-t)b, Jx_h - Jy_{h,i}\right\rangle \leq \|x_h\|^2 - \left\|y_{h,i}\right\|^2 + \alpha_{h,i}M + \xi_{h,i}, \quad \forall t \in (0,1). \tag{3.22}$$

Since $C_{h,i}$ is convex, we see that $ta + (1 - t)b \in C_{h,i}$. Notice that (3.22) is equivalent to

$$\phi\left(ta + (1-t)b, y_{h,i}\right) \leq \phi\left(ta + (1-t)b, x_h\right) + \alpha_{h,i}M + \xi_{h,i}.$$

This proves that $C_{h+1,i}$ is convex. This completes the proof that C_n is closed, and convex for all $n \geq 1$.

Next, we show that $\cap_{i\in\Delta} F(T_i) \subset C_n$, for every $n \geq 1$. It suffices to claim that $\cap_{i\in\Delta} F(T_i) \subset C_{n,i}$, for every $n \geq 1$, and for every $i \geq \Delta$. Note that $\cap_{i\in\Delta} F(T_i) \subset C_{1,i} = C$. Suppose that $\cap_{i\in\Delta} F(T_i) \subset C_{h,i}$ for some h, and for every $i \in \Delta$. Then, for $\forall w \in \cap_{i\in\Delta} F(T_i) \subset C_{h,i}$, we obtain from the restriction $\mu_{n,i} \leq \frac{\alpha_{n,i}}{1-\alpha_{n,i}}$ that

$$\begin{aligned}
&\phi(w, y_{h,i}) \\
&= \phi\left(w, J^{-1}\left(\alpha_{h,i}Jx_1 + (1-\alpha_{h,i})J\left(T_i^h x_h\right)\right)\right) \\
&= \|w\|^2 - 2\left\langle w, \alpha_{h,i}Jx_1 + (1-\alpha_{h,i})J\left(T_i^h x_h\right)\right\rangle + \left\|\alpha_{h,i}Jx_1 + (1-\alpha_{h,i})J\left(T_i^h\right)\right\|^2 \\
&\leq \|w\|^2 - 2\alpha_{h,i}\langle w, Jx_1\rangle - 2\left(1-\alpha_{h,i}\right)\left\langle w, J\left(T_i^h x_h\right)\right\rangle + \alpha_{h,i}\|x_1\|^2 + \left(1-\alpha_{h,i}\right)\left\|T_i^h x_h\right\|^2 \\
&= \alpha_{h,i}\phi(w, x_1) + (1-\alpha_{h,i})\phi\left(w, T_i^h x_h\right) \\
&\leq \phi(w, x_h) + \alpha_{h,i}\phi(w, x_1) - \left(\alpha_{h,i} - (1-\alpha_{h,i})\mu_{h,i}\right)\phi(w, x_h) + \xi_{h,i} \\
&\leq \phi(w, x_h) + \alpha_{h,i}M + \xi_{h,i}
\end{aligned}$$

where $M = \sup_{z\in\cap_{i\in\Delta}F(T_i)}\left\{\phi(z, x_1)\right\}$. This shows that $w \in C_{h+1,i}$. This implies that $\cap_{i\in\Delta} F(T_i) \subset C_n$, for every $n \geq 1$. This completes the proof that $\cap_{i\in\Delta} F(T_i) \subset C_n$, for every $n \geq 1$.

In the light of the proof of Step 4 of Theorem 3.1, we find that $\{x_n\}$ is bounded. It follows the proof of Step 5 of Theorem 3.1 that $x_n \to \bar{x} \in C as\ n \to \infty$. Next, we show that $\bar{x} \in \cap_{i\in\Delta}F(T_i)$. In view of the proof of Step 6 of Theorem 3.1, we find that

$$\lim_{n\to\infty} \phi\left(x_{n+1}, x_n\right) = 0. \tag{3.23}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we arrive at

$$\phi\left(x_{n+1}, y_{n,i}\right) \leq \phi\left(x_{n+1}, x_n\right) + \alpha_{n,i}M + \xi_{n,i}.$$

This in turn implies that

$$\lim_{n\to\infty} \phi\left(x_{n+1}, y_{n,i}\right) = 0. \tag{3.24}$$

In view of the proof of Step 6 of Theorem 3.1, we find that

$$\lim_{n\to\infty} \left\|Jx_n - Jy_{n,i}\right\| = 0. \tag{3.25}$$

Notice from (ϒϒ) that

$$\left\| J\left(T_i^n x_n\right) - Jx_n \right\| \leq \frac{1}{1-\alpha_{n,i}} \left\| Jy_{n,i} - Jx_n \right\| + \frac{\alpha_{n,i}}{1-\alpha_{n,i}} \left\| Jx_n - Jx_1 \right\|.$$

In view of the assumption that $\lim_{n\to\infty} \alpha_{n,i} = 0$, $\forall i \in \Delta$, we find from (3.25) that

$$\lim_{n\to\infty} \left\| J\left(T_i^n x_n\right) - Jx_n \right\| = 0. \tag{3.26}$$

Next, following Steps 6 and 7, we can easily conclude the desired conclusion. This completes the proof of Theorem 3.8.

Remark 3.9. In view of the mappings, and the framework of the spaces, we see that Theorem 3.8 can be viewed as a generalization of the corresponding results announced in Cho et al. [27], Qin et al. [28], and Qin and Su [29].

For a single mapping, we obtain from Theorem 3.8 the following.

Corollary 3.10. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ a closed, asymptotically regular, and generalized asymptotically quasi-φ-nonexpansive mapping with the sequences $\{\mu_n\}$, and $\{\xi_n\}$. Assume that $F(T)$ is nonempty, and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\left(\alpha_n Jx_1 + (1-\alpha_n) J\left(T^n x_n\right)\right), \quad n \geq 1, \\ C_{n+1} = \left\{u \in C_n : \phi\left(u, y_n\right) \leq (u, x_n) + \alpha_n M + \xi_n\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where $M = \sup_{z\in F(T)}\{\varphi(z,x_1)\}$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Assume that $\mu_n \leq \frac{\alpha_n}{1-\alpha_n}$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$, where $\Pi_{F(T)}$ stands for the generalized projection from E onto $F(T)$.

In the framework of Hilbert spaces, Theorem 3.8 is reduced to the following.

Corollary 3.11. *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let Δ be an index set, and $T_i : C \to C$ a closed, asymptotically regular, and generalized asymptotically quasi-nonexpansive mapping with the sequences $\{\mu_{n,i}\}$, and $\{\xi_{n,i}\}$, for every $i \in \Delta$. Assume that $\cap_{i\in\Delta} F(T_i)$ is nonempty, and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = P_{C_1} x_0, \\ y_{n,i} = \alpha_{n,i} x_1 + (1-\alpha_{n,i}) T_i^n x_n, \quad n \geq 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \left\|u - y_{n,i}\right\|^2 \leq \|u - x_n\|^2 + \alpha_{n,i} M + \xi_{n,i}\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where $M = \sup_{z\in\cap_{i\in\Delta}F(T_i)}\left\{\|z - x_1\|^2\right\}$, and $\{\alpha_{n,i}\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_{n,i} = 0$. Assume that $\mu_{n,i} \leq \frac{\alpha_{n,i}}{1-\alpha_{n,i}}$. Then $\{x_n\}$ converges strongly to $P_{\cap_{i\in\Delta}F(T_i)}x_1$, where $P_{\cap_{i\in\Delta}F(T_i)}$ stands for the metric projection from E onto $\cap_{i\in\Delta} F(T_i)$.

Remark 3.12. Comparing with Theorem 3.1 in Martinez-Yanes and Xu [30], we have the following:

(a) improve the mapping from nonexpansive mappings to asymptotically quasi-nonexpansive mappings;
(b) improve the mapping from a single mapping to a family of mappings;
(b) the hybrid projection in Corollary 3.1 is different with the one in [30].

For a single mapping, we obtain from Corollary 3.11 the following.

Corollary 3.13. *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let $T : C \to C$ a closed, asymptotically regular, and generalized asymptotically quasi-nonexpansive mapping with the sequences $\{\mu_n\}$, and $\{\xi_n\}$. Assume that $F(T)$ is nonempty, and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_1 + (1-\alpha_n)T^n x_n, \quad n \geq 1, \\ C_{n+1} = \left\{ u \in C_n : \left\| u - y_n \right\|^2 \leq \|u - x_n\|^2 + \alpha_n M + \xi_n \right\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases}$$

where $M = \sup_{z\in F(T)}\{\|z- x_1\|^2\}$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Assume that $\mu_n \leq \frac{\alpha_n}{1-\alpha_n}$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_1$, where $P_{F(T)}$ stands for the metric projection from E onto $F(T)$.

4.Applications

First, we consider the problem of approximating a common minimizer of a family of proper, lower semicontinuous, and convex functionals.

Let E be a Banach space with the dual E^*. For a proper lower semicontinuous convex function $f : E \to (-\infty,\infty]$, the subdifferential mapping $\partial f \subset E \times E^*$ of f is defined by

$$\partial f(x) = \left\{ x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \forall y \in E \right\}, \quad \forall x \in E.$$

Rockafellar [42] proved that ∂f is a maximal monotone operator. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x\in E} f(x)$.

Theorem 4.1. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let Δ be an index set, and $f_i : C \to C$ a proper, lower semicontinuous, and convex functionals, for every $i \in \Delta$. Assume that $\cap_{i\in\Delta} (\partial f_i)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1}x_0, \\ z_{n,i} = \arg\min_{z\in E} \left\{ f_i(z) + \frac{\|z\|^2}{2r_i} + \frac{\langle z, Jx_n\rangle}{2r_i} \right\}, \\ y_{n,i} = J^{-1}\left(\alpha_{n,i}Jz_{n,i} + (1-\alpha_{n,i})Jx_n\right), \quad n \geq 1, \\ C_{n+1,i} = \left\{ u \in C_{n,i} : \phi(u, y_{n,i}) \leq \phi(u, x_n) \right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $r_i > 0$, $\forall i \in \Delta$, *and* $\{\alpha_{n,i}\}$ *are sequences in* (0,1] *such that* $\liminf_{n\to\infty} \alpha_{n,i} > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Delta}(\partial f_i)^{-1}(0)}x_0$, *where* $\Pi_{\cap_{i\in\Delta}(\partial f_i)^{-1}(0)}$*stands for the generalized projection from* E *onto* $\cap_{i\in\Delta}(\partial f_i)^{-1}(0)$.

Proof. For each $r_i > 0$, and $x \in E$, we see that there exists a unique $x_{r_i} \in D(\partial f_i)$ such that $Jx \in Jx_{r_i} + r_i\partial f_i(x_{r_i})$, where $x_{r_i} = (J + r_i\partial f_i)^{-1}Jx$. Notice that

$$z_{n,i} = \arg\min_{z\in E}\left\{f_i(z) + \frac{\|z\|^2}{2r_i} + \frac{\langle z, Jx_n\rangle}{r_i}\right\},$$

is equivalent to

$$0 \in \partial\left(f_i + \frac{\|\cdot\|^2}{2r_i} + \frac{Jx_n}{r_i}\right)z_{n,i} = \partial f_i(z_{n,i}) + \frac{Jz_{n,i}}{r_i} + \frac{Jx_n}{r_i}.$$

This shows that $z_{n,i} = (J + r_i\partial f_i)^{-1}Jx_n$. In view of the Example 2.3 in Qin et al. [41], we find that $(J + r_i\partial f_i)^{-1}J$ is closed quasi-φ-nonexpansive with $F((J+ r_i\partial f_i)^{-1}J) = (\partial f_i)^{-1}(0)$.

Following the proof of Theorem 3.1, we can immediately conclude the desired conclusion.

Theorem 4.2. *Let* E *be a reflexive, strictly convex, and smooth Banach space such that both* E *and* E^* *have Kadec-Klee property. Let* C *be a nonempty, closed, and convex subset of* E. *Let* Δ *be an index set, and* $f_i : C \to C$ *a proper, lower semicontinuous, and convex functionals, for every* $i \in \Delta$. *Assume that* $\cap_{i\in\Delta}(\partial f_i)^{-1}(0)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta}C_{1,i}, \\ x_1 = \Pi_{C_1}x_0, \\ z_{n,i} = \arg\min_{z\in E}\left\{f_i(z) + \frac{\|z\|^2}{2r_i} + \frac{\langle z, Jx_n\rangle}{2r_i}\right\}, \\ y_{n,i} = J^{-1}\left(\alpha_{n,i}Jx_1 + (1-\alpha_{n,i})Jx_{n,i}\right), \quad n \geq 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leq \alpha_{n,i}\phi(u, x_n) + (1-\alpha_{n,i})\phi(u, x_n)\right\}, \\ C_{n+1} = \cap_{i\in\Delta}C_{n+1,i} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $r_i > 0$, *and* $\{\alpha_{n,i}\}$ *are sequences in* (0,1) *such that* $\lim_{n\to\infty} \alpha_{n,i} = 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\cap_{i\in\Delta}(\partial f_i)^{-1}(0)}x_1$, *where* $\Pi_{\cap_{i\in\Delta}(\partial f_i)^{-1}(0)}$*stands for the generalized projection from* E *onton* $\cap_{i\in\Delta}(\partial f_i)^{-1}(0)$.

Proof. We easily find from Theorems 3.8 and 4.1 the conclusion.

Second, we consider the problem of approximating a solution of a family of variational inequalities.

Let C be a nonempty, closed, and convex subset of a Banach space E. Let E^* be the dual space of E. let $A : C \to E^*$ be a single valued monotone operator which is hemicontinuous; that is, continuous along each line segment in C with respect to the weak* topology of E^*.

Consider the following variational inequality problem of finding a point $x \in C$ such that

$$\langle y - x, Ax\rangle \geq 0, \quad \forall y \in C.$$

In this chapter, we use $VI(C, A)$ to denote the solution set of the variational inequality involving A. The symbol $N_C(x)$ stand for the normal cone for C at a point $x \in C$; that is,

$$N_C(x) = \left\{x^* \in E^* : \left\langle y - x, x^*\right\rangle \le 0,\ \forall y \in C\right\}.$$

Theorem 4.3. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let Δ be an index set, and $A_i : C \to E^*$ a single valued, monotone and hemicontinuous operator. Assume that $\cap_{i\in\Delta} VI(C, A_i)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i} \\ x_1 = \Pi_{C_1} x_0, \\ z_{n,i} = VI\left(C, A_i + \frac{1}{r_i}(J - Jx_n)\right), \\ y_{n,i} = J^{-1}\left(\alpha_{n,i} J z_{n,i} + (1-\alpha_{n,i}) J x_n\right), \quad n \ge 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \le \phi(u, x_n)\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases}$$

where $\{\alpha_{n,i}\}$ are sequences in (0,1] *such that* $\liminf_{n\to\infty} \alpha_{n,i} > 0$. *Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta} VI(C,A_i)} x_0$, where $\Pi_{\cap_{i\in\Delta} VI(C,A_i)}$stands for the generalized projection from E onto $\cap_{i\in\Delta} VI(C,A_i)$.*

Proof. Define a mapping $T_i \subset E \times E^*$ by

$$T_i x = \begin{cases} A_i x + N_C r, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

By Rockafellar [42], we know that T_i is maximal monotone, and $T_i^{-1}(0) = VI(C, A_i)$. For each $r_i > 0$, and $x \in E$, we see that there exists a unique $x_{r_i} \in D(T_i)$ such that $Jx \in Jx_{r_i} + r_i T_i\left(x_{r_i}\right)$, where $x_{r_i} = \left(J + r_i T_i\right)^{-1} Jx$. Notice that

$$z_{n,i} = VI\left(C, A_i + \frac{1}{r_i}(J - Jx_n)\right),$$

which is equivalent to

$$\left\langle y - z_{n,i}, A_i z_{n,i} + \frac{1}{r_i}\left(Jz_{n,i} - Jx_n\right)\right\rangle \ge 0, \quad \forall y \in C,$$

that is,

$$-A_i z_{n,i} + \frac{1}{r_i}\left(Jx_n - Jz_{n,i}\right) \in N_C(z_{n,i}).$$

This implies that $z_{n,i} = (J + r_i T_i)^{-1} Jx_n$. In view of the Example 2.3 in Qin et al. [41], we find that $\left(J + r_i \partial f_i\right)^{-1} J$ is closed quasi-φ-nonexpansive with $F\left(\left(J + r_i \partial f_i\right)^{-1} J\right) = T_i^{-1}(0)$.

Following the proof of Theorem 3.1, we can immediately conclude the desired conclusion.

Theorem 4.4. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let Δ be an index set, and $A_i : C \to E^*$ a single valued, monotone and hemicontinuous operator. Assume that $\cap_{i\in\Delta} VI(C, A_i)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, \\ C_1 = \cap_{i\in\Delta} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ z_{n,i} = VI\left(C, A_i + \frac{1}{r_i}(J - Jx_n)\right), \\ y_{n,i} = J^{-1}\left(\alpha_{n,i} Jx_1 + (1-\alpha_{n,i}) Jz_{n,i}\right), \quad n \geq 1, \\ C_{n+1,i} = \left\{u \in C_{n,i} : \phi(u, y_{n,i}) \leq \alpha_{n,i}\phi(u, x_1) + (1-\alpha_{n,i})\phi(u, x_n)\right\}, \\ C_{n+1} = \cap_{i\in\Delta} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where $r_i > 0$, and $\{\alpha_{n,i}\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_{n,i} = 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Delta} VI(C,A_i)} x_0$, where $\Pi_{\cap_{i\in\Delta} VI(C,A_i)}$ stands for the generalized projection from E onto $\cap_{i\in\Delta} VI(C, A_i)$.

Proof. We easily find from Theorems 3.8 and 4.3 the conclusion.

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